ON RESIDUALITIES IN THE SET OF MARKOV OPERATORS ON $C_1$

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Abstract. We show that the set of those Markov operators on the Schatten class $C_1$ such that $\lim_{n \to \infty} \|P^n - Q\| = 0$, where $Q$ is one-dimensional projection, is norm open and dense. If we require that the limit projections must be on strictly positive states, then such operators $P$ form a norm dense $G_δ$. Surprisingly, for the strong operator topology operators the situation is quite the opposite.

1. Introduction

The Baire category theorem has a long history in ergodic theory. The first proof (see [11] for all the details) that there are nonmixing but weakly mixing transformations was based on this theorem (other constructive methods followed but were more complicated). Some time later Baire methods were successfully applied (compare [7], [8], [18]) in the ergodic theory of Markov operators defined on $L^1(\mu)$ (i.e. such that $Pf \geq 0$ and $\int Pfd\mu = \int fd\mu$ for all nonnegative $f$). Baire type considerations usually bring easy answers to existence problems (see for instance [4], [5], [12], [13], [14] for recent applications). Typical questions concern the size of a specific class of operators (ergodic, conservative, with convergent iterations). Similarly as in [13], we will show in a noncommutative environment that the answers depend heavily on the point of view, i.e. on the choice of topology. We shall see that the set of mixing operators is meager in the strong operator topology but it is residual in the norm topology (even if we require a very fast, exponential, rate of mixing in the operator norm).

We begin our paper by introducing Markov operators on the simplest noncommutative von Neumann algebra of all bounded operators on a separable Hilbert space. As was pointed out by the referee, this topic has recently attracted the attention of specialists and some of our results should hold true for general von Neumann algebras with separable preduals. The reader is referred to [1], [9] and [10] for details and further references.

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a separable (infinite-dimensional) complex Hilbert space. As usual the norm is denoted by $\| \cdot \|$ and the Banach algebra of linear and bounded
operators on \((\mathcal{H}, \| \cdot \|)\) is denoted by \(\mathcal{L}(\mathcal{H})\). Without confusion the operator norm in \(\mathcal{L}(\mathcal{H})\) will be denoted by \(\| \cdot \|\) too. This paper is devoted to linear operators acting on an ordered Banach space of trace-class operators on \(\mathcal{H}\). Let us recall some standard concepts and notation from the theory of linear operators on Hilbert spaces. The reader is referred to any standard book on operators on Hilbert spaces (for instance \([6], [15], [16], [17]\) or \([20]\)). The adjoint operator to \(A\) is denoted by \(A^*\). An operator \(A \in \mathcal{L}(\mathcal{H})\) is called hermitian if \(A = A^*\), i.e. \(\langle Ax, y \rangle = \langle x, Ay \rangle\) holds for all \(x, y \in \mathcal{H}\). Equivalently, an operator \(A\) is hermitian if \(\langle Ax, x \rangle \in \mathbb{R}\) for any \(x \in \mathcal{H}\) (see \([6]\)). Moreover, if \(\langle Ax, x \rangle \in [0, \infty)\) holds for all \(x \in \mathcal{H}\), then we say that \(A\) is positive. Clearly, positive operators on \(\mathcal{H}\) may be easily verified that whenever \(\|\cdot\|\) is preserved:

\[
\forall x, y \in \mathcal{H}, \quad \|Ax + By\| = \|A\| \|x\| + \|B\| \|y\|.
\]

Moreover, if \(\langle Ax, x \rangle\) is finite for all \(x \in \mathcal{H}\), then \(A\) is called a modulus of \(\|\cdot\|\), i.e. \(\|Ax\| \leq \|A\| \|x\|\) for all \(x \in \mathcal{H}\). An operator \(A\) is positive if \(\|Ax\|^2 = \langle Ax, x \rangle\) is finite for all \(x \in \mathcal{H}\). Clearly, positive operators on \(\mathcal{H}\) may be easily verified that whenever \(\|\cdot\|\) is preserved:

\[
\forall x, y \in \mathcal{H}, \quad \|Ax + By\| = \|A\| \|x\| + \|B\| \|y\|.
\]

We say that an operator \(A\) is compact if \(A = A^*\) and it is called a modulus of \(\|\cdot\|\), i.e. \(\|Ax\| \leq \|A\| \|x\|\) for all \(x \in \mathcal{H}\). An operator \(A\) is called hermitian if \(\langle Ax, y \rangle = \langle x, Ay \rangle\) holds for all \(x, y \in \mathcal{H}\). Equivalently, an operator \(A\) is hermitian if \(\langle Ax, x \rangle \in \mathbb{R}\) for any \(x \in \mathcal{H}\) (see \([6]\)). Moreover, if \(\langle Ax, x \rangle \in [0, \infty)\) holds for all \(x \in \mathcal{H}\), then we say that \(A\) is positive. Clearly, positive operators on \(\mathcal{H}\) may be easily verified that whenever \(\|\cdot\|\) is preserved:

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\[
\forall x, y \in \mathcal{H}, \quad \|Ax + By\| = \|A\| \|x\| + \|B\| \|y\|.
\]
**Definition 1.1.** A positive operator $X$ from $C_1$ is called a state if $\text{tr}(X) = 1$. The set of all states is denoted by $S$.

It is easy to verify that $S$ is a convex and closed subset of $C_1$ for the weak topology (hence for both operator and trace norms). By a direct inspection it can be shown that it is not closed for the weak* topology (if $\dim H = \infty$).

**Definition 1.2.** A bounded linear operator $P : C_1 \to C_1$ is said to be positive if $P(C_1) \subseteq C_1$. A positive operator $P$ is called Markov (markovian) if for every $X \in C_1$, we have $\|P(X)\|_1 = \|X\|_1$ (equivalently we may say that $P(S) \subseteq S$). The set of all markovian operators on $C_1$ is denoted by $S$.

There are several natural topologies used in studying the geometry of the set $S$ (and its subsets). First of all we have a norm operator topology inherited from the Banach space $\mathcal{L}(C_1, C_1)$ of all bounded linear operators on $C_1$. Again the norm in $\mathcal{L}(C_1, C_1)$ is denoted simply by $\|\cdot\|$. Additionally we have

1. The strong operator topology (s.o.t.) is defined by the base sets
   \[
   \{P \in \mathcal{L}(C_1, C_1) : \|P(X_j) - P_0(X_j)\|_1 < \varepsilon, \quad j = 1, \ldots, n\}, \quad n \in \mathbb{N},
   \]
   where $X_1, X_2, \ldots$ are dense in $C_1$.

2. The weak operator topology (w.o.t.) is defined by the base sets
   \[
   \{P \in \mathcal{L}(C_1, C_1) : |\text{tr}(P(X_j) - P_0(X_j)A_j)| < \varepsilon, \quad j = 1, \ldots, n\}, \quad n \in \mathbb{N},
   \]
   where $X_1, X_2, \cdots \in C_1$ and $A_1, A_2, \cdots \in \mathcal{L}(H)$ are dense.

3. The weak* operator topology (w*.o.t.) is defined by the base sets
   \[
   \{P \in \mathcal{L}(C_1, C_1) : |\text{tr}(A_j(P(X_j) - P_0(X_j)))| < \varepsilon, \quad j = 1, \ldots, n\}, \quad n \in \mathbb{N},
   \]
   where $X_1, X_2, \cdots \in C_1$ and $A_1, A_2, \cdots \in C_0$ are dense.

The following result is obvious, so it is left without a proof. Namely:

**Lemma 1.3.** The set $S$ of all markovian operators on $C_1$ is a convex and w.o.t. closed subsemigroup of $\mathcal{L}(C_1, C_1)$. However it is not closed for the w*.o.t.

Now we give a few examples of Markov operators.

**Example 1.4.** Let $U$ be a unitary operator on $\mathcal{H}$. Define $P(X) = U^*XU$ and $Q(X) = UXU^*$. Clearly both $P$ and $Q$ are markovian. Moreover, they are invertible isometries of $C_1$.

**Example 1.5.** Let $V$ be a linear contraction (onto) of $\mathcal{H}$ such that $V^*$ is isometric. Similarly as above we define $R(X) = V^*XV$. It is easy to check that $R$ is a markovian (noninvertible in general) operator on $C_1$.

**Example 1.6.** It follows from the above lemma that any convex combination
\[
\sum_j \alpha_j P_j + \sum_k \beta_k Q_k + \sum_l \gamma_l R_l
\]
is markovian as long as $\alpha_j, \beta_k, \gamma_l \geq 0$ and $\sum_j \alpha_j + \sum_k \beta_k + \sum_l \gamma_l = 1$. A slight modification gives
\[
\int P(s)d\nu(s) \in S
\]
whenever all $P(s) \in S$ and the integral over a probabilistic measure $\nu$ is properly defined.
2. Norm residuality

This paper is devoted to geometric properties of sets of operators \( P \in S \) such that \( \lim_{n \to \infty} P^n \) exists and is rank one (i.e. \( P \) is mixing). Of course we have different kinds of mixing depending on considered topologies. We begin with the strongest case, the norm mixing. Moreover, the convergence holds with an exponential rate. Even though the ideas for our first result come from \cite{19} (see also \cite{3} and \cite{4}), for the completeness of the paper (and convenience of the reader) we have decided to include a detailed proof.

Lemma 2.1. Let \( P \) be a Markov operator on \( C_1 \). Then the following conditions are equivalent:

(i) there exist a one-dimensional projection \( Q_{X_*} \in S \) (i.e. \( Q_{X_*}(X) = \text{tr}(X)X_* \) for some \( X_* \in S \)) and constants \( C > 0, 0 < a < 1 \) such that

\[
\|P^n - Q_{X_*}\| < Ca^n \quad \text{for} \quad n \in \mathbb{N},
\]

(ii) there exists a one-dimensional projection \( Q_{X_*} \in S \) such that

\[
\lim_{n \to \infty} \|P^n - Q_{X_*}\| = 0,
\]

(iii) for each \( \varepsilon > 0 \) there exists an index \( n_0 \) such that for all \( X_1, X_2 \in S \) we have

\[
\|P^{n_0}(X_1) - P^{n_0}(X_2)\|_1 < \varepsilon,
\]

(iv) there exists an index \( n_0 \) such that

\[
\lambda = \sup_{X_1, X_2 \in S} \|P^{n_0}(X_1) - P^{n_0}(X_2)\|_1 < 2.
\]

Proof. We easily check that \( \|Q_{X_*}(X)\|_1 = \|\text{tr}(X)X_*\|_1 = \|\text{tr}(X)\|_1 = 1 \) and \( \langle Q_{X_*}(X)x, x \rangle = \langle \text{tr}(X)X_*x, x \rangle = \langle \text{tr}(X)x, x \rangle \geq 0 \) for all \( x \in \mathcal{H} \) and \( X \in C_1. \) Therefore \( Q_{X_*} \in S. \)

(i) \( \Rightarrow \) (ii) Since \( 0 \leq \|P^n - Q_{X_*}\| \leq Ca^n \) and \( a < 1, \) thus the convergence \( \lim_{n \to \infty} \|P^n - Q_{X_*}\| = 0 \) is obvious.

(ii) \( \Rightarrow \) (iii) Given \( \varepsilon > 0, \) let \( \varepsilon_1 = \frac{\varepsilon}{2}. \) There exists \( n_0 \) such that for all \( n \geq n_0 \) we have \( \|P^n - Q_{X_*}\| < \varepsilon_1. \) Now, let \( X_1, X_2 \in S \) be arbitrary. Then

\[
\|P^{n_0}(X_1) - P^{n_0}(X_2)\|_1 = \|P^{n_0}(X_1) - Q_{X_*}(X_1) + Q_{X_*}(X_2) - Q_{X_*}(X_2)\|_1 \\
\leq \|P^{n_0}(X_1) - Q_{X_*}(X_1)\|_1 + \|P^{n_0}(X_2) - Q_{X_*}(X_2)\|_1 + \|Q_{X_*}(X_1) - Q_{X_*}(X_2)\|_1 \\
< \varepsilon_1 + \varepsilon_1 = \varepsilon.
\]

(iii) \( \Rightarrow \) (iv) is obvious.

(iv) \( \Rightarrow \) (i) In this part of our proof we repeat some arguments used in \cite{3} and now adapted to the noncommutative case.

Let us note that for all states \( X_1, X_2 \in S \) we have

\[
\text{tr}(X_1 - X_2)_+ - \text{tr}(X_1 - X_2)_- = \text{tr}(X_1 - X_2) = \text{tr}(X_1) - \text{tr}(X_2) = 0.
\]

It follows from the above that

\[
\|X_1 - X_2\|_1 = \text{tr}|X_1 - X_2| \\
= \text{tr}(X_1 - X_2)_+ + \text{tr}(X_1 - X_2)_- = \text{tr}|(X_1 - X_2)_+| + \text{tr}|(X_1 - X_2)_-| \\
= 2\text{tr}|(X_1 - X_2)_+| = 2\|(X_1 - X_2)_+\|_1.
\]
Because \( P^{n_0}(X_1), P^{n_0}(X_2) \in S \) we get
\[
||P^{n_0}(X_1) - P^{n_0}(X_2)||_1 = 2||P^{n_0}(X_1) - P^{n_0}(X_2)||_1.
\]
Therefore, \( ||P^{n_0}(X_1) - P^{n_0}(X_2)||_1 = \frac{1}{2}||P^{n_0}(X_1) - P^{n_0}(X_2)||_1 \) for all \( X_1, X_2 \in S \). Since all iterates \( P^n \) are contractions, thus
\[
\begin{align*}
||P^{2n_0}(X_1) - P^{2n_0}(X_2)||_1 &= \frac{1}{2}||P^{n_0}(P^{n_0}(X_1)) - P^{n_0}(P^{n_0}(X_2))||_1 \\
&= \frac{1}{2}||P^{n_0}(P^{n_0}(X_1) - P^{n_0}(X_2))||_1 \\
&= \frac{1}{2}||P^{n_0}([P^{n_0}(X_1) - P^{n_0}(X_2)]_+ - [P^{n_0}(X_1) - P^{n_0}(X_2)]_-)||_1 \\
&= \frac{1}{2}||P^{n_0}(X_1) - P^{n_0}(X_2)||_1 \\
&\leq \lambda ||P^{n_0}(X_1) - P^{n_0}(X_2)||_1 \\
&\leq \frac{\lambda}{2}||X_1 - X_2||_1 = \beta ||X_1 - X_2||_1,
\end{align*}
\]
where by (iv) \( \beta = \frac{\lambda}{2} < 1 \). We conclude that (so far only on \( S \)) the mapping \( P \) is eventually a strict contraction. Clearly \( S \) is a complete metric space, as it is a closed subset of a Banach space \( C_1 \). Applying the Banach fixed point theorem, there exists a unique \( P \)-invariant state \( X_* \in S \), such that
\[
\lim_{n \to \infty} ||P^n(X) - X_*||_1 = 0,
\]
where \( X \in S \) is arbitrary. Let us take a one-dimensional Markov operator (projection) \( Q_{X_*}(X) = \text{tr}(X)X_* \). We note that \( PQ_{X_*} = Q_{X_*}P = Q_{X_*} \). For an arbitrary \( k \in \mathbb{N} \) the above method yields
\[
||P^{nk}(X_1) - P^{nk}(X_2)||_1 \\
\leq ||P^{nk}(X_1) - P^{nk}(X_2)||_1 \cdot \|P^{n(k-1)}(Y_1) - P^{n(k-1)}(Y_2)||_1 \\
\leq \beta \cdot ||P^{n(k-1)}(Y_1) - P^{n(k-1)}(Y_2)||_1,
\]
where
\[
Y_1 = \frac{[P^{n_0}(X_1) - P^{n_0}(X_2)]_+}{||P^{n_0}(X_1) - P^{n_0}(X_2)||_1} \quad \text{and} \quad Y_2 = \frac{[P^{n_0}(X_1) - P^{n_0}(X_2)]_-}{||P^{n_0}(X_1) - P^{n_0}(X_2)||_1}.
\]
Substituting \( X_2 = X_* \) and iterating the above estimation we easily get
\[
||P^n(X_1) - Q_{X_*}(X_1)||_1 = ||P^n(X_1) - X_*||_1 \\
= ||P^n(X_1) - P^n(X_*)||_1 \leq C\alpha^n,
\]
for all \( X_1 \in S \), where \( C = \frac{2}{\beta} \) and \( \alpha = \beta^{1/k_0} \).
Finally let us consider general $Z \in C_1$. Note that $Z = Z_1 + iZ_2$ where $Z_1, Z_2$ are self-adjoint and $Z_1 = Z_{1+} - Z_{1-}, Z_2 = Z_{2+} - Z_{2-}$. Obviously $\|Z_1\|_1 + \|Z_2\|_1 \leq 2\|Z\|_1$. Now
\[
P^n(Z) = P^n(Z_{1+}) - P^n(Z_{1-}) + iP^n(Z_{2+}) - iP^n(Z_{2-})
\]
\[
= \|Z_{1+}\|_1 P^n \left( \frac{Z_{1+}}{\|Z_{1+}\|_1} \right) - \|Z_{1-}\|_1 P^n \left( \frac{Z_{1-}}{\|Z_{1-}\|_1} \right) + i\|Z_{2+}\|_1 P^n \left( \frac{Z_{2+}}{\|Z_{2+}\|_1} \right) - i\|Z_{2-}\|_1 P^n \left( \frac{Z_{2-}}{\|Z_{2-}\|_1} \right).
\]

To obtain (i) we write
\[
\|P^n(Z) - Q_{X*}(Z)\|_1 \leq \|P^n(Z_{1+}) - Q_{X*}(Z_{1+})\|_1 + \|P^n(Z_{1-}) - Q_{X*}(Z_{1-})\|_1
+ \|P^n(Z_{2+}) - Q_{X*}(Z_{2+})\|_1 + \|P^n(Z_{2-}) - Q_{X*}(Z_{2-})\|_1
\]
\[
= \|Z_{1+}\|_1 \|P^n \left( \frac{Z_{1+}}{\|Z_{1+}\|_1} \right) - X_\star\|_1 + \|Z_{1-}\|_1 \|P^n \left( \frac{Z_{1-}}{\|Z_{1-}\|_1} \right) - X_\star\|_1
+ \|Z_{2+}\|_1 \|P^n \left( \frac{Z_{2+}}{\|Z_{2+}\|_1} \right) - X_\star\|_1 + \|Z_{2-}\|_1 \|P^n \left( \frac{Z_{2-}}{\|Z_{2-}\|_1} \right) - X_\star\|_1
\]
\[
\leq 2C\alpha^n \|Z\|_1.
\]

\[\square\]

**Definition 2.2.** We say that a Markov operator $P \in S$ is norm mixing if one of the conditions of Lemma 2.1 is satisfied for some $n$ and some $\varepsilon < 2$. The family of all norm mixing Markov operators is denoted by $S_{nm}$.

**Lemma 2.3.** The set $S_{nm}$ is a norm operator topology dense subset of $S$.

**Proof.** Given a state $Y$, an arbitrary $P \in S$, and $0 < \varepsilon < 1$, consider a convex combination
\[
P_\varepsilon(X) = (1 - \varepsilon)P(X) + \varepsilon Q_Y(X).
\]
By convexity $P_\varepsilon \in S$. We will show that $P_\varepsilon \in S_{nm}$. For any pair of states $X_1, X_2$ we have
\[
\|P_\varepsilon(X_1 - X_2)\|_1 = \|P_\varepsilon(X_1) - P_\varepsilon(X_2)\|_1
= \|(1 - \varepsilon)P(X_1) + \varepsilon tr(X_1)Y - ((1 - \varepsilon)P(X_2) + \varepsilon tr(X_2)Y)\|_1
= \|(1 - \varepsilon)(P(X_1) - P(X_2)) + \varepsilon Y - \varepsilon Y\|_1 = (1 - \varepsilon)\|P(X_1) - P(X_2)\|_1
\]
\[
\leq (1 - \varepsilon)\|X_1 - X_2\|_1.
\]

By Lemma 2.1 (iv) we easily obtain $P_\varepsilon \in S_{nm}$. \[\square\]

Combining these two results we instantly get:

**Theorem 2.4.** The set $S_{nm}$ of all norm mixing Markov operators is a norm dense and open subset of $S$.

**Proof.** It remains to show that $S_{nm}$ is norm open. For this we notice that the set $S_{nm} = \{P \in S : \sup_{X_1, X_2 \in S} \|P^n(X_1) - P^n(X_2)\|_1 < \varepsilon\}$ is norm open. \[\square\]
In the next theorem we study the set of mixing Markov operators with strictly positive invariant states. Let us recall:

**Definition 2.5.** We say that a state \( X \in \mathcal{S} \) is strictly positive if for each nonzero \( x \in \mathcal{H} \) we have \( \langle Xx, x \rangle > 0 \) (or equivalently that eigenvectors of \( X \) span the whole space \( \mathcal{H} \), or that \( X \) is “1-1”). The set of all strictly positive states is denoted by \( \mathcal{S}_+ \). The set of all norm mixing Markov operators possessing a strictly positive invariant state is denoted by \( \mathcal{S}_{nm+} \).

In the next result we show that the set \( \mathcal{S}_{nm+} \) is still a large set in \( \mathcal{S} \). Namely we have

**Theorem 2.6.** The set \( \mathcal{S}_{nm+} \) is a dense \( G_δ \) subset of \( \mathcal{S} \) for the norm operator topology.

**Proof.** Let us choose an orthonormal basis \( e_1, e_2, \ldots \) in \( \mathcal{H} \). For a strictly positive state \( X \) we fix an orthonormal basis consisting of its eigenvectors \( e_1^X, e_2^X, \ldots \). By \( \pi_l^X \) we denote the orthonormal projection onto \( \text{lin}\{e_1^X, \ldots, e_l^X\} \). Let

\[
J_{k,X} = \min \left\{ 1 : \|\pi_l^X(x)\| > 1 - 1/k \text{ for all } x \in \text{lin}\{e_1, e_2, \ldots, e_k\}, \|x\| = 1 \right\}.
\]

We define

\[
\varepsilon_k(X) = \inf \left\{ \langle Xx, x \rangle : \|\pi_l^X(x)\| > 1/2, \|x\| = 1 \right\}.
\]

Note that \( \varepsilon_k(X) \to 0 \), as long as \( X \) is strictly positive, where \( k \) is arbitrary. Now let us consider an operator norm \( G_δ \) set

\[
\mathcal{A} = \bigcap_{N} \bigcup_{XN \in \mathcal{S}_+} \bigcup_{n \geq N} \{ P \in \mathcal{S} : \|P^n - Q_{X_N}\| < \frac{\varepsilon_N(X_N)}{2} \}.
\]

We will prove that \( \mathcal{A} = \mathcal{S}_{nm+} \). The inclusion \( \supseteq \) is obvious. In fact, if \( P \in \mathcal{S}_{nm+} \), then for any \( N \in \mathbb{N} \) it is sufficient to take \( X_N = X_\ast \), where \( X_\ast \) is a unique \( P \)-invariant and strictly positive state.

For the proof of the opposite inclusion \( \subseteq \) we first note that \( \mathcal{A} \subseteq \mathcal{S}_{nm} \). In fact, following our Lemma 2.1 it is enough to observe

\[
\sup_{X,Y \in \mathcal{S}}\|P^n(X) - P^n(Y)\|_1 = \sup_{X,Y \in \mathcal{S}}\|(P^n(X) - X_N) - (P^n(Y) - X_N)\|_1
\]

\[
= \sup_{X,Y \in \mathcal{S}}\|(P^n(X) - Q_{X_N}(X)) - (P^n(Y) - Q_{X_N}(Y))\|_1 \leq 2\frac{\varepsilon_N(X_N)}{2} \to 0.
\]

Let \( X_\ast = \lim_{n \to \infty} P^n(X) \) be a \( P \)-invariant state. Suppose that, on the contrary, there exists a normalized \( \psi \in \mathcal{H} \) such that \( X_\ast \psi = 0 \). It follows from our definition of \( J_{k,X} \) that

\[
\lim_{N \to \infty} \|\pi_{J_{N,X}_N}^X(\psi)\| = 1.
\]

Hence

\[
0 < \|X_\ast(\psi)\| = \|X_\ast(\psi) - X_N(\psi)\| \leq \|X_\ast - X_N\|_1 = \|(P^n - Q_{X_N})(X_\ast)\|_1
\]

\[
\leq \|P^n - Q_{X_N}\| < \frac{1}{2}\inf\{\langle X_N(x), x \rangle : \|x\| = 1, \|\pi_{J_{N,X}_N}^X(x)\| > 1/2\} \leq \frac{1}{2}\|X_N(\psi)\|
\]

when \( N \) is large enough, which is a contradiction. It follows that \( X_\ast \) is strictly positive.
3. Category for the strong operator topology

In this section we consider strong operator topology mixing. In comparison with the norm topology, here Markov operators with iterates converging to one-dimensional projections form a meager set. We start with

**Definition 3.1.** We say that a Markov operator $P$ on $S$ is almost mixing in the strong operator topology if for each pair of states $X_1, X_2 \in S$ we have

$$\lim_{n \to \infty} \|P^n(X_1) - P^n(X_2)\|_1 = 0.$$ 

The set of all almost mixing operators is denoted by $S_{sam}$. If moreover there exists $X^* \in S$ such that for all $X_1 \in S$ we have

$$\lim_{n \to \infty} \|P^n(X_1) - X^*\|_1 = 0,$$

then the operator $P$ is called strong operator topology mixing. The set of all s.o.t. mixing Markov operators is denoted by $S_{sm}$.

Clearly $S_{nm} \subset S_{sm} \subset S_{sam}$. Since the operator norm topology is stronger than the strong operator topology, thus we easily get the following.

**Lemma 3.2.** The set $S_{sam}$ is a strong operator topology dense $G_δ$ subset of $S$.

*Proof.* It remains to prove that $S_{sam}$ is an s.o.t. $G_δ$. For this recall that whenever $\mathcal{H}$ is separable (has countable orthonormal basis), then $C_1$ is separable too (finite-dimensional operators are $\| \cdot \|_1$ dense in $C_1$; see [17] or [20] for all the details). Choose $Y_1, Y_2, \ldots$ a countable dense family of states in $S$. It follows from the contraction argument that

$$\lim_{n \to \infty} \|P^n(X_1) - P^n(X_2)\|_1 = 0$$

holds for every pair of $X_1, X_2 \in S$ if and only if

$$\lim_{n \to \infty} \|P^n(Y_i) - P^n(Y_j)\|_1 = 0$$

for any pair $Y_i, Y_j$. Again, since all $P^n$ are contractions, thus the above convergence holds if and only if it holds for some subsequence. To end the proof it is enough to write

$$S_{sam} = \bigcap_{i,j} \bigcap_{k} \bigcap_{N} \bigcup_{n \geq N} \{ P \in S : \|P^n(Y_i) - P^n(Y_j)\|_1 < \frac{1}{k} \}.$$ 

\[ \square \]

The next two results show that for most of the operators in $S_{sam}$ their iterates do not converge (in the w*-o.t.) to a Markov operator.

**Lemma 3.3.** The set $S_0 = \{ P \in S : P^n \to 0 \text{ in the weak}^* \text{ operator topology} \}$ is a strong operator topology dense subset of $S$.

*Proof.* Let us consider an orthonormal basis $e_1, e_2, e_3, \ldots$ in $\mathcal{H}$. Set $E_N$ to be the orthogonal projection onto $\text{lin}\{e_1, \ldots, e_N\}$. Given $Y_1, Y_2, \ldots, Y_s \in S$ and fixed $P \in S$ we denote by

$$\mathcal{U} = \{ \tilde{P} \in S : \forall 1 \leq j \leq s \|P(Y_j) - \tilde{P}(Y_j)\|_1 < \varepsilon \}$$

an s.o.t. open neighborhood of $P$. We will find $\tilde{P} \in \mathcal{U}$ such that $\tilde{P}^n \to 0$ in the weak* operator topology. Let us choose $N \geq 1$ large enough and $\varepsilon'$ small enough so
that \(\|Y_j - E_N Y_j E_N\|_1 < \varepsilon',\) for all \(j = 1, \ldots, s\) and such that the Markov operator defined below belongs to \(U\). Namely, let us define
\[
\widetilde{P}(X) = (1 - \frac{\varepsilon}{3})E_N P(E_N X E_N) E_N \\
+ (1 - \frac{\varepsilon}{3}) \text{tr}((I - E_N)P(E_N X E_N)(I - E_N))(\cdot, e_{N+1}) e_{N+1} \\
+ \frac{\varepsilon}{3} \text{tr}(P(E_N X E_N))(\cdot, e_{N+1}) e_{N+1} + \sigma(I - E_N)X(I - E_N)\sigma^*,
\]
where \(\sigma : \mathcal{H} \to \mathcal{H}\) is given by \(\sigma(e_j) = e_{j+1}\) and \(\sigma^* = \begin{cases} 0, & j = 1, \\ e_{j-1}, & j > 1. \end{cases}\)

Denote \(\alpha(X) = \text{tr}((I - E_N)X(I - E_N)), \) for \(X \in \mathcal{S}.\) We have
\[
1 \geq \alpha(\widetilde{P}^{n+1}(X)) = (1 - \frac{\varepsilon}{3}) \text{tr}((I - E_N)E_N P(E_N \widetilde{P}^n(X) E_N) E_N(I - E_N)) \\
+ (1 - \frac{\varepsilon}{3}) \text{tr}((I - E_N)P(E_N \widetilde{P}^n(X) E_N)(I - E_N)) \cdot 1 \\
+ \frac{\varepsilon}{3} \text{tr}(P(E_N \widetilde{P}^n(X) E_N)) \cdot 1 \\
+ \text{tr}[(I - E_N)\sigma(I - E_N)\widetilde{P}^n(X)(I - E_N)\sigma^*(I - E_N)] \\
\geq \frac{\varepsilon}{3} \text{tr}(E_N \widetilde{P}^n(X) E_N) + \alpha(\widetilde{P}^n(X)) \\
= \frac{\varepsilon}{3} [1 - \text{tr}((I - E_N)\widetilde{P}^n(X)(I - E_N))] + \alpha(\widetilde{P}^n(X)) \\
= \frac{\varepsilon}{3} (1 - \alpha(\widetilde{P}^n(X))) + \alpha(\widetilde{P}^n(X)) = \frac{\varepsilon}{3} + (1 - \frac{\varepsilon}{3}) \alpha(\widetilde{P}^n(X)).
\]

Now by an elementary argument the sequence \(\alpha(\widetilde{P}^n(X)) \to 1\) if \(n \to \infty.\) We introduce \(\beta_{i,k,n} = (\widetilde{P}^n(X)e_i, e_k),\) where \(i, k \geq 1.\) The next three properties easily follow from our definitions of \(\widetilde{P}\) and the coefficient \(\alpha.\) The fourth case is discussed in full detail.

Case 1. For \(1 \leq i, k \leq N\) we have \(\lim_{n \to \infty} \beta_{i,k,n} = 0.\)

Case 2. For \(1 \leq i \leq N, k > N, n \geq 1\) we have \(\beta_{i,k,n} = 0.\)

Case 3. For \(1 \leq k \leq N, i > N, n \geq 1\) we have \(\beta_{i,k,n} = 0.\)

Case 4. If \(k, i > N\) we start with
\[
\langle \widetilde{P}^n(X)e_i, e_k \rangle = \langle (1 - \frac{\varepsilon}{3})E_N P(E_N \widetilde{P}^{n-1}(X) E_N) E_N e_i, e_k \rangle \\
+ (1 - \frac{\varepsilon}{3}) \text{tr}((I - E_N)P(E_N \widetilde{P}^{n-1}(X) E_N)(I - E_N))(e_i, e_{N+1}) (e_{N+1}, e_k) \\
+ \frac{\varepsilon}{3} \text{tr}(P(E_N \widetilde{P}^{n-1}(X) E_N))(e_i, e_{N+1}) (e_{N+1}, e_k) \\
+ \langle \sigma(I - E_N)\widetilde{P}^{n-1}(X)(I - E_N)\sigma^* e_i, e_k \rangle.
\]

It follows from cases 1–3 that in the above sum the first three terms tend to 0. It remains to show that
\[
\langle \sigma(I - E_N)\widetilde{P}^{n-1}(X)(I - E_N)\sigma^* e_i, e_k \rangle \to 0 \text{ when } n \to \infty.
\]

For this,
\[
\langle \sigma(I - E_N)\widetilde{P}^{n-1}(X)(I - E_N)\sigma^* e_i, e_k \rangle \\
= \langle \widetilde{P}^{n-1}(X)(I - E_N)e_{i-1}, (I - E_N)e_{k-1} \rangle \\
= \langle \widetilde{P}^{n-1}(X)e_{i-1}, e_{k-1} \rangle = \beta_{i-1,k-1,n-1}.
\]
Iterating this procedure we obtain
\[ \langle \tilde{P}^n(X)e_i, e_k \rangle = \beta_{i-l,k-l,n-t} + \gamma_{n,t}, \]
where \( \lim_{n \to \infty} \gamma_{n,t} = 0 \) for each \( l \geq 1 \). We have already noticed that
\[
\lim_{n \to \infty} \beta_{i-l,k-l,n-t} = 0, \quad \text{whenever } l \geq (i - N) \lor (k - N).
\]
The proof is completed. \( \square \)

**Lemma 3.4.** Let \( e_1, e_2, e_3, \ldots \) be a fixed orthonormal basis and \( Y_1, Y_2, \ldots \) a countable and dense (with respect to the norm \( \| \cdot \|_1 \)) family of states. Then the set
\[
\bigcap_k \bigcap N \bigcup_{n \geq N} \{ P \in S : \sum_{j=1}^N \langle P^n(Y_k)e_j, e_j \rangle < \frac{1}{N} \}
\]
is a w*-o.t. \( G_\delta \) and an s.o.t. dense subset of \( S \).

**Proof.** It follows directly from its representation that the above set is a w*-o.t. (hence s.o.t.) \( G_\delta \) subset of \( S \). By Lemma 3.3 it is dense in the strong operator topology. \( \square \)

Clearly the Markov operators from the above lemma do not enjoy invariant states. The final results of our paper show that the s.o.t. and norm topologies on \( S \) are very different from the perspective of Baire category. Markov operators with convergent iterates are residual for the norm topology but for the s.o.t. they are rare. Using terminology from the commutative ergodic theory we may say that “sweeping” Markov operators are generic for the s.o.t. Combining the last three lemmas we obtain:

**Theorem 3.5.** The set
\[
\{ P \in S_{sam} : \exists_{n_j} \lim_{j \to \infty} P^n_{n_j} = 0 \text{ in w*-o.t.} \}
\]
is a strong operator topology dense \( G_\delta \) subset of \( S \).

**Proof.** By Lemma 3.3 the above set is an s.o.t. dense subset of \( S \). Lemma 3.4 guarantees that it is s.o.t. \( G_\delta \). \( \square \)

We end the paper with

**Theorem 3.6.** The set \( \{ P \in S_{sam} : P \text{ has no invariant state} \} = S_{sam} \setminus S_{sm} \) is a strong operator topology dense \( G_\delta \) subset of \( S \).

**Proof.** It is obvious that an almost mixing (in the s.o.t.) Markov operator \( P \) has no invariant states if and only if \( P \) is not mixing. Since (by Lemma 3.2) \( S_{sam} \) is strong operator topology dense \( G_\delta \), thus the result follows from
\[
S_{sam} \setminus S_{sm} = S_{sam} \cap \bigcap_k \bigcap N \bigcup_{n \geq N} \{ P \in S : \sum_{j=1}^N \langle P^n(Y_k)e_j, e_j \rangle < \frac{1}{N} \},
\]
where \( e_1, e_2, \ldots \) and \( Y_1, Y_2, \ldots \) are the same as in Lemma 3.4. \( \square \)
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References


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