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CLASSIFICATION OF QUASIFINITE MODULES OVER LIE ALGEBRAS OF MATRIX DIFFERENTIAL OPERATORS ON THE CIRCLE

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Abstract. We prove that an irreducible quasifinite module over the central extension of the Lie algebra of $N \times N$-matrix differential operators on the circle is either a highest or lowest weight module or else a module of the intermediate series. Furthermore, we give a complete classification of indecomposable uniformly bounded modules.

1. Introduction

Kac [5] introduced the notion of conformal algebras. Conformal algebras play important roles in quantum field theory and vertex operator algebras (e.g. [5]), whose study has drawn much attention in the literature (e.g. [1], [5]–[7], [16]–[19]). It is pointed out in [7] that the infinite rank associative conformal algebra Cend_N and the Lie conformal algebra gc_N (the *general Lie conformal algebra*) play the same important roles in the theory of conformal algebras as End_N and gl_N play in the theory of associative and Lie algebras.

There is a one-to-one correspondence between Lie conformal algebras and maximal formal distribution Lie algebras [1, 6, 7]. The Lie algebra $\mathcal{D}^N$ of $N \times N$-matrix differential operators on the circle is a formal distribution Lie algebra associated to the general Lie conformal algebra gc_N. Let $\widehat{\mathcal{D}^N}$ be the universal central extension of $\mathcal{D}^N$. In particular when $N = 1$, $\widehat{\mathcal{D}^1}$ is also known as the W-infinity algebra $\mathcal{W}_{1+\infty}$. Thus one may expect that the representation theory of $\mathcal{D}^N$ and $\widehat{\mathcal{D}^N}$ is very interesting and important (e.g. [2], [4], [8]–[10], [13]).

As is pointed out in [8, 10], although $\widehat{\mathcal{D}^N}$ is a $\mathbb{Z}$-graded Lie algebra, each grading space is still infinite dimensional (cf. (2.4)), and the classification of quasifinite modules is thus a nontrivial problem. The classification of irreducible quasifinite highest weight modules over $\widehat{\mathcal{D}^N}$ was given by Kac and Radul [8] for the case $N = 1$. For the general N, the classification was obtained by Boyallian, Kac, Liberati and Yan [2]. In [13], the author obtained the classification of the irreducible quasifinite modules and indecomposable uniformly bounded modules over $\widehat{\mathcal{D}^1}$. We would like

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to take this chance to point out that there is a slight gap in the proof of Proposition 2.2 of [13]; this gap has been filled in this paper (see Remark 3.4).

In this paper, we generalize the results in [13] to the general case: Precisely, we obtain that an irreducible quasifinite module over $\widehat{\mathcal{D}^N}$ (thus also over $\mathcal{D}^N$) is either a highest or lowest weight module or else a module of the intermediate series. Furthermore, we give a complete classification of indecomposable uniformly bounded modules (Theorem 2.2).

2. Notation and main theorem

Let $N \geq 1$ be an integer. Let $\mathbb{C}[t, t^{-1}]$ be the Laurent polynomial algebra over the variable t, let $\mathbb{C}[D]$ be the polynomial algebra over $D = t\frac{d}{dt}$, and let gl_N be the space of $N \times N$ matrices. The *Lie algebra $\mathcal{D}^N$ of $N \times N$-matrix differential operators on the circle* is the tensor product space $\mathcal{D}^N = \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[D] \otimes gl_N$, spanned by $\{t^i D^j A \mid i \in \mathbb{Z},\, j \in \mathbb{Z}_+,\, A \in gl_N\}$, with the Lie bracket:

$$[t^i D^j A, t^k D^l B] = (t^i D^j A) \cdot (t^k D^l B) - (t^k D^l B) \cdot (t^i D^j A) \tag{2.1}$$

and

$$(t^i D^j A) \cdot (t^k D^l B) = t^{i+k}(D+k)^j D^l AB = \sum_{s=0}^{j} \binom{j}{s} k^s t^{j+k} D^{j+l-s} AB, \tag{2.2}$$

for $i, j \in \mathbb{Z}$, $j, l \in \mathbb{Z}_+$, where $\binom{i}{s} = \frac{i(i-1)\cdots(i-s+1)}{s!}$ if $s \geq 0$ and $\binom{i}{s} = 0$ if $s < 0$ is the binomial coefficient. The associative algebra with the underlined vector space $\mathcal{D}^N$ and the product (2.2) is denoted by $\mathcal{D}^N_{\text{as}}$.

It is proved in [11] that $\mathcal{D}^N$ has a unique nontrivial central extension. The *universal central extension $\widehat{\mathcal{D}^N}$ of $\mathcal{D}^N$* is defined as follows (cf. [2]): The Lie bracket (2.1) is replaced by (cf. (3.12))

$$\begin{aligned}[t^i [D]_j A, t^k [D]_l B] = {} & (t^i [D]_j A) \cdot (t^k [D]_l B) - (t^k [D]_l B) \cdot (t^i [D]_j A) \\ & + \delta_{i,-k} (-1)^j j! l! \binom{i+j}{j+l+1} \operatorname{tr}(AB) C,\end{aligned} \tag{2.3}$$

for $i, j \in \mathbb{Z}$, $j, l \in \mathbb{Z}_+$, where $[D]_j = D(D-1)\cdots(D-j+1) = t^j (\frac{d}{dt})^j$ and $\operatorname{tr}(A)$ is the trace of the matrix A, and where C is a nonzero central element of $\widehat{\mathcal{D}^N}$.

For $m, n \in \mathbb{Z}$, we denote $[m, n] = \{m, m+1, ..., n\}$. Let $\{E_{p,q} \mid p, q \in [1, N]\}$ be the standard basis of gl_N, where $E_{p,q}$ is the matrix with entry 1 at (p, q) and 0 otherwise. Then $\widehat{\mathcal{D}^N}$ has the *principal $\mathbb{Z}$-gradation* $\widehat{\mathcal{D}^N} = \bigoplus_{i \in \mathbb{Z}} (\widehat{\mathcal{D}^N})_i$ with the grading space (cf. [2])

$$(\widehat{\mathcal{D}^N})_i = \operatorname{span}\{t^k D^j E_{p,q} \mid k \in \mathbb{Z},\, j \in \mathbb{Z}_+,\, p, q \in [1, N],\, kN + p - q = i\} \oplus \delta_{i,0} \mathbb{C} C, \tag{2.4}$$

for $i \in \mathbb{Z}$. In particular, $(\widehat{\mathcal{D}^N})_0 = \operatorname{span}\{D^j E_{p,p}, C \mid j \in \mathbb{Z}_+, p \in [1, N]\}$ is a commutative subalgebra. Note that t has degree N.

Definition 2.1. A $\widehat{\mathcal{D}^N}$-module (or a $\mathcal{D}^N_{\text{as}}$-module) V is called a *quasifinite module* (e.g. [2]) if $V = \bigoplus_{j \in \mathbb{Z}} V_j$ is $\mathbb{Z}$-graded such that $(\widehat{\mathcal{D}^N})_i V_j \subset V_{i+j}$ and $\dim V_j < \infty$ for $i, j \in \mathbb{Z}$. This is equivalent to saying that a quasifinite module is a module having finite-dimensional generalized weight spaces with respect to the commutative subalgebra $(\widehat{\mathcal{D}^N})_0$. A quasifinite module V is called a *module of the intermediate series* if $\dim V_j \leq 1$ for $j \in \mathbb{Z}$. It is called a *uniformly bounded module* if there exists

an integer $K > 0$ such that $\dim V_j \leq K$ for $j \in \mathbb{Z}$. A module V is a *trivial module* if $\widehat{\mathcal{D}^N}$ acts trivially on V (i.e., $\widehat{\mathcal{D}^N} V = 0$).

Clearly a $\mathcal{D}^N_{\text{as}}$-module is also a $\mathcal{D}^N$-module (but the converse is not necessarily true), and a $\mathcal{D}^N$-module is a $\widehat{\mathcal{D}^N}$-module (with central charge 0). Thus it suffices to consider $\widehat{\mathcal{D}^N}$-modules.

We shall define 2 families of modules $V(\alpha), \overline{V}(\alpha)$, $\alpha \in \mathbb{C}$, of the intermediate series below. For a fixed $\alpha \in \mathbb{C}$, the obvious representation of $\widehat{\mathcal{D}^N}$ (with trivial action of the central element C) on the space $V(\alpha) = \mathbb{C}^N[t, t^{-1}]t^\alpha$ defines an irreducible module $V(\alpha)$. Let $\{\varepsilon_p = (\delta_{p1}, ..., \delta_{pN})^{\mathrm{T}} \,|\, p \in [1, N]\}$ be the standard basis of $\mathbb{C}^N$, where the superscript "T" means the transpose of vectors or matrices (thus we regard elements of $\mathbb{C}^N$ as column vectors). Then the action of $\widehat{\mathcal{D}^N}$ on $V(\alpha)$ is

$$(t^i D^j E_{p,q})(t^{k+\alpha}\varepsilon_r) = \delta_{q,r}(k+\alpha)^j t^{i+k+\alpha}\varepsilon_p, \tag{2.5}$$

for $i, k \in \mathbb{Z}$, $j \in \mathbb{Z}_+$, $p, q, r \in [1, N]$. For $j \in \mathbb{Z}$, let

$$V(\alpha)_j = \mathbb{C}t^{k+\alpha}\varepsilon_r, \tag{2.6}$$

where $k \in \mathbb{Z}$, $r \in [1, N]$ are unique such that $j + 1 = kN + r$. Then $V(\alpha) = \bigoplus_{j\in\mathbb{Z}} V(\alpha)_j$ is a $\mathbb{Z}$-graded space such that $\dim V(\alpha)_j = 1$ for $j \in \mathbb{Z}$. Thus $V(\alpha)$ is a module of the intermediate series.

For $v \in \mathbb{C}^N$, $k \in \mathbb{Z}$, denote $v_k = t^{k+\alpha}v \in V(\alpha)$ (note that v_k is in general not a homogeneous vector). For $A \in gl_N$, define $Av_k = (Av)_k$, where Av is the natural action of A on v defined linearly by $E_{p,q}\varepsilon_r = \delta_{q,r}\varepsilon_p$ (i.e., the action is defined by the matrix-vector multiplication). Then (2.5) can be rewritten as

$$(t^i D^j A)v_k = (k+\alpha)^j A v_{i+k}, \tag{2.7}$$

for $i, k \in \mathbb{Z}$, $j \in \mathbb{Z}_+$, $A \in gl_N$, $v \in \mathbb{C}^N$. Clearly $V(\alpha)$ is also a $\mathcal{D}^N_{\text{as}}$-module.

Note that (cf. [15]) there exists a Lie algebra isomorphism $\sigma : \mathcal{D}^N \cong \mathcal{D}^N$ such that

$$\sigma(t^i D^j A) = (-1)^{j+1} t^i (D+i)^j A^{\mathrm{T}}, \tag{2.8}$$

for $i \in \mathbb{Z}$, $j \in \mathbb{Z}_+$, $A \in gl_N$. Using this isomorphism, we have another $\widehat{\mathcal{D}^N}$-module $\overline{V}(\alpha)$ (again with trivial action of C), called the *twisted module of* $V(\alpha)$, defined by

$$(t^i D^j A)v_k = (-1)^{j+1}(i+k+\alpha)^j A^{\mathrm{T}} v_{i+k}, \tag{2.9}$$

for $i, k \in \mathbb{Z}$, $j \in \mathbb{Z}_+$, $A \in gl_N$, $v \in \mathbb{C}^N$. In fact, $\overline{V}(\alpha)$ is the dual module of $V(-\alpha)$: If we define a bilinear form on $\overline{V}(\alpha) \times V(-\alpha)$ by $\langle t^{i+\alpha}\varepsilon_p, t^{j-\alpha}\varepsilon_q\rangle = \delta_{i+j,0}\delta_{p,q}$, then

$$\langle xv, v\rangle = -\langle v, xv\rangle,$$

for $x \in \widehat{\mathcal{D}^N}$, $v \in \overline{V}(\alpha)$, $v \in V(-\alpha)$. Note that a $\mathbb{Z}$-gradation of $\overline{V}(\alpha)$ can be defined by (2.6) with k, r satisfying the relation $j = kN + N - r$. Obviously, $\overline{V}(\alpha)$ is not a $\mathcal{D}^N_{\text{as}}$-module.

Now we can generalize the above modules $V(\alpha)$ and $\overline{V}(\alpha)$ as follows: Let $m > 0$ be an integer, and let α be an indecomposable linear transformation on $\mathbb{C}^m$ (thus up to equivalences, α is uniquely determined by its unique eigenvalue λ). Let gl_N, α act on $\mathbb{C}^N \otimes \mathbb{C}^m$ defined by $A(u \otimes v) = Au \otimes v$, $\alpha(u \otimes v) = u \otimes \alpha v$ for $A \in gl_N$, $u \in \mathbb{C}^N$, $u \in \mathbb{C}^m$. Then by allowing v to be in $\mathbb{C}^N \otimes \mathbb{C}^m$ in (2.7) and (2.9), we obtain 2 families of indecomposable uniformly bounded modules $V(m, \alpha) = V(\alpha) \otimes \mathbb{C}^m$, $\overline{V}(m, \alpha) = \overline{V}(\alpha) \otimes \mathbb{C}^m$.

The main result of this paper is the following theorem.

Theorem 2.2. (i) *An irreducible quasifinite module over $\widehat{\mathcal{D}^N}$ is either a highest or lowest weight module or else a module of the intermediate series. A nontrivial module of the intermediate series is a module $V(\alpha)$ or $\overline{V}(\alpha)$ for some $\alpha \in \mathbb{C}$.*

(ii) *A nontrivial indecomposable uniformly bounded module over $\widehat{\mathcal{D}^N}$ is a module $V(m,\alpha)$ or $\overline{V}(m,\alpha)$ for some $m \in \mathbb{Z}_+\backslash\{0\}$ and some indecomposable linear transformation α of $\mathbb{C}^m$.*

Thus in particular we obtain that a nontrivial indecomposable uniformly bounded module over $\mathcal{D}^N$ is simply a $\mathcal{D}^N_{\text{as}}$-module (if the central element $I = t^0D^0I$ acts by 1, where I is the $N \times N$ identity matrix) or its twist (if I acts by -1), and that there is an equivalence between the category of uniformly bounded $\mathcal{D}^N_{\text{as}}$-modules without the trivial composition factor and the category of linear transformations on finite-dimensional vector spaces.

Since irreducible quasifinite highest weight modules have been classified in [2] and irreducible lowest weight modules are simply dual modules of irreducible highest weight modules, this theorem in fact classifies all irreducible quasifinite modules over $\widehat{\mathcal{D}^N}$ and over $\mathcal{D}^N$.

Note that in the language of conformal algebras, this theorem in particular also gives proofs of Theorems 6.1 and 6.2 of [7] on the classification of finite indecomposable modules over the conformal algebras Cend_N and gc_N.

The analogous results of this theorem for affine Lie algebras, the Virasoro algebra, higher-rank Virasoro algebras and Lie algebras of Weyl type were obtained in [3], [12]–[14].

Note that the space $\mathcal{H} = \text{span}\{C, D, E_{p,p} \,|\, p \in [1,N]\}$ is a Cartan subalgebra of $\widehat{\mathcal{D}^N}$ and that the definition of quasifiniteness does not require that V is a *weight module* (i.e., the actions of elements of $\mathcal{H}$ on V are diagonalizable). If we require V to be a weight module, then the linear transformation α is diagonalizable, and thus all uniformly bounded modules are completely reducible.

We shall prove the above theorem in the next section.

3. Proof of Theorem 2.2

We shall keep the notation of the previous section. We denote by I the $N \times N$ identity matrix. When the context is clear, we often omit the symbol I; for instance, $t = t^1D^0I \in \mathcal{D}^N$. Denote $\mathcal{H} = \text{span}\{C, D, E_{p,p} \,|\, p \in [1,N]\}$, a Cartan subalgebra of $\widehat{\mathcal{D}^N}$.

For any $\mathbb{Z}$-graded vector space U, we use the notation U_+, U_- and $U_{[p,q)}$ to denote the subspaces spanned by elements of degree k with $k > 0$, $k < 0$ and $p \le k < q$ respectively. Then $\widehat{\mathcal{D}^N}$ has a triangular decomposition $\widehat{\mathcal{D}^N} = (\widehat{\mathcal{D}^N})_+ \oplus (\widehat{\mathcal{D}^N})_0 \oplus (\widehat{\mathcal{D}^N})_-$. Observe that $(\widehat{\mathcal{D}^N})_+$ is generated by the adjoint action of t on $(\widehat{\mathcal{D}^N})_{[0,N)}$ and that ad_t is locally nilpotent on $\widehat{\mathcal{D}^N}$.

Suppose $V = \bigoplus_{i\in\mathbb{Z}} V_i$ is a quasifinite module over $\mathcal{W}$. A homogeneous nonzero vector $v \in V$ is called a *highest* (resp., *lowest*) *weight vector* if

$$(\widehat{\mathcal{D}^N})_0 v \subset \mathbb{C}v, \quad \text{and} \quad (\widehat{\mathcal{D}^N})_+ v = 0 \ \ (\text{resp., } (\widehat{\mathcal{D}^N})_- v = 0\,).$$

We divide the proof of Theorem 2.2 into 3 lemmas.

Lemma 3.1. *Suppose V is an irreducible quasifinite $\widehat{\mathcal{D}^N}$-module without highest and lowest weight vectors. Then $t|_{V_i} : V_i \to V_{i+N}$ and $t^{-1}|_{V_i} : V_i \to V_{i-N}$ are*

injective and thus bijective for all $i \in \mathbb{Z}$ (recall (2.4) that t has degree N). In particular, by letting $K = \max\{\dim V_p \mid p \in [1, N]\}$, we have $\dim V_i \leq K$ for $i \in \mathbb{Z}$; thus V is uniformly bounded.

Proof. Say $tv_0 = 0$ for some $0 \neq v_0 \in V_i$. By shifting the grading index if necessary, we can suppose $i = 0$. Since $x|_{V_0} : V_0 \to V_{[0,N)}$ for $x \in (\widehat{\mathcal{D}^N})_{[0,N)}$ are linear maps on finite-dimensional vector spaces, there exists a finite subset $S \subset (\widehat{\mathcal{D}^N})_{[0,N)}$ such that all $x|_{V_0}, x \in (\widehat{\mathcal{D}^N})_{[0,N)}$ are linear combinations of $S|_{V_0} = \{y|_{V_0} \mid y \in S\}$. This implies that $(\widehat{\mathcal{D}^N})_{[0,N)}v_0 = (\operatorname{span} S)v_0$. Recall that ad_t is locally nilpotent such that $(\widehat{\mathcal{D}^N})_j \subset \operatorname{ad}_t^k((\widehat{\mathcal{D}^N})_{[0,N)})$ for $j > 0$, where $k \geq 0$ is the integer such that $0 \leq j - kN < N$. Choose $n > 0$ such that $\operatorname{ad}_t^n(S) = 0$. Let $j \geq nN$. Then $k \geq n$ and we have

$$\begin{aligned}(\widehat{\mathcal{D}^N})_j v_0 \subset (\operatorname{ad}_t^k((\widehat{\mathcal{D}^N})_{[0,N)}))v_0 &= t^k(\widehat{\mathcal{D}^N})_{[0,N)}v_0 = t^k(\operatorname{span} S)v_0 \\ &= (\operatorname{ad}_t^k(\operatorname{span} S))v_0 = 0.\end{aligned}$$

This means that $(\widehat{\mathcal{D}^N})_{[nN,\infty)}v_0 = 0$.

The rest of the proof is exactly like that of Proposition 2.1 in [13]. □

Lemma 3.2. *A nontrivial irreducible uniformly bounded module V is a module $V(\alpha)$ or $\overline{V}(\alpha)$ for some $\alpha \in \mathbb{C}$.*

Proof. Let $V' = \operatorname{span}\{v \in V \mid \mathcal{H}v \subset \mathbb{C}v\}$ be the space spanned by weight vectors. Clearly V' is a submodule. Since $\dim V_i < \infty$, there exists at least a common eigenvector (i.e., a weight vector) of $\mathcal{H}$ in V_i for $i \in \mathbb{Z}$, i.e., $V' \neq 0$. Thus $V = V'$ is a weight module. Since C, I are central elements, we have (cf. Remark 3.4 below)

$$C|_V = c_0 \cdot \mathbf{1}_V, \quad I|_V = c_1 \cdot \mathbf{1}_V,$$

for some $c_0, c_1 \in \mathbb{C}$. Let

$$\mathrm{Vir} = \operatorname{span}\{t^i D, C \mid i \in \mathbb{Z}\}, \quad \mathcal{W} = \operatorname{span}\{t^i D^j, C \mid i \in \mathbb{Z},\, j \in \mathbb{Z}_+\}$$

be subalgebras of $\widehat{\mathcal{D}^N}$ isomorphic to the Virasoro algebra and the Lie algebra $\widehat{\mathcal{D}^1}$ respectively. For $m \in [0, N-1]$, set

$$V[m] = \bigoplus_{i \in \mathbb{Z}} V_{iN+m}. \tag{3.1}$$

Clearly, $V[m]$ is a uniformly bounded module over Vir or $\mathcal{W}$, and $V = \bigoplus_{m=0}^{N-1} V[m]$. Since $V \neq 0$, we have $V[m] \neq 0$ for some m. Say, $V[0] \neq 0$. Obviously, a composition factor of the Vir-module $V[0]$ is a Vir-module of the intermediate series, on which the central element C must act trivially (cf. [12, 14]); thus $c_0 = 0$ (and so we can omit C in the following discussion).

By the structure of uniformly bounded $\mathcal{W}$-modules in [13], we have $c_1 = 0, \pm 1$. If $c_1 = 0$, by [13], each $V[m]$ must be a trivial $\mathcal{W}$-module; thus V is trivial as a $\mathcal{W}$-module. But since $[\mathcal{W}, \widehat{\mathcal{D}^N}] = \widehat{\mathcal{D}^N}$, we obtain that V is a trivial $\widehat{\mathcal{D}^N}$-module, contradicting the assumption of the lemma. Thus $c_1 \neq 0$. If necessary, by using the isomorphism σ in (2.8) (which interchanges $V(\alpha)$ with $\overline{V}(\alpha)$), we can always suppose $c_1 \neq -1$ (since $\sigma(I) = -I$). Thus $c_1 = 1$. Thus again by [13], each $\mathcal{W}$-module $V[m]$ must have the form $A_{p,G}$ defined in [13], i.e., there exist $K_m \geq 0$

(might depend on m) and a $K_m \times K_m$ diagonal matrix G_m such that we can choose a suitable basis $Y_{kN+m} = (y^{(1)}_{kN+m}, ..., y^{(K_m)}_{kN+m})$ of V_{kN+m} for $k \in \mathbb{Z}$ satisfying

$$(t^i D^j) Y_{kN+m} = Y_{(i+k)N+m}(k + G_m)^j, \tag{3.2}$$

for $i, k \in \mathbb{Z}$, $j \in \mathbb{Z}_+$, where the right-hand side is the vector-matrix multiplication by regarding Y_{kN+m} as a row vector, and where, here and below, when the context is clear, in an expression involving matrices, we always identify a scalar (such as k in the right-hand side) with the corresponding scalar matrix of the suitable rank.

Note that for $p, q \in [1, N]$, $tE_{p,q}$ has degree $N + p - q$; thus $(tE_{p,q})V_{kN+m} \subset V_{(k+1)N+m+p-q}$. Let k_1, m_1 be integers such that $m_1 \in [0, N-1]$ and

$$kN + m + p - q = k_1 N + m_1.$$

Then we can write

$$(tE_{p,q})Y_{kN+m} = Y_{(k_1+1)N+m_1} P^{(k,m)}_{p,q}, \tag{3.3}$$

where $P^{(k,m)}_{p,q}$ is some $K_{m_1} \times K_m$ matrix. Applying $[t^i, tE_{p,q}] = 0$ and $[D, E_{p,q}] = 0$ to Y_{kN+m}, using (3.2) and (3.3), we obtain that $P^{(k,m)}_{p,q}$ does not depend on k (and denote it by $P^{(m)}_{p,q}$) and

$$G_{m_1} P^{(m)}_{p,q} = P^{(m)}_{p,q} G_m. \tag{3.4}$$

Applying $[t^i D^j, tE_{p,q}] = \sum_{s=1}^{j} \binom{j}{s} t^{i+1} D^{j-s} E_{p,q}$ to Y_{kN+m}, by induction on j, we obtain

$$(t^i D^j E_{p,q}) Y_{kN+m} = Y_{(k_1+i)N+m_1} P^{(m)}_{p,q} (k + G_m)^j. \tag{3.5}$$

Let $K = \sum_{m=0}^{N-1} K_m$ and let $U = \bigoplus_{m=0}^{N-1} V_m$ be a subspace of V of dimension K. Then

$$(Y_0, Y_1, ..., Y_{N-1}) = (y^{(1)}_0, ..., y^{(K_0)}_0, y^{(1)}_1, ..., y^{(K_1)}_1, ..., y^{(1)}_{N-1}, ..., y^{(K_{N-1})}_{N-1})$$

is a basis of U. For each $E_{p,q} \in gl_N$, we define a linear transformation $\rho(E_{p,q})$ of U as follows:

$$\rho(E_{p,q}) Y_m = Y_{m_1} P^{(m)}_{p,q}, \tag{3.6}$$

for $m \in [0, N-1]$, where $m_1 \in [0, N-1]$ such that $m_1 \equiv m + p - q \pmod N$. This uniquely defines a linear map $\rho : gl_N \to \mathrm{End}(U)$. We prove that $\rho(E_{p,q}E_{p',q'}) = \rho(E_{p,q})\rho(E_{p',q'})$. By shifting the index of V_k if necessary, it suffices to prove

$$\rho(E_{p,q})\rho(E_{p',q'})Y_0 = \rho(E_{p,q}E_{p',q'})Y_0 = \delta_{q,p'}\rho(E_{p,q'})Y_0. \tag{3.7}$$

Note from (2.1) and (2.2) that

$$\begin{aligned} [DE_{p,q}, tE_{p',q'}] &= tE_{p,q}E_{p',q'} + tD[E_{p,q}, E_{p',q'}] \\ &= \delta_{q,p'} tE_{p,q'} + \delta_{q,p'} tDE_{p,q'} - \delta_{q',p} tDE_{p',q}. \end{aligned} \tag{3.8}$$

First suppose $p \geq q$, $p' \geq q'$, applying (3.8) to Y_{kN}, by (3.5), we obtain

$$P^{(p'-q')}_{p,q}(\overline{k} + 1)P^{(0)}_{p',q'} - P^{(p-q)}_{p',q'} P^{(0)}_{p,q} \overline{k} = \delta_{q,p'} P^{(0)}_{p,q'} + (\delta_{q,p'} P^{(0)}_{p,q'} - \delta_{q',p} P^{(0)}_{p',q})\overline{k}, \tag{3.9}$$

where we denote $\overline{k} = k + G_0$. Since $\overline{k}$ commutes with $P^{(0)}_{p',q'}$ by (3.4), regarding expressions in (3.9) as polynomials on $\overline{k}$, by comparing the coefficient of $\overline{k}^0$, we obtain

$$P^{(p'-q')}_{p,q} P^{(0)}_{p',q'} = \delta_{q,p'} P^{(0)}_{p,q'}, \tag{3.10}$$

which is equivalent to (3.7). By symmetry, we also have (3.7) if $p \leq q$, $p' \leq q'$. Now suppose $p < q$, $p' \geq q'$. Again applying (3.8) to Y_{kN}, we have (3.9) with $P^{(p-q)}_{p',q'}$

replaced by $P^{(N+p-q)}_{p',q'}$. Thus we still have (3.10), which is again equivalent to (3.7). Finally suppose $p \geq q$, $p' < q'$. Then we have (3.9) and (3.10) with $P^{(p'-q')}_{p,q}$ replaced by $P^{(N+p'-q')}_{p,q}$ and again we have (3.7).

Thus ρ is a representation of the simple associative algebra gl_N ($= \mathrm{End}_N$). Thus $U = \bigoplus_{s=1}^n U^{(s)}$ is decomposed as a direct sum of simple gl_N-submodules $U^{(s)}$ such that each $U^{(s)}$ is either the natural gl_N-module ($\cong \mathbb{C}^N$) or the trivial module. Since $I|_U$ is the identity map, we have that each $U^{(s)}$ is the natural module. Since $[D, gl_N] = 0$ and $D|_U$ is diagonalizable, we can choose submodules $U^{(s)}$ such that $D(U^{(s)}) \subset U^{(s)}$ and $D|_{U^{(s)}}$ is a scalar map. Now clearly $U^{(1)}$ generates a simple $\widehat{\mathcal{D}^N}$-submodule of V of the form $V(\alpha)$ (cf. (3.5) and (3.6)). Since V is irreducible, we have $V = V(\alpha)$ for some $\alpha \in \mathbb{C}$ (if we have used the isomorphism σ in (2.8) in the above proof, then V is the module $\overline{V}(\alpha)$). □

Lemma 3.3. *A nontrivial indecomposable uniformly bounded module V is a module of the form $V(m, \alpha)$ or $\overline{V}(m, \alpha)$.*

Proof. First note that a central element, while not necessarily acting by a scalar on an indecomposable module, nevertheless has only one eigenvalue (cf. Remark 3.4 below). Let c_0 and c_1 be the eigenvalues of C and I respectively. As in the arguments of the proof of Lemma 3.2, we have $c_0 = 0$ (thus C acts nilpotently on V) and we can suppose $c_1 = 1$ (by making use of the isomorphism σ in (2.8)). Thus each composition factor of V has the form $V(\alpha)$. Therefore V has a finite number m of composition factors ($m = \dim V_0$). By induction on m, it suffices to consider the case when $m = 2$. In this case V is not irreducible but indecomposable.

First suppose

$$C|_V = 0, \quad I|_V = \mathbf{1}_V. \tag{3.11}$$

Following the proof of Lemma 3.2 (now G_m is not necessarily diagonal), we have $U = U^{(1)} \oplus U^{(2)}$, and both $U^{(1)}$ and $U^{(2)}$ are the natural gl_N-modules. Since $D(U) \subset U$, $[D, gl_N] = 0$ and V is not irreducible, the subspace $U' = \{u \in U \mid Du \in \mathbb{C}u\}$ of eigenvectors of D is a proper (and thus simple) gl_N-submodule of U (isomorphic to $\mathbb{C}^N$ as a gl_N-module) and $D|_{U'}$ is a scalar map λ for some $\lambda \in \mathbb{C}$. Thus $U = U' \oplus U''$, where U'' is another copy of U' such that $Du'' = \lambda u'' + u'$ for $u'' \in U''$, where $u' \in U'$ is the corresponding copy of u''. Therefore $U \cong \mathbb{C}^N \otimes \mathbb{C}^2$ such that gl_N acts on $\mathbb{C}^N$ and D acts on $\mathbb{C}^2$ (and $\alpha = D|_{\mathbb{C}^2}$ is an indecomposable linear transformation on $\mathbb{C}^2$), and we obtain $V = V(2, \alpha)$ (if we have used the isomorphism σ in (2.8) in the above proof, then V is the module $\overline{V}(2, \alpha)$).

It remains to prove (3.11). We shall use (2.3), which can be rewritten as follows:

$$\begin{aligned}
[t^{i+j}(\tfrac{d}{dt})^j A, t^{k+l}(\tfrac{d}{dt})^l B] = & \sum_{s=0}^{j} \binom{j}{s} [k+l]_s t^{i+j+k+l-s} (\tfrac{d}{dt})^{j+l-s} AB \\
& - \sum_{s=0}^{l} \binom{l}{s} [i+j]_s t^{i+j+k+l-s} (\tfrac{d}{dt})^{j+l-s} BA \\
& + \delta_{i,-k} (-1)^j j! l! \binom{i+j}{j+l+1} \mathrm{tr}(AB) C,
\end{aligned} \tag{3.12}$$

where $[k]_j = k(k-1)\cdots(k-j+1)$ is a similar notation to $[D]_j$ in (2.3).

Consider the $\mathcal{W}$-module $V[0]$ (cf. (3.1)). As in the proof of Proposition 2.2 of [13], we can choose a basis $X_0 = (x_0^{(1)}, x_0^{(2)})$ of V_0 and define a basis $X_n = (x_n^{(1)}, x_n^{(2)})$ of

V_{nN} by induction on $|n|$ such that $tX_n = X_{n+1}$ for $n \in \mathbb{Z}$. Assume that

$$CX_n = X_nC_n, \quad (t^{i+j}(\tfrac{d}{dt})^j)X_n = X_{n+i}P_{i,j,n},$$

for some 2×2 matrices $C_n, P_{i,j,n}$. Using $[C,t] = 0$, we obtain $C_n = C_0$. Using $[t^{i+j}(\frac{d}{dt})^j, t] = jt^{i+j}(\frac{d}{dt})^{j-1}$, we obtain $P_{i,j,n+1} - P_{i,j,n} = jP_{i+1,j-1,n}$. Thus induction on j gives

$$P_{i,0,n} = P_i, \quad P_{i,1,n} = \bar{n}P_{i+1} + Q_i,$$

$$P_{i,2,n} = [\bar{n}]_2P_{i+2} + 2\bar{n}Q_{i+1} + R_i, \quad P_{i,3,n} = [\bar{n}]_3P_{i+3} + 3[\bar{n}]_2Q_{i+2} + 3\bar{n}R_{i+1} + S_i,$$

for some 2×2 matrices P_i, Q_i, R_i, S_i, where $\bar{n} = n+G_0$ for some fixed 2×2 matrix G_0 (cf. (3.2)), $[\bar{n}]_j$ is again a similar notation to $[D]_j$, and $Q_0 = 0$. (We use the notation $\bar{n} = n + G_0$ in order to ensure that $Q_0 = 0$. Note from $[t\frac{d}{dt}, t^{i+j}(\frac{d}{dt})^j] = it^{i+j}(\frac{d}{dt})^j$ that G_0 commutes with all matrices C_0, P_i, Q_i, R_i, S_i.)

By choosing a composition series of $V[0]$, we can assume that all these matrices are upper triangular matrices. Furthermore, by the structure of modules of the intermediate series, we see that

$$P_i \text{ have the form } \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right), \text{ and } C_0, Q_i, R_i, S_i \text{ have the form } \left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right). \tag{3.13}$$

Thus all matrices in (3.13) are commutative.

Applying $[t^{i+1}\frac{d}{dt}, t^k] = kt^{i+k} - \delta_{i,-k}\binom{i+1}{2}NC$ (cf. (3.12)) to X_n, we obtain $kP_{i+1}P_k = kP_{i+k} - \delta_{i,-k}\binom{i+1}{2}NC_0$, from which we obtain $P_i = P_0^{1-i} = (1-i)P_0 + i$ for $i \in \mathbb{Z}$ (using (3.13), we have $(P_0 - 1)^2 = 0$) and $C_0 = 0$ (thus $C|_{V[0]} = 0$ and similarly $C|_{V[m]} = 0$ for $m \in [0, N-1]$, and so in the following, we can omit C). Applying $[t^2(\frac{d}{dt})^2, t^k] = 2kt^{k+1}\frac{d}{dt} + [k]_2t^k$ to X_n, by comparing the coefficient of $\overline{n}^0$, we obtain $([k]_2P_2 + 2kQ_1)P_k = 2kQ_k + [k]_2P_k$. Thus

$$Q_k = Q_1P_k + \tfrac{k-1}{2}(P_2 - 1)P_k = Q_1 + \tfrac{k-1}{2}(1 - P_0), \tag{3.14}$$

if $k \neq 0$ (using (3.13), we have $Q_1P_k = Q_1$, $(P_2 - 1)P_k = P_2 - 1$). Letting $j = 2, l = 0$ and applying (3.12) to X_n, we see that (3.14) also holds for $k = 0$. Since $Q_0 = 0$, we obtain $Q_k = \frac{k}{2}(1 - P_0)$. Similarly, letting $i = l = 0, j = 3$ and applying (3.12) to X_n, we obtain $([k]_3P_3 + 3[k]_2Q_2 + 3kR_1)P_k = 3kR_k + 3[k]_2Q_k + [k]_3P_k$, from which we obtain

$$R_k = R_1 + \tfrac{(k-1)(k-2)}{3}(P_3 - 1) + (k-1)(Q_2 - Q_k) = R_1 + \tfrac{(k-1)(k-2)}{6}(1 - P_0).$$

Finally letting $j = 3, l = 0$ and applying (3.12) to X_n, we obtain $P_0 = 1$. Thus $I|_{V[0]} = \mathbf{1}_{V[0]}$. Similarly $I|_{V[m]} = \mathbf{1}_{V[m]}$. This proves (3.11), thus the lemma. □

Theorem 2.2 now follows from Lemmas 3.1–3.3.

Remark 3.4. We would like to point out that a central element does not necessarily act as a scalar on an indecomposable module since we do not assume that a central element acts diagonalizably. Thus there is a gap in the assertion in (2.1) of [13]. This gap has been filled in the above proof of Lemma 3.3.

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