CLASSIFICATION OF QUASIFINITEN MODULES OVER LIE ALGEBRAS OF MATRIX DIFFERENTIAL OPERATORS ON THE CIRCLE

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Abstract. We prove that an irreducible quasifinite module over the central extension of the Lie algebra of $N \times N$-matrix differential operators on the circle is either a highest or lowest weight module or else a module of the intermediate series. Furthermore, we give a complete classification of indecomposable uniformly bounded modules.

1. Introduction

Kac [5] introduced the notion of conformal algebras. Conformal algebras play important roles in quantum field theory and vertex operator algebras (e.g. [5]), whose study has drawn much attention in the literature (e.g. [1], [2]–[7], [10]–[19]). It is pointed out in [7] that the infinite rank associative conformal algebra $\text{Cend}_N$ and the Lie conformal algebra $\text{gc}_N$ (the general Lie conformal algebra) play the same important roles in the theory of conformal algebras as $\text{End}_N$ and $\text{gl}_N$ play in the theory of associative and Lie algebras.

There is a one-to-one correspondence between Lie conformal algebras and maximal formal distribution Lie algebras [1, 6, 7]. The Lie algebra $\mathcal{D}^N$ of $N \times N$-matrix differential operators on the circle is a formal distribution Lie algebra associated to the general Lie conformal algebra $\text{gc}_N$. Let $\hat{\mathcal{D}}^N$ be the universal central extension of $\mathcal{D}^N$. In particular when $N = 1$, $\hat{\mathcal{D}}^1$ is also known as the $W$-infinity algebra $\mathcal{W}_{1+\infty}$. Thus one may expect that the representation theory of $\mathcal{D}^N$ and $\hat{\mathcal{D}}^N$ is very interesting and important (e.g. [2], [4], [8]–[10], [13]).

As is pointed out in [8], although $\mathcal{D}^N$ is a $\mathbb{Z}$-graded Lie algebra, each grading space is still infinite dimensional (cf. (2.4)), and the classification of quasifinite modules is thus a nontrivial problem. The classification of irreducible quasifinite highest weight modules over $\hat{\mathcal{D}}^N$ was given by Kac and Radul [8] for the case $N = 1$. For the general $N$, the classification was obtained by Boyallian, Kac, Liberati and Yan [2]. In [13], the author obtained the classification of the irreducible quasifinite modules and indecomposable uniformly bounded modules over $\hat{\mathcal{D}}^1$. We would like
to take this chance to point out that there is a slight gap in the proof of Proposition 2.2 of [13]; this gap has been filled in this paper (see Remark 3.4).

In this paper, we generalize the results in [13] to the general case: Precisely, we obtain that an irreducible quasifinite module over $D^N$ (thus also over $D^N$) is either a highest or lowest weight module or else a module of the intermediate series. Furthermore, we give a complete classification of indecomposable uniformly bounded modules (Theorem 2.2).

2. Notation and main theorem

Let $N \geq 1$ be an integer. Let $\mathbb{C}[t, t^{-1}]$ be the Laurent polynomial algebra over the variable $t$, let $\mathbb{C}[D]$ be the polynomial algebra over $D = t\frac{d}{dt}$, and let $gl_N$ be the space of $N \times N$ matrices. The Lie algebra $D^N$ of $N \times N$-matrix differential operators on the circle is the tensor product space $D^N = \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[D] \otimes gl_N$, spanned by $\{t^{i}D^{j}A \mid i \in \mathbb{Z}, j \in \mathbb{Z}_+, A \in gl_N\}$, with the Lie bracket:

$$[t^{i}D^{j}A, t^{k}D^{l}B] = (t^{i}D^{j}A) \cdot (t^{k}D^{l}B) - (t^{k}D^{l}B) \cdot (t^{i}D^{j}A)$$

and

$$\langle t^{i}D^{j}A \rangle \cdot \langle t^{k}D^{l}B \rangle = t^{i+k}(D+k)^{i}D^{j}AB = \sum_{s=0}^{j} \binom{j}{s} k^{s}j^{i+k}D^{j+l-s}AB,$$

for $i, j \in \mathbb{Z}$, $j \in \mathbb{Z}_+$, and $\binom{i}{j} = \frac{i(i-1)\cdots(i-s+1)}{s!}$ if $s \geq 0$ and $\binom{i}{j} = 0$ if $s < 0$ is the binomial coefficient. The associative algebra with the underlined vector space $D^N$ and the product (2.2) is denoted by $D^N_{as}$.

It is proved in [11] that $D^N$ has a unique nontrivial central extension. The universal central extension $\overline{D^N}$ of $D^N$ is defined as follows (cf. [2]): The Lie bracket (2.1) is replaced by (cf. (3.12))

$$[t^{i}D_{j}A, t^{k}D_{l}B] = (t^{i}D_{j}A) \cdot (t^{k}D_{l}B) - (t^{k}D_{l}B) \cdot (t^{i}D_{j}A)$$

$$+ \delta_{i,-k}(-1)^{i}j!! \left(\begin{array}{c} i+j \\ j+l+1 \end{array}\right) \text{tr}(AB)C,$$

for $i, j \in \mathbb{Z}$, $j \in \mathbb{Z}_+$, and $\delta_{i,-k} = \text{tr}(AD^{-1})$. The trace of the matrix $A$, and where $C$ is a nonzero central element of $\overline{D^N}$.

For $m, n \in \mathbb{Z}$, we denote $[m,n] = \{m, m+1, \ldots, n\}$. Let $\{E_{p,q} \mid p, q \in [1,N]\}$ be the standard basis of $gl_N$, where $E_{p,q}$ is the matrix with entry 1 at $(p, q)$ and 0 otherwise. Then $\overline{D^N}$ has the principal $\mathbb{Z}$-gradation $\overline{D^N} = \bigoplus_{i \in \mathbb{Z}} \overline{D^N}_i$, with the grading space (cf. [2])

$$\overline{D^N}_i = \text{span}\{t^{k}D^{j}E_{p,q} \mid k \in \mathbb{Z}, j \in \mathbb{Z}_+, p, q \in [1, N], kN + p - q = i\} \oplus \delta_{i,0}CC,$$

for $i \in \mathbb{Z}$. In particular, $\overline{D^N}_0 = \text{span}\{t^{k}D^{j}E_{p,p} \mid j \in \mathbb{Z}_+, p \in [1, N]\}$ is a commutative subalgebra. Note that $t$ has degree $N$.

**Definition 2.1.** A $\overline{D^N}$-module (or a $D^N_{as}$-module) $V$ is called a quasifinite module (e.g. [2]) if $V = \bigoplus_{j \in \mathbb{Z}} V_j$ is $\mathbb{Z}$-graded such that $\overline{D^N}_i V_j \subset V_{i+j}$ and $\dim V_j < \infty$ for $i, j \in \mathbb{Z}$. This is equivalent to saying that a quasifinite module is a module having finite-dimensional generalized weight spaces with respect to the commutative subalgebra $\overline{D^N}_0$. A quasifinite module $V$ is called a module of the intermediate series if $\dim V_j \leq 1$ for $j \in \mathbb{Z}$. It is called a uniformly bounded module if there exists
an integer $K > 0$ such that $\dim V_j \leq K$ for $j \in \mathbb{Z}$. A module $V$ is a \textit{trivial module} if $\mathcal{D}^N$ acts trivially on $V$ (i.e., $\mathcal{D}^N V = 0$).

Clearly a $\mathcal{D}_{as}^N$-module is also a $\mathcal{D}^N$-module (but the converse is not necessarily true), and a $\mathcal{D}^N$-module is a $\mathcal{D}_\infty^N$-module (with central charge 0). Thus it suffices to consider $\mathcal{D}^N$-modules.

We shall define 2 families of modules $V(\alpha), \mathbf{V}(\alpha)$, $\alpha \in \mathbb{C}$, of the intermediate series below. For a fixed $\alpha \in \mathbb{C}$, the obvious representation of $\widehat{\mathcal{D}}^N$ (with trivial action of the central element $C$) on the space $V(\alpha) = \mathbb{C}^N[t, t^{-1}]^{t\alpha}$ defines an irreducible module $V(\alpha)$. Let $\{\varepsilon_p = (\delta_{p1}, ..., \delta_{pN})^T | p \in [1, N]\}$ be the standard basis of $\mathbb{C}^N$, where the superscript "T" means the transpose of vectors or matrices (thus we regard elements of $\mathbb{C}^N$ as column vectors). Then the action of $\widehat{\mathcal{D}}^N$ on $V(\alpha)$ is

\begin{equation}
(t^i D^j E_{p,q})(t^{k+\alpha} \varepsilon_r) = \delta_{q,r}(k + \alpha) t^{i+k+\alpha} \varepsilon_p,
\end{equation}

for $i, k \in \mathbb{Z}, j \in \mathbb{Z}_+, p, q, r \in [1, N]$. For $j \in \mathbb{Z}$, let

\begin{equation}
V(\alpha)_j = \mathbb{C} t^{k+\alpha} \varepsilon_r,
\end{equation}

where $k \in \mathbb{Z}, r \in [1, N]$ are unique such that $j + 1 = kN + r$. Then $V(\alpha) = \bigoplus_{j \in \mathbb{Z}} V(\alpha)_j$ is a $\mathbb{Z}$-graded space such that $\dim V(\alpha)_j = 1$ for $j \in \mathbb{Z}$. Thus $V(\alpha)$ is a module of the intermediate series.

For $v \in \mathbb{C}^N, k \in \mathbb{Z},$ denote $v_k = t^{k+\alpha} v \in V(\alpha)$ (note that $v_k$ is in general not a homogeneous vector). For $A \in gl_N$, define $Av_k = (Av_k)$, where $Av$ is the natural action of $A$ on $v$ defined linearly by $E_{p,q} \varepsilon_r = \delta_{q,r} \varepsilon_p$ (i.e., the action is defined by the matrix-vector multiplication). Then (2.5) can be rewritten as

\begin{equation}
(t^i D^j A)v_k = (k + \alpha)^j Av_{i+k},
\end{equation}

for $i, k \in \mathbb{Z}, j \in \mathbb{Z}_+, A \in gl_N, v \in \mathbb{C}^N$. Clearly $V(\alpha)$ is also a $\mathcal{D}_{as}^N$-module.

Note that (cf. [5]) there exists a Lie algebra isomorphism $\sigma : \mathcal{D}^N \cong \mathcal{D}^N$ such that

\begin{equation}
\sigma(t^i D^j A) = (-1)^{i+1} t^i (D + i)^j A^T,
\end{equation}

for $i \in \mathbb{Z}, j \in \mathbb{Z}_+, A \in gl_N$ Using this isomorphism, we have another $\widehat{\mathcal{D}}^N$-module $\mathbf{V}(\alpha)$ (again with trivial action of $C$), called the twisted module of $V(\alpha)$, defined by

\begin{equation}
(t^i D^j A)v_k = (-1)^{i+1} (i + k + \alpha)^j A^T v_{i+k},
\end{equation}

for $i, k \in \mathbb{Z}, j \in \mathbb{Z}_+, A \in gl_N, v \in \mathbb{C}^N$. In fact, $\mathbf{V}(\alpha)$ is the dual module of $V(-\alpha)$: If we define a bilinear form on $\mathbf{V}(\alpha) \times V(-\alpha)$ by $\langle t^{i+\alpha} \varepsilon_p, t^{j-\alpha} \varepsilon_q \rangle = \delta_{i+j, \alpha} \delta_{p,q}$, then

$\langle xv, v \rangle = -\langle v, xv \rangle$,

for $x \in \widehat{\mathcal{D}}^N, v \in \mathbf{V}(\alpha), v \in V(-\alpha)$. Note that a $\mathbb{Z}$-gradation of $\mathbf{V}(\alpha)$ can be defined by (2.6) with $k, r$ satisfying the relation $j = kN + N - r$. Obviously, $\mathbf{V}(\alpha)$ is not a $\mathcal{D}_{as}^N$-module.

Now we can generalize the above modules $V(\alpha)$ and $\mathbf{V}(\alpha)$ as follows: Let $m > 0$ be an integer, and let $\alpha$ be an indecomposable linear transformation on $\mathbb{C}^m$ (thus up to equivalences, $\alpha$ is uniquely determined by its unique eigenvalue $\lambda$). Let $gl_N$, $\alpha$ act on $\mathbb{C}^N \otimes \mathbb{C}^m$ defined by $A(u \otimes v) = Au \otimes v, \alpha(u \otimes v) = u \otimes \alpha v$ for $A \in gl_N, u \in \mathbb{C}^N, v \in \mathbb{C}^m$. Then by allowing $v$ to be in $\mathbb{C}^N \otimes \mathbb{C}^m$ in (2.7) and (2.9), we obtain 2 families of indecomposable uniformly bounded modules $V(m, \alpha) = V(\alpha) \otimes \mathbb{C}^m$, $
\mathbf{V}(m, \alpha) = \mathbf{V}(\alpha) \otimes \mathbb{C}^m$. 

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The main result of this paper is the following theorem.

**Theorem 2.2.** (i) An irreducible quasifinite module over $\widehat{D}^N$ is either a highest or lowest weight module or else a module of the intermediate series. A nontrivial module of the intermediate series is a module $V(\alpha)$ or $\overline{V}(\alpha)$ for some $\alpha \in \mathbb{C}$.

(ii) A nontrivial indecomposable uniformly bounded module over $\widehat{D}^N$ is a module $V(m, \alpha)$ or $\overline{V}(m, \alpha)$ for some $m \in \mathbb{Z}_+ \setminus \{0\}$ and some indecomposable linear transformation $\alpha$ of $\mathbb{C}^m$.

Thus in particular we obtain that a nontrivial indecomposable uniformly bounded module over $\widehat{D}^N$ is simply a $D_{ac}^N$-module (if the central element $I = t^0D^0 I$ acts by 1, where $I$ is the $N \times N$ identity matrix) or its twist (if $I$ acts by $-1$), and that there is an equivalence between the category of uniformly bounded $D_{ac}^N$-modules without the trivial composition factor and the category of linear transformations on finite-dimensional vector spaces.

Since irreducible quasifinite highest weight modules have been classified in [2] and irreducible lowest weight modules are simply dual modules of irreducible highest weight modules, this theorem in fact classifies all irreducible quasifinite modules over $\widehat{D}^N$ and over $D^N$.

Note that in the language of conformal algebras, this theorem in particular also gives proofs of Theorems 6.1 and 6.2 of [7] on the classification of finite indecomposable modules over the conformal algebras $\text{C}_{\text{end}}$ and $\text{gc}_N$.

The analogous results of this theorem for affine Lie algebras, the Virasoro algebra, higher-rank Virasoro algebras and Lie algebras of Weyl type were obtained in [3], [12]–[14].

Note that the space $\mathcal{H} = \text{span}\{C, D, E_{p,p} | p \in [1, N]\}$ is a Cartan subalgebra of $\widehat{D}^N$ and that the definition of quasifiniteness does not require that $V$ is a weight module (i.e., the actions of elements of $\mathcal{H}$ on $V$ are diagonalizable). If we require $V$ to be a weight module, then the linear transformation $\alpha$ is diagonalizable, and thus all uniformly bounded modules are completely reducible.

We shall prove the above theorem in the next section.

### 3. Proof of Theorem 2.2

We shall keep the notation of the previous section. We denote by $I$ the $N \times N$ identity matrix. When the context is clear, we often omit the symbol $I$; for instance, $t = t^1D^1 I \in D^N$. Denote $\mathcal{H} = \text{span}\{C, D, E_{p,p} | p \in [1, N]\}$, a Cartan subalgebra of $\widehat{D}^N$.

For any $\mathbb{Z}$-graded vector space $U$, we use the notation $U_+, U_-$ and $U_{[p,q]}$ to denote the subspaces spanned by elements of degree $k$ with $k > 0$, $k < 0$ and $p \leq k < q$ respectively. Then $\widehat{D}^N$ has a triangular decomposition $\widehat{D}^N = (\widehat{D}^N)_+ \oplus (\widehat{D}^N)_0 \oplus (\widehat{D}^N)_-$.

Observe that $(\widehat{D}^N)_+$ is generated by the adjoint action of $t$ on $(\widehat{D}^N)_{[0,N]}$ and that $\text{ad}_t$ is locally nilpotent on $\widehat{D}^N$.

Suppose $V = \bigoplus_{i \in \mathbb{Z}} V_i$ is a quasifinite module over $\mathcal{W}$. A homogeneous nonzero vector $v \in V$ is called a highest (resp., lowest) weight vector if $(\widehat{D}^N)_0 v \subseteq \mathbb{C}v$, and $(\widehat{D}^N)_+ v = 0$ (resp., $(\widehat{D}^N)_- v = 0$).

We divide the proof of Theorem 2.2 into 3 lemmas.

**Lemma 3.1.** Suppose $V$ is an irreducible quasifinite $\widehat{D}^N$-module without highest and lowest weight vectors. Then $t|_{V_i} : V_i \to V_{i+N}$ and $t^{-1}|_{V_i} : V_i \to V_{i-N}$ are
injective and thus bijective for all $i \in \mathbb{Z}$ (recall (2.4) that $t$ has degree $N$). In particular, by letting $K = \max\{\dim V_p \mid p \in [1, N]\}$, we have $\dim V_i \leq K$ for $i \in \mathbb{Z}$; thus $V$ is uniformly bounded.

**Proof.** Say $tv_0 = 0$ for some $0 \neq v_0 \in V_i$. By shifting the grading index if necessary, we can suppose $i = 0$. Since $x|_{V_0} : V_0 \rightarrow V_{0,N}$ for $x \in (\mathring{D}^N)_{0,N}$ are linear maps on finite-dimensional vector spaces, there exists a finite subset $S \subset (\mathring{D}^N)_{0,N}$ such that all $x|_{V_0}, x \in (\mathring{D}^N)_{0,N}$ are linear combinations of $S|_{V_0} = \{y|_{V_0} \mid y \in S\}$. This implies that $(\mathring{D}^N)_{0,N}v_0 = (\text{span } S)v_0$. Recall that $\text{ad}_t$ is locally nilpotent such that $(\mathring{D}^N)_j \subset \text{ad}_t^k((\mathring{D}^N)_{0,N})$ for $j > 0$, where $k \geq 0$ is the integer such that $0 \leq j - kN < N$. Choose $n > 0$ such that $\text{ad}_t^n(S) = 0$. Let $j \geq n$. Then $k \geq n$ and we have

$$(\mathring{D}^N)_j v_0 \subset (\text{ad}_t^k((\mathring{D}^N)_{0,N})j)v_0 = t^k(\mathring{D}^N)_{0,N}v_0 = t^k(\text{span } S)v_0 = (\text{ad}_t^k(\text{span } S))v_0 = 0.$$

This means that $(\mathring{D}^N)_{n,N,\infty}v_0 = 0$.

The rest of the proof is exactly like that of Proposition 2.1 in [13]. \hfill \Box

**Lemma 3.2.** A nontrivial irreducible uniformly bounded module $V$ is a module $V(\alpha)$ or $\mathring{V}(\alpha)$ for some $\alpha \in \mathbb{C}$.

**Proof.** Let $V' = \text{span}\{v \in V \mid Hv \subset Cv\}$ be the space spanned by weight vectors. Clearly $V'$ is a submodule. Since $\dim V_i < \infty$, there exists at least a common eigenvector (i.e., a weight vector) of $H$ in $V_i$ for $i \in \mathbb{Z}$, i.e., $V' \neq 0$. Thus $V = V'$ is a weight module. Since $C, I$ are central elements, we have (cf. Remark 3.4 below)

$$C|_V = c_0 \cdot 1_V, \quad I|_V = c_1 \cdot 1_V,$$

for some $c_0, c_1 \in \mathbb{C}$. Let

$$\text{Vir} = \text{span}\{t^i D, C \mid i \in \mathbb{Z}\}, \quad \mathcal{W} = \text{span}\{t^i D^1, C \mid i \in \mathbb{Z}, j \in \mathbb{Z}_+\}$$

be subalgebras of $\mathring{D}^N$ isomorphic to the Virasoro algebra and the Lie algebra $\mathring{D}^1$ respectively. For $m \in [0, N - 1]$, set

$$V[m] = \bigoplus_{i \in \mathbb{Z}} V_{i,N+m}.$$  \hfill (3.1)

Clearly, $V[m]$ is a uniformly bounded module over $\text{Vir}$ or $\mathcal{W}$, and $V = \bigoplus_{m=0}^{N-1} V[m]$. Since $V \neq 0$, we have $V[m] \neq 0$ for some $m$. Say, $V[0] \neq 0$. Obviously, a composition factor of the Vir-module $V[0]$ is a Vir-module of the intermediate series, on which the central element $C$ must act trivially (cf. [12, 14]); thus $c_0 = 0$ (and so we can omit $C$ in the following discussion).

By the structure of uniformly bounded $\mathcal{W}$-modules in [13], we have $c_1 = 0, \pm 1$. If $c_1 = 0$, by [13], each $V[m]$ must be a trivial $\mathcal{W}$-module; thus $V$ is trivial as a $\mathcal{W}$-module. But since $[\mathcal{W}, \mathring{D}^N] = \mathring{D}^N$, we obtain that $V$ is a trivial $\mathring{D}^N$-module, contradicting the assumption of the lemma. Thus $c_1 \neq 0$. If necessary, by using the isomorphism $\sigma$ in (2.8) (which interchanges $V(\alpha)$ with $\mathring{V}(\alpha)$), we can always suppose $c_1 \neq -1$ (since $\sigma(I) = -I$). Thus $c_1 = 1$. Thus again by [13], each $\mathcal{W}$-module $V[m]$ must have the form $A_{p,G}$ defined in [13], i.e., there exist $K_m \geq 0$
(might depend on m) and a $K_m \times K_m$ diagonal matrix $G_m$ such that we can choose a suitable basis $Y_{N+k+1} = (y_{(1)}^{K_m+1}, ..., y_{(K_m+1)}^{(K_m)})$ of $V_{N+k+1}$ for $k \in \mathbb{Z}$ satisfying

$$
(t^i D^j)Y_{N+k+1} = Y_{(i+k)N+m}(k + G_m)^j,
$$

for $i, k \in \mathbb{Z}$, $j \in \mathbb{Z}_+$, where the right-hand side is the vector-matrix multiplication by regarding $Y_{N+k+1}$ as a row vector, and where, here and below, when the context is clear, in an expression involving matrices, we always identify a scalar (such as $p$ in the right-hand side) with the corresponding scalar matrix of the suitable rank.

Note that for $p, q \in [1, N]$, $tE_{p,q}$ has degree $N + p - q$; thus $(tE_{p,q})Y_{N+k+m} \subset \langle Y_{N+k+m+p-q} \rangle$. Let $k_1, m_1$ be integers such that $m_1 \in [0, N - 1]$ and $kN + m + p - q = k_1 N + m_1$.

Then we can write

$$
(tE_{p,q})Y_{N+k+m} = Y_{(k_1+1)N+m_1} P^{(p,q)},
$$

where $P^{(p,q)}$ is some $K_{m_1} \times K_{m}$ matrix. Applying $[t^i, tE_{p,q}] = 0$ and $[D, E_{p,q}] = 0$ to $Y_{N+k+m}$, using (3.2) and (3.3), we obtain that $P^{(p,q)}$ does not depend on $k$ (and denote it by $P^{(m)})$ and

$$
G_m P^{(m)} = P^{(m)} G_m.
$$

Applying $[t^i D^j, tE_{p,q}] = \sum_{s=1}^{j} [t^i D^{j-s}] E_{p,q}$ to $Y_{N+k+m}$, by induction on $j$, we obtain

$$
(t^i D^j E_{p,q})Y_{N+k+m} = Y_{(k_1+i)N+m_1} P^{(m)} (k + G_m)^j.
$$

Let $K = \sum_{m=0}^{N-1} K_m$ and let $U = \bigoplus_{m=0}^{N-1} V_m$ be a subspace of $V$ of dimension $K$. Then

$$
(Y_0, Y_1, ..., Y_{N-1}) = (y_0^{(1)}, ..., y_0^{(K_m)}, y_1^{(1)}, ..., y_1^{(K_m)}, ..., y_{N-1}^{(1)}, ..., y_{N-1}^{(K_m-1)})
$$

is a basis of $U$. For each $E_{p,q} \in gl_N$, we define a linear transformation $\rho(E_{p,q})$ of $U$ as follows:

$$
\rho(E_{p,q})Y_m = Y_{m_1} P^{(m)},
$$

for $m \in [0, N - 1]$, where $m_1 \in [0, N - 1]$ such that $m_1 \equiv m + p - q \pmod{N}$. This uniquely defines a linear map $\rho : gl_N \to \text{End}(U)$. We prove that $\rho(E_{p,q})\rho(E_{p',q'}) = \rho(E_{p',q'})\rho(E_{p,q})$. By shifting the index of $V_k$ if necessary, it suffices to prove

$$
\rho(E_{p,q})\rho(E_{p',q'}) Y_0 = \rho(E_{p,q})\rho(E_{p',q'}) Y_0 = \delta_{p,p'}\rho(E_{p,q}) Y_0.
$$

Note from (2.1) and (2.2) that

$$
[DE_{p,q}, tE_{p,q}] = tE_{p,q} E_{p',q'} + tD[E_{p,q}, E_{p,q'}]
$$

$$
= \delta_{q,p} tE_{p,q} + \delta_{q,p'} tDE_{p,q} - \delta_{q,p} tDE_{p,q'}.
$$

First suppose $p \geq q$, $p' \geq q'$, applying (3.8) to $Y_{N+k}$, by (3.5), we obtain

$$
P^{(p-q)'}(k_{p',q'}) = \delta_{q,p} P^{(0)}_{p,q} + \delta_{q,p'} P^{(0)}_{p',q} - \delta_{q,p} P^{(0)}_{p,q'} - \delta_{q,p'} P^{(0)}_{p',q'},
$$

where we denote $k = k + G_0$. Since $k$ commutes with $P^{(0)}_{p',q'}$, by (3.4), regarding expressions in (3.9) as polynomials on $k$, by comparing the coefficient of $k^0$ we obtain

$$
P^{(p-q)'}(p',q') = \delta_{q,p} P^{(0)}_{p,q'},
$$

which is equivalent to (3.7). By symmetry, we also have (3.7) if $p \leq q$, $p' \leq q'$. Now suppose $p < q$, $p' \geq q'$. Again applying (3.8) to $Y_{N+k}$, we have (3.9) with $P^{(p-q)'}_{p',q'}$. 
Finally suppose $p \geq q$, $p' < q'$. Then we have (3.9) and (3.10) with $P_{p,q}$ replaced by $P_{p',q'}$ and again we have (3.7).

Thus $\rho$ is a representation of the simple associative algebra $gl_N$ (= End$_N$).

Thus $U = \bigoplus_{r=1}^{m} U^{(r)}$ is decomposed as a direct sum of simple $gl_N$-submodules $U^{(r)}$ such that each $U^{(r)}$ is either the natural $gl_N$-module ($\cong \mathbb{C}^N$) or the trivial module. Since $I|_U$ is the identity map, we have that each $U^{(r)}$ is the natural module. Since $[D, gl_N] = 0$ and $D|_U$ is diagonalizable, we can choose submodules $U^{(s)}$ such that $D(U^{(s)}) \subset U^{(s)}$ and $D|_{U^{(s)}}$ is a scalar map. Now clearly $U^{(1)}$ generates a simple $\mathbb{C}^N$-submodule of $V$ of the form $V(\alpha)$ (cf. (3.5) and (3.6)). Since $V$ is irreducible, we have $V = V(\alpha)$ for some $\alpha \in \mathbb{C}$ (if we have used the isomorphism $\sigma$ in (2.8) in the above proof, then $V$ is the module $\nabla(\alpha)$). 

**Lemma 3.3.** A nontrivial indecomposable uniformly bounded module $V$ is a module of the form $V(m, \alpha)$ or $\nabla(m, \alpha)$.

**Proof.** First note that a central element, while not necessarily acting by a scalar on an indecomposable module, nevertheless has only one eigenvalue (cf. Remark 3.4 below). Let $c_0$ and $c_1$ be the eigenvalues of $C$ and $I$ respectively. As in the arguments of the proof of Lemma 3.2, we have $c_0 = 0$ (thus $C$ acts nilpotently on $V$) and we can suppose $c_1 = 1$ (by making use of the isomorphism $\sigma$ in (2.8)). Thus each composition factor of $V$ has the form $V(\alpha)$. Therefore $V$ has a finite number $m$ of composition factors ($m = \dim V_0$). By induction on $m$, it suffices to consider the case when $m = 2$. In this case $V$ is not irreducible but indecomposable.

First suppose

\[
C|_V = 0, \quad I|_V = 1_V.
\]

Following the proof of Lemma 3.2 (now $G_m$ is not necessarily diagonal), we have $U = U^{(1)} \oplus U^{(2)}$, and both $U^{(1)}$ and $U^{(2)}$ are the natural $gl_N$-modules. Since $D(U) \subset U$, $[D, gl_N] = 0$ and $V$ is not irreducible, the subspace $U' = \{u \in U | Du \in Cu\}$ of eigenvectors of $D$ is a proper (and thus simple) $gl_N$-module of $U$ (isomorphic to $\mathbb{C}^N$ as a $gl_N$-module) and $D|_{U'}$ is a scalar map $\lambda$ for some $\lambda \in \mathbb{C}$. Thus $U = U' \oplus U''$, where $U''$ is another copy of $U'$ such that $Du'' = \lambda u'' + u'$ for $u'' \in U''$, where $u' \in U'$ is the corresponding copy of $u''$. Therefore $U \cong \mathbb{C}^N \otimes \mathbb{C}^2$ such that $gl_N$ acts on $\mathbb{C}^N$ and $D$ acts on $\mathbb{C}^2$ (and $\alpha = D|_{\mathbb{C}^2}$ is an indecomposable linear transformation on $\mathbb{C}^2$), and we obtain $V = V(2, \alpha)$ (if we have used the isomorphism $\sigma$ in (2.8) in the above proof, then $V$ is the module $\nabla(2, \alpha)$).

It remains to prove (3.11). We shall use (2.3), which can be rewritten as follows:

\[
[t^{i+j}\left(\frac{d}{dt}\right)^j A, t^{k+l}\left(\frac{d}{dt}\right)^l B] = \sum_{s=0}^{j} \binom{j}{s} [k+l]_s t^{i+j+k+l-s} \left(\frac{d}{dt}\right)^{j+l-s} AB
\]

\[
-\sum_{s=0}^{l} \binom{l}{s} [i+j]_s t^{i+j+k+l-s} \left(\frac{d}{dt}\right)^{j+l-s} BA
\]

\[
+ \delta_{i-k} (-1)^{j+l}! \binom{i+j}{i+j+k+l} \operatorname{tr}(AB) C,
\]

where $[k]_s = k(k-1) \cdots (k-s+1)$ is a similar notation to $[D]_s$ in (2.3).

Consider the $W$-module $V[0]$ (cf. (3.1)). As in the proof of Proposition 2.2 of [13], we can choose a basis $X_0 = (x_0^{(1)}, x_0^{(2)})$ of $V_0$ and define a basis $X_n = (x_n^{(1)}, x_n^{(2)})$ of
V_{n,N} by induction on |n| such that tX_n = X_{n+1} for n \in \mathbb{Z}. Assume that

\[ CX_n = P_i C_{n+1}, \quad (t^{i+j}(\frac{\delta}{\delta t})^j)X_n = X_{n+i}P_{i,j,n}, \]

for some 2 \times 2 matrices C_{n}, P_{i,j,n}. Using \[ C, t \] = 0, we obtain C_{n} = C_{0}. Using \[ t^{i+j}(\frac{\delta}{\delta t})^j, t \] = \( j \) \( t^{i+j}(\frac{\delta}{\delta t})^j \)^{-1}, we obtain \( P_{i,j,n+1} - P_{i,j,n} = jP_{i+1,j-1,n} \). Thus

induction on \( j \) gives

\[ P_{i,0,n} = P_i, \quad P_{i,1,n} = nP_{i+1} + Q_i, \]

\[ P_{i,2,n} = [\bar{n}]_2P_{i+2} + 2\bar{n}Q_{i+1} + R_i, \]

\[ P_{i,3,n} = [\bar{n}]_3P_{i+3} + 3[\bar{n}]_2Q_{i+2} + 3\bar{n}R_{i+1} + S_i, \]

for some 2 \times 2 matrices \( P_i, Q_i, R_i, S_i \), where \( \bar{n} = n + G_0 \) for some fixed 2 \times 2 matrix \( G_0 \) (cf. (3.2)). \( [\bar{n}]_j \) is again a similar notation to \([D]_j\), and \( Q_0 = 0 \). (We use the notation \( \bar{n} = n + G_0 \) in order to ensure that \( Q_0 = 0 \). Note from \([t, t^{i+j}(\frac{\delta}{\delta t})^j] = (i + j) t^{i+j}(\frac{\delta}{\delta t})^j \) that \( G_0 \) commutes with all matrices \( C_0, P_i, Q_i, R_i, S_i \).

By choosing a composition series of \( V[0] \), we can assume that all these matrices are upper triangular matrices. Furthermore, by the structure of modules of the intermediate series, we see that

\[ (3.13) \quad P_i \text{ have the form } (I_0^1 0^1), \quad \text{and } C_0, Q_i, R_i, S_i \text{ have the form } (I_0^0 0^0). \]

Thus all matrices in (3.13) are commutative.

Applying \([t^{i+j}(\frac{\delta}{\delta t})^j, t^k] = k(t^{i+j} - \delta_i - k(t^{i+1})NC \) (cf. (3.12)) to \( X_n \), we obtain \( kP_i + P_k = kP_{i+k} - \delta_i - k(t^{i+1})NC_0 \), from which we obtain \( P_i = P_0^{i+1} = (1 - i)P_0 + i \) for \( i \in \mathbb{Z} \) (using (3.13), we have \( (P_0 - 1)^2 = 0 \) and \( C_0 = 0 \) (thus \( C|_V|_0 = 0 \) and similarly \( C|_V|_m = 0 \) for \( m \in [0, N - 1] \)), and so in the following, we can omit \( C \).

Applying \([t^2(\frac{\delta}{\delta t})^2, t^k] = 2k(t^{i+1} + \delta_i - k(t^{i+1})NC_0 \), we obtain \((k)_2P_2 + 2kQ_1P_2 = 2kQ_k + [k]_2P_k\). Thus

\[ (3.14) \quad Q_k = Q_1P_k + \frac{k-1}{k}(P_2 - 1)P_k = Q_1 + \frac{k-1}{k}(1 - P_0), \]

if \( k \neq 0 \) (using (3.13), we have \( Q_1P_k = Q_1, (P_2 - 1)P_k = P_2 - 1 \)). Letting \( j = 2, l = 0 \) and applying (3.12) to \( X_n \), we see that (3.14) also holds for \( k = 0 \). Since \( Q_0 = 0 \), we obtain \( Q_k = \frac{k}{2}(1 - P_0) \). Similarly, letting \( i = l = 0, j = 3 \) and applying (3.12) to \( X_n \), we obtain \((k)_3P_3 + 3[\bar{n}]_2Q_2 + 3\bar{n}R_1)P_k = 3\bar{n}R_k + 3[\bar{n}]_2Q_k + [\bar{n}]_3P_k\), from which we obtain

\[ R_k = R_1 + \frac{(k-1)(k-2)}{6}(P_3 - 1) + (k - 1)(Q_2 - Q_k) = R_1 + \frac{(k-1)(k-2)}{6}(1 - P_0). \]

Finally letting \( j = 3, l = 0 \) and applying (3.12) to \( X_n \), we obtain \( P_0 = 1 \). Thus \( I|_V|_0 = 1 \). Similarly \( I|_V|_m = \mathbf{1}_V|_m \). This proves (3.11), thus the lemma. □

Theorem 2.2 now follows from Lemmas 3.1–3.3.

Remark 3.4. We would like to point out that a central element does not necessarily act as a scalar on an indecomposable module since we do not assume that a central element acts diagonalizable. Thus there is a gap in the assertion in (2.1) of [13]. This gap has been filled in the above proof of Lemma 3.3.

References


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