SPACES OF TYPE BLO FOR NON-DOUBLING MEASURES

YINSHENG JIANG

(Communicated by Andreas Seeger)

Abstract. The spaces of type BLO for the positive Radon measures satisfying a growth condition on $\mathbb{R}^d$ are introduced. It is shown that some properties which hold for the classical space BLO when $\mu$ is a doubling measure remain valid for the spaces of type BLO introduced in this paper, without assuming $\mu$ doubling.

Let $d, n$ be some fixed integers with $d \geq 2, 1 \leq n \leq d$, and let $\mu$ be a positive Radon measure on $\mathbb{R}^d$ satisfying the growth condition

$$\mu(B(x,r)) \leq C_0 r^n \quad \text{for all } x \in \mathbb{R}^d, r > 0.$$ 

We do not assume that $\mu$ is doubling.

A kernel $k(\cdot, \cdot) \in L^1_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, y) : x = y\})$ is called a Calderón-Zygmund kernel if

1. $|k(x, y)| \leq C |x - y|^n$,
2. there exists $0 < \delta \leq 1$ such that

$$|k(x, y) - k(x', y)| + |k(y, x) - k(y, x')| \leq C \frac{|x - x'|^{\delta}}{|x - y|^{n+\delta}} \quad \text{if } |x - x'| \leq |x - y|/2.$$ 

The Calderón-Zygmund operator associated to the kernel $k$ and the measure $\mu$ is formally defined as

$$Tf(x) = \int_{\mathbb{R}^d} k(x, y)f(y)d\mu(y).$$

We say that $T$ is bounded on $L^p(\mu)$ if the truncated operators

$$T_\epsilon f(x) = \int_{|x - y| > \epsilon} k(x, y)f(y)d\mu(y)$$

are bounded on $L^p(\mu)$ uniformly on $\epsilon > 0$.

The maximal operator $T_*$ associated with the Calderón-Zygmund operator $T$ is defined as

$$T_* f(x) = \sup_{\epsilon > 0} |T_\epsilon f(x)|$$

for $f \in L^p(\mu), p \in [1, \infty)$. 

The author was supported in part by NSFC Grant #10261007.

Received by the editors November 22, 2003 and, in revised form, March 24, 2004.

2000 Mathematics Subject Classification. Primary 42B20, 42B25, 42B35.

Key words and phrases. Non-doubling measure, RBLO($\mu$), RBMO($\mu$), Calderón-Zygmund maximal operator, maximal operator.
When \( \mu \) is a positive Radon measure satisfying the growth condition (1) and \( T \) is bounded on \( L^2(\mu) \), the weak type \((1,1)\) and \( L^p(\mu) \) estimates for \( T_\ast \) were obtained by F. Nazarov, S. Treil and A. Volberg in [4].

It is well known that if \( \mu \) is Lebesgue measure and \( T \) is bounded on \( L^2(\mathbb{R}^d) \), then \( T_\ast \) is bounded from \( L^\infty(\mathbb{R}^d) \) into \( BMO(\mathbb{R}^d) \) [5], and furthermore, is bounded from \( L^\infty(\mathbb{R}^d) \) into \( BLO(\mathbb{R}^d) \) [3]. For the positive Radon measure \( \mu \) satisfying (1), X. Tolsa [6] has introduced the spaces \( RBMO(\mu) \), which are the suitable substitutes for the classical spaces \( BMO(\mathbb{R}^d) \), and proved that if \( T \) is bounded on \( L^2(\mu) \), then \( T \) is bounded from \( L^\infty(\mu) \) into \( RBMO(\mu) \).

In this note, for the positive Radon measure \( \mu \) satisfying (1) we will introduce the spaces \( RBLO(\mu) \) as the substitutes for the classical spaces \( BLO(\mathbb{R}^d) \) defined by R. R. Coifman and R. Rochberg [2], and will show that if \( T \) is bounded on \( L^2(\mu) \), then \( T_\ast \) is bounded from \( L^\infty(\mu) \) into \( RBLO(\mu) \).

We take notation and definitions from [6]. By a cube \( Q \subset \mathbb{R}^d \) we will mean a \((4\sqrt{d},(4\sqrt{d})^{n+1})\)-doubling cube. Given a cube \( Q \subset \mathbb{R}^d \), let \( N \) be the smallest integer \( \geq 0 \) such that \( \alpha^N Q \) is doubling. Denote this doubling cube by \( \tilde{Q} \). If \( \alpha \) and \( \beta \) are not specified, by a doubling cube we will mean a \((4\sqrt{d},(4\sqrt{d})^{n+1})\)-doubling cube. Given two cubes \( Q \subset R, \) we set

\[
K_{Q,R} = 1 + \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^k Q)}{(2^k Q)^n},
\]

where \( N_{Q,R} \) is the first integer \( k \) such that \( 2^k Q \supset R \).

Let \( \rho > 1 \) be some fixed constant. Say that \( f \in L^1_{loc}(\mu) \) belongs to \( RBMO(\mu) \) if there exists some constant \( C_1 \) such that for any cube \( Q \),

\[
\frac{1}{\mu(\rho Q)} \int_Q |f - m_Q(f)| d\mu \leq C_1
\]

for any doubling cubes \( Q \subset R \),

\[
|m_Q(f) - m_R(f)| \leq C_1 K_{Q,R}
\]

for any two doubling cubes \( Q \subset R \),

where \( m_Q(f) = \mu(Q)^{-1} \int_Q f d\mu \). The best constant \( C_1 \) is the \( RBMO(\mu) \) norm of \( f \), which we denote as \( \|f\|_* \) and is independent of \( \rho > 1 \). It is shown by X. Tolsa [6] that one can replace (3) by

\[
\frac{1}{\mu(Q)} \int_Q |f - m_Q(f)| d\mu \leq C_2
\]

for any doubling cube \( Q \),

or by

\[
\left( \frac{1}{\mu(\rho Q)} \int_Q |f - m_Q(f)|^p d\mu \right)^{1/p} \leq C_3
\]

for \( 1 \leq p < \infty \) and any cube \( Q \),

and the best constants \( C_2 \) and \( C_3 \) are comparable to \( \|f\|_* \).

We are ready to define the spaces of type \( BLO \) for the positive Radon measure \( \mu \) satisfying (1).
Definition 1. We say that \( f \in L_{\text{loc}}^1(\mu) \) belongs to RBLO(\( \mu \)) if there exists some constant \( C_4 \) such that for any doubling cube \( Q \),
\[
\text{m}_Q(f) - \text{essinf}_{x \in Q} f(x) \leq C_4
\]
and
\[
m_Q(f) - m_R(f) \leq C_4 K_{Q,R} \text{ for any two doubling cubes } Q \subset R.
\]
The smallest constant \( C_4 \) will be denoted by \( \| f \|_{RBLO} \).

Observe that (7) and (8) are equivalent to (1) and (4). It is easy to check that \( L^\infty(\mu) \subset RBLO(\mu) \) with \( \| f \|_{RBLO} \leq 2 \| f \|_{L^\infty(\mu)} \) and \( RBLO(\mu) \subset RBMO(\mu) \) with \( \| f \|_p \leq 2 \| f \|_{RBLO} \).

C. Bennett [1] has obtained a criterion for the classical spaces \( BLO(\mathbb{R}^d) \). To give out the \( RBLO(\mu) \) criterion we consider the non-centered doubling maximal function
\[
Mf(x) = \sup_{Q\ni x, Q \text{ doubling}} m_Q(f) = \sup_{Q\ni x, Q \text{ doubling}} \frac{1}{\mu(Q)} \int_Q f \, d\mu,
\]
where the supremum extends over all doubling cubes \( Q \) containing \( x \). By the Lebesgue differentiation theorem, \( Mf(x) \geq f(x) \mu\text{-a.e. } x \in \mathbb{R}^d \). Moreover, \( |Mf(x)| \leq M(|f|)(x) := Nf(x) \). The maximal operator \( N \) is weak type \((1,1)\) and bounded on \( L^p(\mu) \) for \( 1 < p \leq \infty \) [6], so is \( M \).

Lemma 1. \( f \in RBLO(\mu) \) if and only if \( Mf - f \in L^\infty(\mu) \) and \( f \) satisfies (8).
Furthermore,
\[
\| Mf - f \|_{L^\infty(\mu)} \sim \| f \|_{RBLO}.
\]

Proof. Suppose first that \( f \in RBLO(\mu) \). Then \( f \) satisfies (8). Observe that (see [5] Remark 2.3), by the Lebesgue differentiation theorem, for \( \mu\text{-a.e. } x \in \mathbb{R}^d \) one can find a sequence of doubling cubes \( \{Q_k\}_k \) centered at \( x \) with \( l(Q_k) \to 0 \) such that
\[
\lim_{k \to \infty} \frac{1}{\mu(Q_k)} \int_{Q_k} f \, d\mu = f(x).
\]
Let \( x \) be any such point, and let \( Q \) be any doubling cube containing \( x \). Then \( f(x) \geq \text{essinf}_{x \in Q} f(x) \) and so
\[
m_Q(f) - f(x) \leq m_Q(f) - \text{essinf}_{x \in Q} f(x) \leq \| f \|_{RBLO}.
\]
Taking the supremum over all doubling cubes containing \( x \), we get
\[
Mf(x) - f(x) \leq \| f \|_{RBLO}.
\]
Hence \( Mf - f \in L^\infty(\mu) \) and \( \| Mf - f \|_{L^\infty(\mu)} \leq \| f \|_{RBLO} \).

Conversely, suppose \( Mf - f \in L^\infty(\mu) \) and let \( Q \) be any doubling cube in \( \mathbb{R}^d \). If any \( x \in Q \) is such that
\[
f(x) < m_Q(f) - \| Mf - f \|_{L^\infty(\mu)},
\]
then
\[
Mf(x) - f(x) \geq m_Q(f) - f(x) > \| Mf - f \|_{L^\infty(\mu)}.
\]
So, for \( \mu\text{-a.e. } x \in Q \),
\[
f(x) \geq m_Q(f) - \| Mf - f \|_{L^\infty(\mu)},
\]
and consequently,
\[ \operatorname{essinf}_{x \in Q} f(x) \geq m_Q(f) - \|Mf - f\|_{L^\infty(\mu)}. \]

Since \( f \) satisfies (5), we get that \( f \in \text{RBLO}(\mu) \) and
\[ \|f\|_{\text{RBLO}} \leq C\|Mf - f\|_{L^\infty(\mu)}. \]

Our main results are as follows.

**Theorem 1.** If the Calderón-Zygmund operator \( T \) is bounded on \( L^2(\mu) \), then the maximal operator \( T_* \) is bounded from \( L^\infty(\mu) \) into \( \text{RBLO}(\mu) \).

**Theorem 2.** If \( f \in \text{RBMO}(\mu) \) and \( Mf \) satisfies (5), then \( Mf \in \text{RBLO}(\mu) \) and
\[ \|Mf\|_{\text{RBLO}} \leq C\|f\|_{\text{RBMO}}. \]

In particular, \( M \) is bounded on \( \text{RBLO}(\mu) \).

**Theorem 3.** A locally integrable function \( f \) belongs to \( \text{RBLO}(\mu) \) if and only if there exist \( h \in L^\infty(\mu) \) and \( F \in \text{RBMO}(\mu) \) with \( MF \) satisfying (5) such that
\[ f = MF + h. \]

Furthermore,
\[ \|f\|_{\text{RBLO}} \sim \inf(\|F\|_{\text{RBMO}} + \|h\|_{L^\infty(\mu)}) \]
where the infimum extends over all representations of the form (11).

*The proof of Theorem 1.* Let \( x \in \mathbb{R}^d \cap \text{supp}(\mu) \) and \( Q \) be any doubling cube containing \( x \). For each fixed cube \( Q \) let \( B \) be the smallest ball centered at \( x \) which contains \( Q \). Then \( 2B \subset 4\sqrt{d}Q \). If \( f \in L^\infty(\mu) \cap L^{p_0}(\mu) \) for some \( p_0 \in [1, \infty) \), by \( L^2(\mu) \)-boundedness of \( T_* \) [3], we have
\[
\frac{1}{\mu(Q)} \int_Q T_*(f\chi_{2B})d\mu \leq \frac{1}{\mu(Q)^{1/2}} \left\{ \int_{\mathbb{R}^d} [T_*(f\chi_{2B})]^2 d\mu \right\}^{1/2} 
\leq \frac{C\|\mu(2B)^{1/2}\|}{\mu(Q)^{1/2}} \|f\|_{L^\infty(\mu)} 
\leq \frac{C\|\mu(4\sqrt{d}Q)^{1/2}\|}{\mu(Q)^{1/2}} \|f\|_{L^\infty(\mu)} 
\leq C\|f\|_{L^\infty(\mu)}. 
\]

(13)

From (2) it follows that
\[ T_*(f\chi_{R^d \setminus 2B})(x) \leq T_*f(x). \]

By this and the conditions of the Calderón-Zygmund kernel, for all \( y \in Q \), we have
\[
T_*(f\chi_{R^d \setminus 2B})(y) \leq |T_*(f\chi_{R^d \setminus 2B})(y) - T_*(f\chi_{R^d \setminus 2B})(x)| + T_*(f\chi_{R^d \setminus 2B})(x) 
\leq C\|f\|_{L^\infty(\mu)} + T_*f(x). 
\]

Therefore,
\[
\frac{1}{\mu(Q)} \int_Q T_*(f\chi_{R^d \setminus 2B})d\mu \leq C\|f\|_{L^\infty(\mu)} + T_*f(x). 
\]
From this and \[13\), we get

\[
\frac{1}{\mu(Q)} \int_Q T_s f \, d\mu \leq C \|f\|_{L^\infty(\mu)} + T_s f(x).
\]

So,

\[
\|M(T_s f) - T_s f\|_{L^\infty(\mu)} \leq C \|f\|_{L^\infty(\mu)}.
\]

From this and Lemma \[8\), it suffices to show that \(T_s f\) satisfies \(8\). We apply the argument analogous to \(9\) pp. 104-105]. Let \(Q \subset R\) be any two doubling cubes. Recall that \(N_{Q,R}\) is the first integer \(k\) such that \(2^k Q \supset R\). We denote \(Q_R = 2^{N_{Q,R}+1} Q\). Thus, for \(x \in Q\) and \(y \in R\), we set

\[
T_s f(x) = T_s(f \chi_{2Q})(x) + \sum_{k=1}^{N_{Q,R}} T_s(f \chi_{2^{k+1}Q \setminus 2^k Q})(x) + T_s(f \chi_{R \setminus Q_R})(x)
\]

\[
- \left( T_s(f \chi_{Q_R})(y) + T_s(f \chi_{R \setminus Q_R})(y) \right) + T_s f(y).
\]

Since

\[
|T_s(f \chi_{R \setminus Q_R})(x) - T_s(f \chi_{R \setminus Q_R})(y)| \leq C \|f\|_{L^\infty(\mu)},
\]

we get

\[
T_s f(x) \leq T_s(f \chi_{2Q})(x) + C \left( 1 + \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^{k+1}Q)}{l(2^{k+1}Q)^n} \right) \|f\|_{L^\infty(\mu)} + T_s(f \chi_{Q_R})(y) + T_s f(y).
\]

Now we take the mean over \(Q\) for \(x\), and over \(R\) for \(y\). We write

\[
m_R(T_s(f \chi_{Q_R})) \leq m_R(T_s(f \chi_{Q_R \setminus 2R})) + m_R(T_s(f \chi_{Q_R \setminus 2R})).
\]

Similar to the previous estimate \(13\) we obtain

\[
m_Q(T_s(f \chi_{2Q})) \leq C \|f\|_{L^\infty(\mu)}
\]

and

\[
m_R(T_s(f \chi_{Q_R \setminus 2R})) \leq C \|f\|_{L^\infty(\mu)}.
\]

On the other hand, since \(l(Q_R) \approx l(R)\), we have

\[
m_R(T_s(f \chi_{Q_R \setminus 2R})) \leq C \|f\|_{L^\infty(\mu)}.
\]

Therefore,

\[
m_Q(T_s f) - m_R(T_s f) \leq CK_{Q,R} \|f\|_{L^\infty(\mu)}.
\]

If \(f \notin L^p(\mu)\) for all \(p \in [1, \infty)\), then the integral \(\int_{|x-y| > \epsilon} k(x,y) f(y) \, d\mu(y)\) may not be convergent. The operator \(T_s\) can be extended to the whole space \(L^\infty(\mu)\) following the standard arguments: Given a cube \(Q_0\) centered at the origin with side length \(l(Q_0) > 3\epsilon\), we write \(f = f_1 + f_2\), with \(f_1 = f \chi_{2Q_0}\). For \(x \in Q_0\), we define

\[
T_s f(x) = T_s f_1(x) + \int_{|x-y| > \epsilon} (k(x,y) - k(y,0)) f_2(y) \, d\mu(y).
\]

Now both integrals in this equation are convergent. With arguments similar to the case \(f \in L^\infty(\mu) \cap L^p(\mu)\) we complete the proof. \(\square\)
To prove Theorem 2 and Theorem 3 we require the following lemma.

**Lemma 2.** If \( f \in RBMO(\mu) \), then for any doubling cube \( Q \),

\[
\frac{1}{\mu(Q)} \int_Q Mf d\mu \leq C\|f\|_* + \essinf_{x \in Q} Mf(x).
\]  
Moreover, if \( Mf \) is finite \( \mu \)-a.e., then

\[
\frac{1}{\mu(Q)} \int_Q Mf d\mu - \essinf_{x \in Q} Mf(x) \leq C\|f\|_*.
\]

*Proof.* Set \( Q \). To prove Theorem 2 and Theorem 3 we require the following lemma.

\[
\int_Q M((f - m_Q(f))\chi_{3Q}) d\mu
\]
\[
\leq \mu(Q)^{1/2} \left\{ \int_{\mathbb{R}^d} |M((f - m_Q(f))\chi_{3Q})|^2 d\mu \right\}^{1/2}
\]
\[
\leq C\mu(Q)^{1/2} \left\{ \int_{\mathbb{R}^d} |(f - m_Q(f))\chi_{3Q}|^2 d\mu \right\}^{1/2}
\]
\[
\leq C\mu(Q)^{1/2} \left( \left\{ \int_{3Q} |f - m_{3\overline{Q}}(f)|^2 d\mu \right\}^{1/2} + \left\{ \int_{3\overline{Q}} |m_Q(f) - m_{3\overline{Q}}(f)|^2 d\mu \right\}^{1/2} \right)
\]
\[
\leq C\mu(Q)^{1/2} \left( \mu((12\sqrt{3}/5)Q)^{1/2}\|f\|_* + \mu(3\overline{Q})^{1/2}\|f\|_* K_{Q,3\overline{Q}} \right)
\]
\[
\leq C\mu(Q)^{1/2} \mu(4\sqrt{d}Q)^{1/2}\|f\|_* \leq C\mu(Q)\|f\|_*.
\]

Next, we shall show that

\[
\frac{1}{\mu(Q)} \int_Q M(m_Q(f)\chi_{3Q} + f\chi_{\mathbb{R}^d \setminus 3Q}) d\mu \leq C\|f\|_* + \essinf_{x \in Q} Mf(x).
\]

It suffices to show that

\[
M(m_Q(f)\chi_{3Q} + f\chi_{\mathbb{R}^d \setminus 3Q})(x) \leq C\|f\|_* + \essinf_{x \in Q} Mf(x) \quad \mu\text{-a.e. } x \in Q.
\]

For this it will be enough to show that

\[
\frac{1}{\mu(R)} \int_R (m_Q(f)\chi_{3Q} + f\chi_{\mathbb{R}^d \setminus 3Q}) d\mu \leq C\|f\|_* + \essinf_{x \in Q} Mf(x)
\]

for any doubling cube \( R \ni x \). If \( R \subset 3Q \), the result follows immediately:

\[
\frac{1}{\mu(R)} \int_R (m_Q(f)\chi_{3Q} + f\chi_{\mathbb{R}^d \setminus 3Q}) d\mu = m_Q(f) \leq \essinf_{x \in Q} Mf(x).
\]

So, suppose \( R \cap (\mathbb{R}^d \setminus 3Q) \neq \emptyset \). Then \( l(R) > l(Q) \) and \( 3Q \subset 5R \). Write

\[
m_Q(f)\chi_{3Q} + f\chi_{\mathbb{R}^d \setminus 3Q} = (m_Q(f) - m_{5\overline{R}}(f))\chi_{3Q} + (f - m_{5\overline{R}}(f))\chi_{\mathbb{R}^d \setminus 3Q} + m_{5\overline{R}}(f).
\]
Obviously, \( m_{5R}^-(f) \leq \text{essinf } Mf(x) \). Further,

\[
\begin{align*}
\int_{5R} \left\{ (m_Q(f) - m_{5R}^-(f)) \chi_{5Q} + (f - m_{5R}^-(f)) \chi_{5R \setminus 3Q} \right\} d\mu \\
\leq \mu(3Q)m_Q(f) - m_{5R}^-(f) + \int_{5R} |f - m_{5R}^-(f)| \chi_{5R \setminus 3Q} d\mu \\
\leq \frac{\mu(4\sqrt{dQ})}{\mu(Q)} \int_Q |f - m_{5R}^-(f)| + \int_{5R \setminus 3Q} |f - m_{5R}^-(f)| d\mu \\
\leq C \int_{5R} |f - m_{5R}^-(f)| d\mu \\
\leq C \mu(A\sqrt{dR}) \|f\|_* \leq C\mu(R) \|f\|_*.
\end{align*}
\]

This establishes (18) and completes the proof of Lemma 2. \( \square \)

The proof of Theorem 2. By Lemma 2, it suffices to prove that if \( f \in RBLO(\mu) \), then \( Mf \) satisfies (8). In fact, applying Lemma 1 for any two doubling cubes \( Q \subset R \) we have

\[
m_Q(Mf) - m_R(Mf) \\
\leq m_Q(Mf) - m_Q(f) + m_R(Mf) - m_R(f) + |m_Q(f) - m_R(f)| \\
\leq 2\|Mf - f\|_{L^\infty(\mu)} + \|f\|_* K_{Q,R} \\
\leq 2\|f\|_{RBLO} + \|f\|_* K_{Q,R} \\
\leq C\|f\|_{RBLO} K_{Q,R}.
\]

As we remarked above, this completes the proof of Theorem 2. \( \square \)

Theorem 3 follows immediately from Lemma 1 and Theorem 2 and we omit the details here.

Acknowledgement

I would like to thank the referee for the helpful comment which improved the presentation of this paper.

References


Department of Mathematics, Xinjiang University, Urumqi, 830046, People’s Republic of China

E-mail address: ysjiang@xju.edu.cn