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COMPOSITION OPERATORS ON BANACH FUNCTION SPACES

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Abstract. We study the boundedness and the compactness of composition operators on some Banach function spaces such as absolutely continuous Banach function spaces on a σ–finite measure space, Lorentz function spaces on a σ–finite measure space and rearrangement invariant spaces on a resonant measure space. In addition, we study some properties of the spectra of a composition operator on the general Banach function spaces.

1. Introduction

Let $\Omega = (\Omega, \Sigma, \mu)$ be a σ-finite measure space. Let $T : \Omega \mapsto \Omega$ be a non–singular measurable transformation, that is, $\mu(T^{-1}(A)) = 0$ for each $A \in \Sigma$ whenever $\mu(A) = 0$. Then T induces a well–defined composition transformation C_T from $L(\mu)$ into itself defined by

$$C_T f(x) = f(T(x)),\ x \in \Omega,\ f \in L(\mu),$$

where $L(\mu)$ denotes the linear space of all equivalence classes of Σ–measurable functions on Ω and we identify any two functions that are equal μ–almost everywhere on Ω. If C_T maps a Banach function space X into itself, then we call C_T a composition operator on X induced by T.

There is a vast literature on composition operators and their applications (see [2], [5], [19] and [21]).

In particular, for composition operators on measurable function spaces and their applications, one can refer to [4], [6], [7], [10], [11], [12], [14], [15], [16], [20], [21], [22] and the references therein.

Let $M_\circ$ be the class of all functions in $L(\mu)$ that are finite μ–a.e. For $f \in M_\circ$, we define the distribution function of $|f|$ on $0 < \lambda < \infty$ by

$$\mu_f(\lambda) = \mu(\{x \in \Omega : |f(x)| > \lambda\}),$$

and the non–increasing rearrangement of f on $(0, \infty)$ by

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \le t\} = \sup\{\lambda > 0 : \mu_f(\lambda) > t\}.$$

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The Banach function space X is defined as

$$X = \{f \in L(\mu) : \|f\|_X < \infty\},$$

where the norm $\|\cdot\|_X$ on X has the following properties:
for each $f, g, f_n \in L(\mu),\ n \geq 1$, we have

(1) $0 \leq |\, g \,| \leq |\, f \,|\ \ \mu-$a.e. $\Rightarrow \|g\|_X \leq \|f\|_X$,
(2) $0 \leq |\, f_n \,| \nearrow |\, f \,|\ \ \mu-$a.e. $\Rightarrow \|f_n\|_X \nearrow \|f\|_X$, and
(3) $E \in \Sigma$ with $\mu(E) < \infty \Rightarrow \|\chi_E\|_X < \infty$ and $\int_E |\, f \,|\, d\mu \leq c_E \|f\|_X$ for some constant $0 < c_E < \infty$ depending on E and the norm $\|.\|_X$ but independent of f.

See [1] for details on Banach function spaces.

A function f in a Banach function space X is said to have absolutely continuous norm if $\|f\chi_{E_n}\|_X \to 0$ for each sequence $\{E_n\}_{n=1}^{\infty}$ satisfying $E_n \to \emptyset\ \mu$–a.e. If each function in X has absolutely continuous norm, then we say X is a Banach function space with absolutely continuous norm.

A rearrangement invariant space is a Banach function space X such that whenever $f \in X$ and g is an equimeasurable function with f, then $g \in X$ and $\|g\|_X = \|f\|_X$. For the sake of completeness of the definition, we recall a result from [1, p. 59].

Proposition 1.1. *Let $(X, \|.\|_X)$ be a rearrangement invariant Banach function space on a resonant measure space (Ω, Σ, μ). Then the associate space X' is also a rearrangement invariant space (under the norm $\|.\|_{X'}$) and these norms are given by*

$$\|g\|_{X'} = \sup\{\int_0^\infty f^*(s)g^*(s)\,ds : \|f\|_X \leq 1\},\ g \in M_\circ,$$

and

$$\|f\|_X = \sup\{\int_0^\infty f^*(s)g^*(s)\,ds : \|g\|_{X'} \leq 1\},\ f \in M_\circ.$$

See [1], [8] and [12] for details on rearrangement invariant spaces. The study of composition operators on Lorentz spaces $L(pq, \mu),\ 1 \leq q \leq p < \infty$, or $p = q = \infty$, has been initiated in [9].

The Lorentz spaces $L^{pq}(\mu)$ are the Banach function spaces continuously embedded in $M_\circ$ under the norms

$$\|f\|_{pq} = \begin{cases} [\int_0^\infty (t^{1/p} f^{**}(t))^q dt/t]^{1/q} & \text{if } 1 < p < \infty, 1 \leq q < \infty, \\ \sup_{0<t<\infty} t^{1/p} f^{**}(t) & \text{if } 1 < p < \infty, q = \infty, \end{cases}$$

where $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds$ is the average function.

We refer to p as the principal index. The Lorentz space $L^{pq}(\mu)$, for $1 < p < \infty,\ 1 \leq q \leq \infty$ is a rearrangement invariant Banach function space with upper and lower Boyd indices both equal to $1/p$. For $1 < p < \infty$, the Lebesgue spaces $L^p(\mu)$ are equivalent to $L^{pp}(\mu)$.

Definition 1.2. Let M and N be two normed linear spaces. Then a mapping $S : M \mapsto N$ is called non–expansive if $\|S(x) - S(y)\| \leq \|x - y\|$, for each $x, y \in M$.

Theorem 1.3 (cf. [1, p. 221])**.** *Let (Ω, Σ, μ) be a resonant measure space. For $1 < p < \infty, 1 \leq q < \infty$ (or $p = q = 1$), the dual as well as the associate space of $L^{pq}(\mu)$ is, up to the equivalence of norms, the Lorentz space $L^{p'q'}(\mu)$, where $1/p + 1/p' = 1/q + 1/q' = 1$.*

The main aim of this paper is to study the boundedness and the compactness of composition operators on Lorentz spaces $L^{pq}(\mu)$, for $1 < p < \infty, 1 \leq q \leq \infty$. In section 1, we study the boundedness of the composition operators on Banach function spaces. In section 2, we discuss the compactness of composition operators on rearrangement invariant spaces. In section 3, we study the spectra of C_T on the general Banach function spaces on a σ-finite measure space and in section 4, we give some examples.

2. Boundedness

Let $\mathcal{K}$ denote the family of all characteristic functions of the members of Σ, that is,

$$\mathcal{K} = \{\chi_A : A \in \Sigma\}.$$

Let $K = \mathcal{K} \cap X$. The next theorem shows under what conditions a bounded linear operator is a composition operator on a general Banach function space X.

Theorem 2.1. *If A is a composition operator on a Banach function space X, then K is invariant under A. Conversely, if (Ω, Σ, μ) is a Borel σ–finite measure space and A is a positive bounded linear operator on an absolutely continuous Banach function space X, then A is a composition operator on X.*

Proof. The proof is along similar lines as in [21, Theorem 2.1.13, p. 23]. □

Corollary 2.2. *Let A be a composition operator on a Banach function space X. Then $A(f \cdot g) = A(f) \cdot A(g)$, whenever $f, g, f \cdot g \in X$. Conversely, if (Ω, Σ, μ) is a σ–finite Borel measure space, A is a positive bounded linear operator on an absolutely continuous Banach function space X and $A(f \cdot g) = A(f) \cdot A(g)$, whenever $f, g, f \cdot g \in X$, then A is a composition operator on X.*

The following theorem gives a necessary and sufficient condition for the continuity of composition operators on the Lorentz spaces $L^{pq}(\mu)$, for $1 < p < \infty, 1 \leq q \leq \infty$.

Theorem 2.3. *Let $T : \Omega \mapsto \Omega$ be a non–singular measurable transformation. Then T induces a composition operator C_T on $L^{pq}(\mu)$, $1 < p < \infty, 1 \leq q \leq \infty$, if and only if there exists some constant $M > 0$ such that*

$$\mu \circ T^{-1}(A) \leq M\mu(A), \text{ for each } A \in \Sigma. \tag{2.1}$$

Moreover,

$$\|C_T\| = \sup_{A \in \Sigma, 0 < \mu(A) < \infty} \left(\frac{\mu \circ T^{-1}(A)}{\mu(A)}\right)^{1/p}. \tag{2.2}$$

Proof. Suppose the condition (2.1) holds. Then

$$\begin{aligned} \mu_{C_T f}(\lambda) &= \mu(\{x \in \Omega : |f(T(x))| > \lambda\}) = \mu(\{T^{-1}(y) \in \Omega : |f(y)| > \lambda\}) \\ &\leq M\mu(\{y \in \Omega : |f(y)| > \lambda\}) \leq M\mu_f(\lambda). \end{aligned}$$

Also, for each $t \geq 0$, we have

$$\begin{aligned} (C_T f)^*(t) &= \sup\{\lambda > 0 : \mu_{C_T f}(\lambda) > t\} \\ &\leq \sup\{\lambda > 0 : M\mu_f(\lambda) > t\} = f^*(t/M), \end{aligned}$$

which implies that

$$(C_T f)^{**}(t) \leq f^{**}(t/M).$$

So for $q \neq \infty$, we have

$$\begin{aligned}\|C_T f\|_{pq} &= [\int_0^\infty (\, t^{1/p}(C_T f)^{**}(t)\,)^q \frac{dt}{t}]^{1/q} \\ &\leq [\int_0^\infty (\, t^{1/p} f^{**}(\frac{t}{M})\,)^q \frac{dt}{t}]^{1/q} \\ &= [\int_0^\infty (\, M^{1/p} t^{1/p} f^{**}(t)\,)^q \frac{dt}{t}]^{1/q} = M^{1/p}\|f\|_{pq}.\end{aligned}$$

Also for $q = \infty$, we have

$$\begin{aligned}\|C_T f\|_{p\infty} &= \sup_{0<t<\infty} t^{1/p}(C_T f)^{**}(t) \\ &\leq \sup_{0<t<\infty} M^{1/p}(t/M)^{1/p} f^{**}(t/M) = M^{1/p}\|f\|_{p\infty}.\end{aligned}$$

Thus, we conclude that C_T is a bounded linear operator on $L^{pq}(\mu)$ and clearly $\|C_T\| \leq M^{1/p}$.

Conversely, suppose C_T is a bounded operator. For $E \in \Sigma$ with $0 < \mu(E) < \infty$, we have $\chi_E \in L^{pq}(\mu)$ as

$$\|\chi_E\|_{pq} = \big(p/q\big)^{1/q}\big(p/(p-1)\big)^{1/p}\big(\,\mu(E)\,\big)^{1/p} < \infty.$$

Since C_T is bounded, we have for $q \neq \infty$, $\|C_T\chi_E\|_{pq} \leq k\|\chi_E\|_{pq}$ for some $k > 0$, i.e.,

$$\big(p/q\big)^{1/q}\big(p/(p-1)\big)^{1/p}\big(\,\mu T^{-1}(E)\,\big)^{1/p} \leq k\big(p/q\big)^{1/q}\big(p/(p-1)\big)^{1/p}\big(\,\mu(E)\,\big)^{1/p}.$$

Hence,

$$\mu \circ T^{-1}(E) \leq k^p \mu(E) \text{ for } E \in \Sigma.$$

Also for $q = \infty$, we have

$$\|\chi_E\|_{p\infty} = \sup_{0<t<\infty} t^{1/p}\chi^{**}(t) = \sup_{0<t<\infty} t^{1/p} = (\mu(E))^{1/p} < \infty.$$

So the inequality holds. Thus $\mu \circ T^{-1}(E) \leq M\mu(E)$, for $E \in \Sigma$ with $M = k^p > 0$. Further, we see that

$$\|C_T\| = \sup_{A\in\Sigma, 0<\mu(A)<\infty} (\frac{\mu \circ T^{-1}(A)}{\mu(A)})^{1/p}.$$

□

Theorem 2.4. *Let X and X' be the rearrangement invariant spaces on a resonant measure space (Ω, Σ, μ) with the fundamental functions φ_X and $\varphi_{X'}$, respectively. Let $T : \Omega \mapsto \Omega$ be a non–singular measurable transformation. Then C_T is a bounded composition operator on X and X' if and only if the condition* (2.1) *holds.*

Proof. By using Theorem 2.3, we have

$$(C_T\, f)^*(t) \leq f^*(t/M), \text{ for } t \geq 0,\ f \in M_\circ.$$

For $M \geq 1$, using the fact that g^* is a decreasing function, we see that

$$\begin{aligned}\int_0^\infty (C_T f)^*(s)\, g^*(s)\, ds &\leq \int_0^\infty f^*(s/M)\, g^*(s)\, ds = \int_0^\infty f^*(u) g^*(Mu) M\, du \\ &\leq M \int_0^\infty f^*(u)\, g^*(u)\, du.\end{aligned}$$

Taking the supremum over all $g \in X'$ with $\|g\|_{X'} \leq 1$, we get

$$\|C_T f\|_X \leq M\|f\|_X, \forall f \in X.$$

Similarly, we have

$$\int_0^\infty f^*(s)(C_T g)^*(s)\, ds \leq M \int_0^\infty f^*(u) g^*(u)\, du.$$

Again, taking the supremum over all $f \in X$ with $\|f\|_X \leq 1$, we get

$$\|C_T g\|_{X'} \leq M\|g\|_{X'}, \forall g \in X'.$$

Thus C_T is a bounded composition operator on X and X'.

Conversely, let $E \in \Sigma$ with $0 < \mu(E) < \infty$. Then by definition $\chi_E \in X$ and $\chi_E \in X'$, and so we have

$$\|C_T \chi_E\|_X \leq k\|\chi_E\|_X, \text{ for some } k > 0,$$

which implies that

$$\varphi_X(\mu T^{-1}(E)) \leq k\varphi_X(\mu(E)), \text{ for some } k > 0.$$

Similarly, we have

$$\varphi_{X'}(\mu T^{-1}(E)) \leq k'\varphi_{X'}(\mu(E)), \text{ for some } k' > 0.$$

Multiplying the above two inequalities and using [1, Theorem 5.2, p. 66], we get

$$\mu T^{-1}(E) \leq kk'\mu(E).$$

Thus $\mu T^{-1}(E) \leq m\mu(E)$, for some $m = kk' > 0$. □

The largest rearrangement invariant space over a resonant measure space (Ω, Σ, μ) with the fundamental function φ_X is also defined as

$$V(X) = \{f \in M_\circ : \|f\|_{V(X)} = \|f\|_\infty \varphi_X(0+) + \int_0^\infty f^*(s)\Phi_X(s)\, ds < \infty\},$$

where the fundamental function φ_X can be represented as the integral of a non–negative, decreasing function, say Φ_X, on $(0, \infty)$.

Theorem 2.5. *For a non–singular measurable transformation $T : \Omega \mapsto \Omega$, we have C_T is bounded on $V(X)$ and $V(X')$ if and only if the condition* (2.1) *holds.*

Proof. Using the fact that Φ_X is decreasing and $M \geq 1$, we have

$$\begin{aligned}
\|C_T f\|_{V(X)} &= \|C_T f\|_\infty \varphi_X(0+) + \int_0^\infty (C_T f)^*(s)\Phi_X(s)\, ds \\
&\leq \|f\|_\infty \varphi_X(0+) + \int_0^\infty f^*(s/M)\Phi_X(s)\, ds \\
&= \|f\|_\infty \varphi_X(0+) + M\int_0^\infty f^*(u)\Phi_X(Mu)\, du \\
&\leq M(\, \|f\|_\infty \varphi_X(0+) + \int_0^\infty f^*(u)\Phi_X(u)\, du\,) \\
&= M\|f\|_{V(X)},
\end{aligned}$$

for each $f \in V(X)$. Thus, C_T is bounded on $V(X)$ and also on $V(X')$.

The converse follows along similar lines as in the converse part of Theorem 2.4. □

The next theorem characterises those maps T for which C_T is non–expansive. A map $T : X \mapsto X$ is said to be μ–expansive, if $\mu(E) \leq \mu(T(E))$ for all $E \in \Sigma$.

Theorem 2.6. *Let (Ω, Σ, μ) be a σ–finite standard Borel measure space. Let $T(E) \in \Sigma$ whenever $E \in \Sigma$. The following are equivalent.*

(1) $\|C_T\| \leq 1$.
(2) *C_T is non–expansive on $L^{pq}(\mu), 1 < p < \infty, 1 \leq q \leq \infty$, or on rearrangement invariant spaces X and X' on a resonant measure space.*
(3) *T is μ–expansive.*
(4) *$\mu(T^{-1}(E)) \leq \mu(E)$, for all $E \in \Sigma$.*

Proof. The proof follows from Theorem 2.3 and Theorem 2.4. □

3. Compactness

In this section, we give a necessary and sufficient condition for the compactness of composition operators on $L^{pq}(\mu)$, $1 < p < \infty$, $1 \leq q \leq \infty$. Also, we discuss the compactness of the composition operators on rearrangement invariant spaces X on a non–atomic measure space.

Theorem 3.1. *Let C_T be a composition operator on the Lorentz spaces $L^{pq}(\mu)$, for $1 < p < \infty$, $1 \leq q \leq \infty$, induced by a non–singular measurable transformation T on Ω and let $\{A_n\}$ be all the atoms of Ω with $\mu(A_n) = a_n > 0$, for each n. Then C_T is compact if and only if μ is purely atomic and*

$$b_n = \frac{\mu T^{-1}(A_n)}{\mu(A_n)} \to 0.$$

Proof. We have $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$, where $\mu_1 = \mu\,|_{\Omega_1}$ is non–atomic and $\mu_2 = \mu\,|_{\Omega_2}$ is atomic so that $\mu = \mu_1 + \mu_2$. Since $\mu \circ T^{-1} \ll \mu$, so by the Radon–Nikodym theorem, there exists $h \in L^\infty(\Omega, \Sigma, \mu)$ such that h is locally integrable on Ω_1 and

$$\mu \circ T^{-1}(E) = \int_E h(t)\, d\mu, \text{ for } E \in \Sigma \cap \Omega_1. \tag{3.1}$$

Let $E = \{x \in \Omega_1 : h(x) > 0\}$. Then $E \in \Sigma \cap \Omega_1$.

Firstly, we prove that $\mu(E) = 0$. Suppose $\mu(E) \neq 0$. So there is some $\epsilon > 0$ such that

$$E_\epsilon = \{x \in X_1 : h(x) \geq \epsilon^p\} \in \Sigma \cap \Omega_1 \text{ with } \mu(E) > 0.$$

Since $(\Omega_1, \Sigma \cap \Omega_1, \mu_1)$ is non–atomic, we have $\mu(F) = \mu_1(F)$ whenever $F \in \Sigma \cap X_1$. So we can choose a contracting sequence $\{E_n\}$ of subsets of E_ϵ such that for some $n_\circ \in \mathbf{N}$, we have

$$0 < \mu(E_n) = \mu_1(E_n) < 1/n, \text{ for all } n > n_\circ.$$

For each $n \in \mathbf{N}$, define

$$f_n(x) = \frac{\chi_{E_n}(x)}{\|\chi_{E_n}\|_{pq}},\ x \in X.$$

Then $f_n \to 0$ weakly, $\|f_n\|_{pq} = 1$ and

$$(C_T f_n)(x) = \frac{\chi_{T^{-1}(E_n)}(x)}{\|\chi_{E_n}\|_{pq}},\ x \in X.$$

For $n > n_\circ$, we have

$$\begin{aligned}\|C_T f_n\|_{pq} &= \frac{\|\chi_{T^{-1}(E_n)}\|_{pq}}{\|\chi_{E_n}\|_{pq}} = \Big(\frac{\mu T^{-1}(E_n)}{\mu(E_n)}\Big)^{1/p} \\ &= \Big(\frac{\int_{E_n} h(t)\,d\mu}{\mu(E_n)}\Big)^{1/p} \geq \epsilon,\end{aligned}$$

which implies that $C_T f_n \not\to 0$ strongly. This contradicts the compactness of C_T. Thus $\mu(E) = 0$. Therefore,

$$\mu \circ T^{-1}(X_1) = 0 = \int_{X_1} h(t)\,d\mu(t) \geq \epsilon^p \mu(X_1).$$

Hence μ is purely atomic.

Next, we claim that $b_n \to 0$. Suppose the contrary. Then there exists some $\epsilon > 0$ such that $b_n \geq \epsilon$, for all $n \in \mathbf{N}$. Let $\Omega = \bigcup_{n=1}^{\infty} A_n$, where each A_n is an atom. For each $n \in \mathbf{N}$, let

$$f_n = \frac{\chi_{A_n}}{\|\chi_{T^{-1}(A_n)}\|_{pq}}.$$

Then for each n, we have

$$\begin{aligned}\|f_n\|_{pq} &= \frac{\|\chi_{A_n}\|_{pq}}{\|\chi_{T^{-1}(A_n)}\|_{pq}} = \Big(\frac{\mu(A_n)}{\mu(T^{-1}(A_n))}\Big)^{1/p} \\ &= 1/b_n^{1/p} \leq 1/\epsilon^{1/p}\end{aligned}$$

and $\|C_T f_n\|_{pq} = 1$.

Also for $n \neq m$, since $C_T f_n$ and $C_T f_m$ have disjoint supports, we have

$$\|C_T f_n - C_T f_m\|_{pq} = \|C_T f_n + C_T f_m\|_{pq}.$$

Thus,

$$\begin{aligned}1 &= \frac{1}{2}\|C_T f_n + C_T f_n\|_{pq} \\ &= \frac{1}{2}\|C_T f_n - C_T f_m + C_T f_n + C_T f_m\|_{pq} \\ &\leq \frac{1}{2}(\|C_T f_n - C_T f_m\|_{pq} + \|C_T f_n + C_T f_m\|_{pq}) \\ &= \|C_T f_n - C_T f_m\|_{pq},\end{aligned}$$

which contradicts the compactness of C_T. Hence $b_n \to 0$.

Conversely, since (Ω, Σ, μ) is atomic with atoms A_n and $b_n \to 0$, note that f and $\sum f(A_n)\chi_{A_n}$ are equal μ–a.e. For each $N \in \mathbf{N}$, define $C_T^{(N)}$ by

$$C_T^{(N)} f = \sum_{n \leq N} f(A_n)\chi_{T^{-1}(A_n)}.$$

Then for each $\lambda > 0$, we have

$$\begin{aligned}\mu_{(C_T - C_T^{(N)})f}(\lambda) &\leq \sum_{n>N, |f(A_n)|>\lambda} \mu(\,T^{-1}(A_n)\,) \\ &\leq (\sup_{n>N} b_n) \sum_{|f(A_n)|>\lambda} \mu(A_n) = (\sup_{n>N} b_n)\mu_f(\lambda).\end{aligned}$$

Therefore

$$\|C_T - C_T^{(N)}\| \leq (\sup_{n>N} b_n)^{1/p} \to 0$$

as $N \to \infty$. Since C_T is the limit of finite rank operators $C_T^{(N)}$, it is compact. □

Theorem 3.2. *Let X be a rearrangement invariant Banach function space over a non–atomic measure space (Ω, Σ, μ) with a concave fundamental function φ_X. Let $T : \Omega \mapsto \Omega$ be a non–singular measurable transformation. Then there is no non–trivail compact composition operator on X.*

Proof. Suppose C_T is a non–trivial compact composition operator on X. Since C_T is bounded and T is non–singular, there exists some $h \in L^\infty(\Omega, \Sigma, \mu)$ such that h is locally integrable on Ω and

$$\mu \circ T^{-1}(E) = \int_E h(t)\, d\mu, \text{ for each } E \in \Sigma.$$

Let $E = \{x \in \Omega : h(x) \geq 0\}$. Then $\mu(E) = 0$. If not, then there is some $\epsilon > 0$ such that

$$E_\epsilon = \{x \in \Omega : h(x) > \epsilon\} \in \Sigma \text{ with } \mu(E_\epsilon) > 0.$$

Without loss of generality, we assume that $\epsilon \in (0, 1)$. Since (Ω, Σ, μ) is non–atomic, we can extract a contracting sequence $\{E_n\}$ of subsets of E_ϵ such that $0 < \mu(E_n) < 1/n$, for each $n > n_\circ$, for some $n_\circ \in \mathbf{N}$. For each $n \geq 1$, set

$$f_n(x) = \frac{\chi_{E_n}(x)}{\|\chi_{E_n}\|_X}, x \in X.$$

Then $f_n \to 0$ weakly, $\|f_n\|_X = 1$ and

$$(C_T f_n)(x) = \frac{\chi_{T^{-1}(E_n)}(x)}{\|\chi_{E_n}\|_X}.$$

For $n > n_\circ$, using the concavity of φ_X we see that

$$\begin{aligned} \|C_T f_n\|_X &= \frac{\|\chi_{T^{-1}(E_n)}\|_X}{\|\chi_{E_n}\|_X} = \frac{\varphi_X(\mu(T^{-1}(E_n)))}{\varphi_X(\mu(E_n))} \\ &= \frac{\varphi_X(\int_{E_n} h(t)\, d\mu)}{\varphi_X(\mu(E_n))} \geq \frac{\varphi_X(\epsilon\mu(E_m))}{\varphi_X(\mu(E_m))} \\ &\geq \epsilon. \end{aligned}$$

This implies that $C_T f_n \not\to 0$ strongly, which contradicts the compactness of C_T. □

4. Spectrum

In this section, we examine the spectrum $\sigma(C_T)$, the approximate spectrum $\sigma_{ap}(C_T)$, the point spectrum $\sigma_p(C_T)$ and the compression spectrum $\sigma_c(C_T)$ of a bounded composition operator C_T on a general Banach function space X induced by a non–singular measurable transformation $T : \Omega \mapsto \Omega$. Let U denote the open unit disc and ∂U be the unit circle in the complex plane $\mathbf{C}$. These results generalise the results of Ridge [17]. Using the fact that the norm $\|.\|_X$ is increasing on X, the following results generalise the results of [17] without any extra effort.

Theorem 4.1. *$\sigma_{ap}(C_T) \cap \partial U$ is a union of subgroups of ∂U.*

Corollary 4.2. *$\sigma_p(C_T) \cap \partial U$ is a union of subgroups of ∂U.*

Theorem 4.3. *$\sigma_{ap}(C_T) \setminus \partial U$ has circular symmetry about* 0.

Corollary 4.4. *$\sigma_p(C_T) \setminus \partial U$ has circular symmetry about* 0.

Corollary 4.5. *The residual spectrum $\sigma_c(C_T) \setminus \sigma_{ap}(C_T) \setminus \partial U$ (and $\sigma(C_T) \setminus \partial U$) has circular symmetry about* 0.

Corollary 4.6. *If C_T is a compact composition operator on X, then $\sigma(C_T) \setminus \{0\}$ is a finite subset of ∂U. So, $C_T = A + N$, where A is of finite rank and N is quasinilpotent.*

Theorem 4.7. *If X is a Banach lattice on a finite measure space, then $\sigma_p(C_T) \setminus \{0\}$ is either connected or is contained in ∂U.*

5. Examples

Example 5.1. Let $T : \mathbf{R} \mapsto \mathbf{R}$ be a monotonic function. Then T induces a composition operator on the Lorentz spaces $L^{pq}(\mu)$, for $1 < p < \infty$, $1 \leq q \leq \infty$ if and only if $\frac{1}{T'}$ is essentially bounded, where T' denotes the derivative of T. In this case, we have

$$\|C_T\| = \|\frac{1}{T'}\|_\infty.$$

Example 5.2. If $T : \mathbf{R} \mapsto \mathbf{R}$ is a polynomial, then $T' \neq 0$ at every point of $\mathbf{R}$ if and only if T is monotonic and $\frac{1}{T'}$ is bounded. Therefore, if $T : \mathbf{R} \mapsto \mathbf{R}$ is a polynomial of even degree, then T does not induce a bounded composition operator on $L^{pq}(\mu)$, for $1 < p < \infty$, $1 \leq q < \infty$.

Example 5.3. Let $\Omega = \mathbf{R}$, $\Sigma =$ Borel σ–algebra, $\mu =$ Lebesgue measure. If $t \in \mathbf{R}$ is fixed, define $T_t : \Omega \mapsto \Omega$ by $T_t(x) = x + t$, for each $x \in \Omega$. Then $\mu \circ T_t^{-1} \ll \mu$ and $\|C_{T_t}\| = 1$ on $L^{pq}(\mu)$, for $1 < p < \infty$, $1 \leq q < \infty$. Using [10, Proposition 3.1] with $E_\circ = [t/2, t]$, we see that $\sigma_p(C_{T_t}) = \{\lambda \in \mathbf{C} : \mid \lambda \mid < 1\}$. Also, it can be easily seen that the family $\{C_{T_t} : t \geq 0\}$ is a strongly continuous semigroup on $L^{pq}(\mu)$; see [13, p. 34].

Example 5.4. Let $\Omega = \mathbf{R}$, $\Sigma =$ Borel σ–algebra, $\mu =$ Lebesgue measure. If $t > 0$ is fixed, we define $T : \Omega \mapsto \Omega$ by

$$T(x) = \begin{cases} x + t & \text{if } x > 0, \\ x - t & \text{if } x \leq 0. \end{cases}$$

Then, again $\mu \circ T^{-1} \ll \mu$ and $\|C_T\| = 1$ on $L^{pq}(\mu)$, for $1 < p < \infty$, $1 \leq q < \infty$ and $\sigma_p(C_T) = \{\lambda \in \mathbf{C} :\mid \lambda \mid < 1\}$.

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