SPACES OF TYPE BLO FOR NON-DOUBLING MEASURES

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Abstract. The spaces of type BLO for the positive Radon measures satisfying a growth condition on \( \mathbb{R}^d \) are introduced. It is shown that some properties which hold for the classical space BLO when \( \mu \) is a doubling measure remain valid for the spaces of type BLO introduced in this paper, without assuming \( \mu \) doubling.

Let \( d, n \) be some fixed integers with \( d \geq 2, 1 \leq n \leq d \), and let \( \mu \) be a positive Radon measure on \( \mathbb{R}^d \) satisfying the growth condition

\[
\mu(B(x, r)) \leq C_0 r^n \quad \text{for all} \quad x \in \mathbb{R}^d, r > 0.
\]

We do not assume that \( \mu \) is doubling.

A kernel \( k(\cdot, \cdot) \in L^1_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, y) : x = y\}) \) is called a Calderón-Zygmund kernel if

1. \( |k(x, y)| \leq \frac{C}{|x - y|^n} \),

2. there exists \( 0 < \delta \leq 1 \) such that

\[
|k(x, y) - k(x', y)| + |k(y, x) - k(y, x')| \leq C \frac{|x - x'|^\delta}{|x - y|^{n+\delta}} \quad \text{if} \quad |x - x'| \leq |x - y|/2.
\]

The Calderón-Zygmund operator associated to the kernel \( k \) and the measure \( \mu \) is formally defined as

\[
Tf(x) = \int_{\mathbb{R}^d} k(x, y)f(y)d\mu(y).
\]

We say that \( T \) is bounded on \( L^p(\mu) \) if the truncated operators

\[
T_\epsilon f(x) = \int_{|x-y|>\epsilon} k(x, y)f(y)d\mu(y)
\]

are bounded on \( L^p(\mu) \) uniformly on \( \epsilon > 0 \).

The maximal operator \( T_\ast \) associated with the Calderón-Zygmund operator \( T \) is defined as

\[
T_\ast f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|
\]

for \( f \in L^p(\mu), p \in [1, \infty) \).
When $\mu$ is a positive Radon measure satisfying the growth condition (1) and $T$ is bounded on $L^2(\mu)$, the weak type $(1,1)$ and $L^p(\mu)$ estimates for $T_\ast$ were obtained by F.Nazarov, S.Treil and A.Volberg in [3].

It is well known that if $\mu$ is Lebesgue measure and $T$ is bounded on $L^2(\mathbb{R}^d)$, then $T_\ast$ is bounded from $L^\infty(\mathbb{R}^d)$ into $\text{BMO}(\mathbb{R}^d)$ [5], and furthermore, is bounded from $L^\infty(\mathbb{R}^d)$ into $\text{BLO}(\mathbb{R}^d)$ [3]. For the positive Radon measure $\mu$ satisfying (1), X.Tolsa [6] has introduced the spaces $\text{RBMO}(\mu)$, which are the suitable substitutes for the classical spaces $\text{BMO}(\mathbb{R}^d)$, and proved that if $T$ is bounded on $L^2(\mu)$, then $T$ is bounded from $L^\infty(\mu)$ into $\text{RBMO}(\mu)$.

In this note, for the positive Radon measure $\mu$ satisfying (1) we will introduce the spaces $\text{RBLO}(\mu)$ as the substitutes for the classical spaces $\text{BLO}(\mathbb{R}^d)$ defined by R.R.Coifman and R.Rochberg [2], and will show that if $T$ is bounded on $L^2(\mu)$, then $T_\ast$ is bounded from $L^\infty(\mu)$ into $\text{RBLO}(\mu)$.

We take notation and definitions from [6]. By a cube $Q \subset \mathbb{R}^d$ we will mean a cube centered at some point $z_Q \in \text{supp}(\mu)$ with sides parallel to the axes. Let $\alpha > 1$ and $\beta > \alpha^n$. A cube with side length $l(Q)$ is said to be $(\alpha, \beta)$-doubling if $\mu(\alpha Q) \leq \beta \mu(Q)$, where $\alpha Q$ denotes the cube concentric with $Q$ and having side length $\alpha l(Q)$. Given a cube $Q \subset \mathbb{R}^d$, let $N$ be the smallest integer $\geq 0$ such that $\alpha^N Q$ is doubling. Denote this doubling cube by $\tilde{Q}$. If $\alpha$ and $\beta$ are not specified, by a doubling cube we will mean a $(4\sqrt{d}, (4\sqrt{d})^{n+1})$-doubling cube. Given two cubes $Q \subset R$, we set

$$K_{Q,R} = 1 + \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^k Q)}{(2^k Q)^n},$$

where $N_{Q,R}$ is the first integer $k$ such that $2^k Q \supset R$.

Let $\rho > 1$ be some fixed constant. Say that $f \in L^1_{\text{loc}}(\mu)$ belongs to $\text{RBLO}(\mu)$ if there exists some constant $C_1$ such that for any cube $Q$,

$$\frac{1}{\mu(\rho Q)} \int_Q |f - m_Q(f)| d\mu \leq C_1,$$

and

$$|m_Q(f) - m_R(f)| \leq C_1 K_{Q,R},$$

for any two doubling cubes $Q \subset R$, where $m_Q(f) = \mu(Q)^{-1} \int_Q f d\mu$. The best constant $C_1$ is the $\text{RBMO}(\mu)$ norm of $f$, which we denote as $\|f\|_*$ and is independent of $\rho > 1$. It is shown by X.Tolsa [6] that one can replace (3) by

$$\frac{1}{\mu(Q)} \int_Q |f - m_Q(f)| d\mu \leq C_2,$$

or by

$$\left(\frac{1}{\mu(\rho Q)} \int_Q |f - m_{\tilde{Q}}(f)|^p d\mu \right)^{1/p} \leq C_3,$$

for $1 \leq p < \infty$ and any cube $Q$, and the best constants $C_2$ and $C_3$ are comparable to $\|f\|_*$.

We are ready to define the spaces of type $\text{BLO}$ for the positive Radon measure $\mu$ satisfying (1).
Definition 1. We say that \( f \in L_{\text{loc}}^1(\mu) \) belongs to \( RBLO(\mu) \) if there exists some constant \( C_4 \) such that for any doubling cube \( Q \),
\[
m_Q(f) - \text{essinf}_{x \in Q} f(x) \leq C_4 \tag{7}
\]
and
\[
m_Q(f) - m_R(f) \leq C_4 K_{Q,R} \tag{8}
\]
for any two doubling cubes \( Q \subset R \).

The smallest constant \( C_4 \) will be denoted by \( \|f\|_{RBLO} \).

Observe that (7) and (8) are equivalent to \( \|f\|_{RBLO} \) and (11). It is easy to check that \( L^\infty(\mu) \subset RBLO(\mu) \) with \( \|f\|_{RBLO} \leq 2\|f\|_{L^\infty(\mu)} \) and \( RBLO(\mu) \subset RBMO(\mu) \) with \( \|f\|_{RBMO} \leq 2\|f\|_{RBLO} \).

C. Bennett [1] has obtained a criterion for the classical spaces \( BLO(\mathbb{R}^d) \). To give out the \( RBLO(\mu) \) criterion we consider the non-centered doubling maximal function
\[
Mf(x) = \sup_{Q \ni x, Q \text{ doubling}} m_Q(f) = \sup_{Q \ni x, Q \text{ doubling}} \frac{1}{\mu(Q)} \int_Q f d\mu,
\]
where the supremum extends over all doubling cubes \( Q \) containing \( x \). By the Lebesgue differentiation theorem, \( Mf(x) \geq f(x) \) \( \mu \)-a.e. \( x \in \mathbb{R}^d \). Moreover, \( |Mf(x)| \leq M|f(x)| := Nf(x) \). The maximal operator \( N \) is weak type \((1,1)\) and bounded on \( L^p(\mu) \) for \( 1 < p \leq \infty \) [5], so is \( M \).

Lemma 1. \( f \in RBLO(\mu) \) if and only if \( Mf - f \in L^\infty(\mu) \) and \( f \) satisfies (8).

Furthermore,
\[
\|Mf - f\|_{L^\infty(\mu)} \sim \|f\|_{RBLO} \tag{10}
\]

Proof. Suppose first that \( f \in RBLO(\mu) \). Then \( f \) satisfies (8). Observe that (see [1] Remark 2.3), by the Lebesgue differentiation theorem, for \( \mu \)-a.e. \( x \in \mathbb{R}^d \) one can find a sequence of doubling cubes \( \{Q_k\}_k \) centered at \( x \) with \( l(Q_k) \to 0 \) such that
\[
\lim_{k \to \infty} \frac{1}{\mu(Q_k)} \int_{Q_k} f d\mu = f(x).
\]

Let \( x \) be any such point, and let \( Q \) be any doubling cube containing \( x \). Then \( f(x) \geq \text{essinf}_{x \in Q} f(x) \) and so
\[
m_Q(f) - f(x) \leq m_Q(f) - \text{essinf}_{x \in Q} f(x) \leq \|f\|_{RBLO}.
\]

Taking the supremum over all doubling cubes containing \( x \), we get
\[
Mf(x) - f(x) \leq \|f\|_{RBLO}.
\]

Hence \( Mf - f \in L^\infty(\mu) \) and \( \|Mf - f\|_{L^\infty(\mu)} \leq \|f\|_{RBLO} \).

Conversely, suppose \( Mf - f \in L^\infty(\mu) \) and let \( Q \) be any doubling cube in \( \mathbb{R}^d \). If any \( x \in Q \) is such that
\[
f(x) < m_Q(f) - \|Mf - f\|_{L^\infty(\mu)},
\]
then
\[
Mf(x) - f(x) \geq m_Q(f) - f(x) > \|Mf - f\|_{L^\infty(\mu)}.
\]

So, for \( \mu \)-a.e. \( x \in Q \),
\[
f(x) \geq m_Q(f) - \|Mf - f\|_{L^\infty(\mu)},
\]
and consequently,
\[
\inf_{x \in Q} f(x) \geq m_Q(f) - \|Mf - f\|_{L^\infty(\mu)}.
\]
Since \( f \) satisfies (3) we get that \( f \in RBLO(\mu) \) and
\[
\|f\|_{RBLO} \leq C\|Mf - f\|_{L^\infty(\mu)}.
\]
\[\square\]

Our main results are as follows.

**Theorem 1.** If the Calderón-Zygmund operator \( T \) is bounded on \( L^2(\mu) \), then the maximal operator \( T_* \) is bounded from \( L^\infty(\mu) \) into \( RBLO(\mu) \).

**Theorem 2.** If \( f \in RBMO(\mu) \) and \( Mf \) satisfies (3), then \( Mf \in RBLO(\mu) \) and
\[
\|Mf\|_{RBLO} \leq C\|f\|_*.
\]
In particular, \( M \) is bounded on \( RBLO(\mu) \).

**Theorem 3.** A locally integrable function \( f \) belongs to \( RBLO(\mu) \) if and only if there exist \( h \in L^\infty(\mu) \) and \( F \in RBMO(\mu) \) with \( MF \) satisfying (3) such that
\[
f = MF + h.
\]
Furthermore,
\[
\|f\|_{RBLO} \sim \inf(\|F\|_* + \|h\|_{L^\infty(\mu)})
\]
where the infimum extends over all representations of the form (11).

**The proof of Theorem 1.** Let \( x \in \mathbb{R}^d \cap \text{supp}(\mu) \) and \( Q \) be any doubling cube containing \( x \). For each fixed cube \( Q \) let \( B \) be the smallest ball centered at \( x \) which contains \( Q \). Then \( 2B \subset 4\sqrt{d}Q \). If \( f \in L^\infty(\mu) \cap L^{p_0}(\mu) \) for some \( p_0 \in [1, \infty) \), by \( L^2(\mu) \) boundedness of \( T_* \) [4], we have
\[
\frac{1}{\mu(Q)} \int_Q T_* (f|_{2B})d\mu \leq \frac{1}{\mu(Q)^{1/2}} \left\{ \int_{\mathbb{R}^d} [T_* (f|_{2B})]^2 d\mu \right\}^{1/2} \\
\leq \frac{C}{\mu(Q)^{1/2}} \left\{ \int_{\mathbb{R}^d} f|_{2B}^2 d\mu \right\}^{1/2} \\
\leq C\frac{\mu(2B)^{1/2}}{\mu(Q)^{1/2}} \|f\|_{L^\infty(\mu)} \\
\leq C\frac{\mu(4\sqrt{d}Q)^{1/2}}{\mu(Q)^{1/2}} \|f\|_{L^\infty(\mu)} \\
\leq C\|f\|_{L^\infty(\mu)}.
\]
(13)
From (2) it follows that
\[
T_* (f|_{\mathbb{R}^d\setminus 2B})(x) \leq T_* f(x).
\]
By this and the conditions of the Calderón-Zygmund kernel, for all \( y \in Q \), we have
\[
T_* (f|_{\mathbb{R}^d\setminus 2B})(y) \leq |T_* (f|_{\mathbb{R}^d\setminus 2B})(y) - T_* (f|_{\mathbb{R}^d\setminus 2B})(x)| + T_* (f|_{\mathbb{R}^d\setminus 2B})(x) \\
\leq C\|f\|_{L^\infty(\mu)} + T_* f(x).
\]
Therefore,
\[
\frac{1}{\mu(Q)} \int_Q T_* (f|_{\mathbb{R}^d\setminus 2B})d\mu \leq C\|f\|_{L^\infty(\mu)} + T_* f(x).
\]
From this and \[13\], we get

\[
\frac{1}{\mu(Q)} \int_Q T_* f \, d\mu \leq C \|f\|_{L^\infty(\mu)} + T_* f(x).
\]

So,

\[
\|M(T_* f) - T_* f\|_{L^\infty(\mu)} \leq C \|f\|_{L^\infty(\mu)}.
\]

From this and Lemma \[1\] it suffices to show that \(T_* f\) satisfies \(8\). We apply the argument analogous to \([1\] pp. 104-105\]. Let \(Q \subset R\) be any two doubling cubes. Recall that \(N_{Q,R}\) is the first integer \(k\) such that \(2^k Q \supset R\). We denote \(Q_R = 2^{N_{Q,R}+1} Q\). Thus, for \(x \in Q\) and \(y \in R\), we set

\[
T_* f(x) = T_*(f \chi_{2Q})(x) + \sum_{k=1}^{N_{Q,R}} T_*(f \chi_{2^{k+1}Q \setminus 2^k Q})(x) + T_*(f \chi_{R \setminus Q_R})(x)
\]

\[
- \left( T_*(f \chi_{Q_{R}}) (y) + T_*(f \chi_{R \setminus Q_R}) (y) \right) + T_* f(y).
\]

Since

\[
|T_*(f \chi_{R \setminus Q_R})(x) - T_*(f \chi_{R \setminus Q_R})(y)| \leq C \|f\|_{L^\infty(\mu)},
\]

we get

\[
T_* f(x) \leq T_*(f \chi_{2Q})(x) + C \left( 1 + \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^{k+1}Q)}{l(2^{k+1}Q)^n} \right) \|f\|_{L^\infty(\mu)}
\]

\[
+ T_*(f \chi_{Q_{R}}) (y) + T_* f(y).
\]

Now we take the mean over \(Q\) for \(x\), and over \(R\) for \(y\). We write

\[
m_R(T_*(f \chi_{Q_{R}})) \leq m_R(T_*(f \chi_{Q_{R} \cap 2R})) + m_R(T_*(f \chi_{Q_{R} \setminus 2R})).
\]

Similar to the previous estimate \([13\] we obtain

\[
m_Q(T_*(f \chi_{2Q})) \leq C \|f\|_{L^\infty(\mu)}
\]

and

\[
m_R(T_*(f \chi_{Q_{R} \cap 2R})) \leq C \|f\|_{L^\infty(\mu)}.
\]

On the other hand, since \(l(Q_{R}) \approx l(R)\), we have

\[
m_R(T_*(f \chi_{Q_{R} \setminus 2R})) \leq C \|f\|_{L^\infty(\mu)}.
\]

Therefore,

\[
m_Q(T_* f) - m_R(T_* f) \leq CK_{Q,R} \|f\|_{L^\infty(\mu)}.
\]

If \(f \notin L^p(\mu)\) for all \(p \in [1, \infty)\), then the integral \(\int_{|x-y|\geq \epsilon} k(x, y) f(y) d\mu(y)\) may not be convergent. The operator \(T_*\) can be extended to the whole space \(L^\infty(\mu)\) following the standard arguments: Given a cube \(Q_0\) centered at the origin with side length \(l(Q_0) > 3\epsilon\), we write \(f = f_1 + f_2\), with \(f_1 = f \chi_{2Q_0}\). For \(x \in Q_0\), we define

\[
T_* f(x) = T_*(f_1)(x) + \int_{|x-y|\geq \epsilon} (k(x, y) - k(0, y)) f_2(y) d\mu(y).
\]

Now both integrals in this equation are convergent. With arguments similar to the case \(f \in L^\infty(\mu) \cap L^p(\mu)\) we complete the proof. \[\square\]
To prove Theorem 2 and Theorem 3 we require the following lemma.

**Lemma 2.** If \( f \in \text{RBMO}(\mu) \), then for any doubling cube \( Q \),

\[
\frac{1}{\mu(Q)} \int_Q Mf \, d\mu \leq C\|f\|_* + \text{ess inf}_{x \in Q} Mf(x).
\]

Moreover, if \( Mf \) is finite \( \mu \text{-a.e.} \), then

\[
\frac{1}{\mu(Q)} \int_Q Mf \, d\mu - \text{ess inf}_{x \in Q} Mf(x) \leq C\|f\|_*.
\]

**Proof.** Set \( \rho = 4\sqrt{d}/5 \) in (3). Fix a doubling cube \( Q \). For \( f \in \text{RBMO}(\mu) \) we write

\[ f = (f - m_Q(f))\chi_{3Q} + (m_Q(f))\chi_{3Q} + f\chi_{\mathbb{R}^d \setminus 3Q}. \]

Since \( M \) is bounded on \( L^2(\mu) \), we have

\[
\int_Q M((f - m_Q(f))\chi_{3Q}) \, d\mu \\
\leq \mu(Q)^{1/2} \left\{ \int_{\mathbb{R}^d} |M((f - m_Q(f))\chi_{3Q})|^2 \, d\mu \right\}^{1/2} \\
\leq C\mu(Q)^{1/2} \left\{ \int_{\mathbb{R}^d} |(f - m_Q(f))\chi_{3Q}|^2 \, d\mu \right\}^{1/2} \\
\leq C\mu(Q)^{1/2} \left\{ \left( \int_{3Q} |f - m_{\tilde{5}Q}(f)|^2 \, d\mu \right)^{1/2} + \left( \int_{3Q} |m_Q(f) - m_{\tilde{5}Q}(f)|^2 \, d\mu \right)^{1/2} \right\} \\
\leq C\mu(Q)^{1/2} \left( \mu((12\sqrt{d}/5)Q)^{1/2}\|f\|_* + \mu(3Q)^{1/2}\|f\|_* K_{Q,3Q} \right) \\
\leq C\mu(Q)^{1/2} (4\sqrt{d}Q)^{1/2}\|f\|_* \leq C\mu(Q)\|f\|_*.
\]

Next, we shall show that

\[
\frac{1}{\mu(Q)} \int_Q M(m_Q(f)\chi_{3Q} + f\chi_{\mathbb{R}^d \setminus 3Q}) \, d\mu \leq C\|f\|_* + \text{ess inf}_{x \in Q} Mf(x).
\]

It suffices to show that

\[ M(m_Q(f)\chi_{3Q} + f\chi_{\mathbb{R}^d \setminus 3Q})(x) \leq C\|f\|_* + \text{ess inf}_{x \in Q} Mf(x) \text{ \( \mu \)-a.e. } x \in Q. \]

For this it will be enough to show that

\[
\frac{1}{\mu(R)} \int_R (m_Q(f)\chi_{3Q} + f\chi_{\mathbb{R}^d \setminus 3Q}) \, d\mu \leq C\|f\|_* + \text{ess inf}_{x \in Q} Mf(x)
\]

for any doubling cube \( R \ni x \). If \( R \subset 3Q \), the result follows immediately:

\[
\frac{1}{\mu(R)} \int_R (m_Q(f)\chi_{3Q} + f\chi_{\mathbb{R}^d \setminus 3Q}) \, d\mu = m_Q(f) \leq \text{ess inf}_{x \in Q} Mf(x).
\]

So, suppose \( R \cap (\mathbb{R}^d \setminus 3Q) \neq \emptyset \). Then \( l(R) > l(Q) \) and \( 3Q \subset 5R \). Write

\[ m_Q(f)\chi_{3Q} + f\chi_{\mathbb{R}^d \setminus 3Q} = (m_Q(f) - m_{\tilde{5}R}(f))\chi_{3Q} + (f - m_{\tilde{5}R}(f))\chi_{\mathbb{R}^d \setminus 3Q} + m_{\tilde{5}R}(f). \]
Obviously, \( m_{5R}(f) \leq \text{essinf}_{x \in Q} Mf(x) \). Further,

\[
\int_{R} \left\{ (m_Q(f) - m_{5R}(f))\chi_{3Q} + (f - m_{5R}(f))\chi_{R \setminus 3Q} \right\} d\mu \\
\leq \mu(3Q)m_Q(f) - m_{5R}(f) |f - m_{5R}(f)| |f - m_{5R}(f)| d\mu \\
\leq \frac{\mu(4\sqrt{dQ})}{\mu(Q)} \int_{Q} |f - m_{5R}(f)| + \int_{3Q \setminus 5R} |f - m_{5R}(f)| d\mu \\
\leq C \int_{5R} |f - m_{5R}(f)| d\mu \\
\leq C \mu(4\sqrt{R}) \|f\|_{*} \leq C \mu(R) \|f\|_{*}.
\]

This establishes (18) and completes the proof of Lemma 2.

\[ \square \]

The proof of Theorem 2. By Lemma 2, it suffices to prove that if \( f \in RBLO(\mu) \), then \( Mf \) satisfies (8). In fact, applying Lemma 1, for any two doubling cubes \( Q \subset R \) we have

\[
m_Q(Mf) - m_{R}(Mf) \\
\leq m_Q(Mf) - m_Q(f) + m_{R}(Mf) - m_{R}(f) + |m_Q(f) - m_{R}(f)| \\
\leq 2 \|Mf - f\|_{L^{\infty}(\mu)} + \|f\|_{*} K_{Q,R} \\
\leq 2 \|f\|_{RBLO} + \|f\|_{*} K_{Q,R} \\
\leq C \|f\|_{RBLO} K_{Q,R}.
\]

As we remarked above, this completes the proof of Theorem 2.

\[ \square \]

Theorem 3 follows immediately from Lemma 1 and Theorem 2, and we omit the details here.

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REFERENCES


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