A TOPOLOGICAL PALEY-WIENER PROPERTY FOR LOCALLY COMPACT GROUPS

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Abstract. We investigate a certain topological Paley-Wiener property and show, for instance, that compact-free nilpotent groups and simply connected solvable groups share this property.

Introduction

Let $f$ be a bounded and compactly supported measurable function on $\mathbb{R}^n$ ($f \in L_c^\infty(\mathbb{R}^n)$). By the classical Paley-Wiener theorem, the Fourier transform $\hat{f}$ of $f$ extends to an entire function on $\mathbb{C}^n$. It follows that $f = 0$ whenever $\hat{f}$ vanishes on a set of positive Lebesgue measure. If, more generally, $G$ is a (second countable) unimodular locally compact group of type I and $\hat{G}$ denotes the dual space of $G$, then $G$ is said to satisfy the weak Paley-Wiener property if the operator-valued Fourier transform $\pi \to \pi(f)$ of a non-zero function $f \in L_c^\infty(G)$ cannot vanish on a set of positive Plancherel measure on $\hat{G}$. This weak Paley-Wiener property has been established by several authors for simply connected nilpotent Lie groups [1], [7], [14], [15] and later for completely solvable Lie groups [8].

In this paper we introduce and study another Paley-Wiener type property which can be defined for arbitrary locally compact groups $G$. We say that $G$ has the topological Paley-Wiener property if for every non-zero $f \in L_c^\infty(G)$, the map $\pi \to \pi(f)$ cannot vanish on any non-empty open subset of the reduced dual $\hat{G}_r$ of $G$. Our main result is an extension theorem which asserts that if $G$ contains a closed normal subgroup $N$ that has the topological Paley-Wiener property, then so does $G$, provided that $G/N$ is abelian and compact-free (Theorem 2.2). As a consequence, compact-free nilpotent locally compact groups and simply connected solvable Lie groups share this topological Paley-Wiener property (Corollaries 2.4 and 2.5).

1. Preliminaries and some basic results

Let $G$ be a locally compact group with fixed left Haar measure, and let $C^*(G)$ be the completion of the convolution algebra $L^1(G)$ with respect to the norm $\|f\|_{C^*(G)} = \sup\{\|\pi(f)\| : \pi \in \pi(G)\}$, where the supremum is taken over all $*$-representations $\pi$ of $L^1(G)$ as an algebra of bounded linear operators in a Hilbert space. Let $P(G)$ be the set of all continuous positive definite functions on $G$, and let $B(G)$ denote the...
linear span of $P(G)$. Then $B(G)$ can be identified with the dual space $C^*(G)^*$ of $C^*(G)$ by the pairing $\langle f, u \rangle = \int_G f(x)u(x)dx$ for $f \in L^1(G)$ and $u \in B(G)$. With pointwise multiplication and the dual norm, $B(G)$ is a commutative Banach algebra, called the Fourier-Stieltjes algebra of $G$. The Fourier algebra $A(G)$ is the norm closure in $B(G)$ of $A_c(G) = B(G) \cap C_c(G)$, the compactly supported functions in $B(G)$. For details regarding $B(G)$ and $A(G)$, see the fundamental paper [3].

As is customary, we shall use the same letter, for example $\pi$, for a unitary representation of $G$ and the corresponding $*$-representations of $L^1(G)$ and $C^*(G)$. For any representation $\pi$ of $G$, let $A_\pi(G)$ denote the closed linear subspace of $B(G)$ generated by all coefficient functions of $\pi$, and $B_\pi(G)$ the $w^*$-closure of $A_\pi(G)$ in $B(G)$. Then $B_\pi(G)$ can be identified with the dual of the $C^*$-algebra $\pi(C^*(G))$.

Note that, when $\rho = \rho_G$ denotes the regular representation of $G$ on $L^2(G)$, then $A_\rho(G) = A(G)$, and $B_\rho(G)$ equals $B(G)$ if and only if $G$ is amenable.

Moreover, we need to recall some definitions from representation theory. If $S$ and $T$ are sets of unitary representations, then $S$ is weakly contained in $T$ ($S \prec T$) if $\bigcap\{\ker \sigma : \sigma \in S\} \supseteq \bigcap\{\ker \tau : \tau \in T\}$, where $\ker \tau$ denotes the $C^*$-kernel of a representation $\tau$. The sets $S$ and $T$ are said to be weakly equivalent ($S \sim T$) if $S \prec T$ and $T \prec S$. Then, for any two representations $\sigma$ and $\tau$, $\sigma \prec \tau$ if and only $B_\sigma(G) \subseteq B_\tau(G)$. The support of a representation $\pi$, supp $\pi$, is the closed subset of $\hat{G}$ consisting of all $\tau \in \hat{G}$ such that $\tau \prec \pi$. Thus $\pi \sim \hat{G}$ if and only if supp $\pi = \hat{G}$. As general references to dual spaces of locally compact groups and weak containment properties, we mention [4] and [6].

**Lemma 1.1.** If $G$ contains a non-trivial compact normal subgroup $K$, then $G$ cannot have the topological Paley-Wiener property.

**Proof.** Let $\mu_K$ denote the (normalized) Haar measure of $K$. Let $\pi$ be an irreducible representation of $G$ such that $\pi \not\in \hat{G}/\hat{K}$. If $\sigma$ is an irreducible subrepresentation of $\pi|_K$, then, since $\sigma \not\neq 1_K$, by the orthogonality relations $\int_K \langle \sigma(k)\xi, \eta \rangle dk = 0$ for all $\xi, \eta \in \mathcal{H}_\sigma$. It follows that $\pi(\mu_K) = 0$. Now, consider $f = g * \mu_K, g \in C_c(G)$. Then $f \in L^\infty(G)$ and $\pi(f) = 0$ for all $\pi \in \mathcal{G}_r \setminus \hat{G}/\hat{K}$, a non-empty open subset of $\hat{G}_r$.

For any $f \in L^1(G)$, let

$$Z_f = \{ \pi \in \mathcal{G}_r : \pi(f) = 0 \},$$

the zero set of $f$ in $\mathcal{G}_r$. Note that when studying the topological Paley-Wiener property, we can always assume that the function $f \in L^\infty_c(G)$ actually belongs to $A_c(G)$ since $f \ast f^*$ is positive definite and $Z_{f\ast f^*} = Z_f$. We say that $f \in A_c(G)$ generates $B_\rho(G)$ in the $w^*$-topology if the linear span of all two-sided translates of $f$ is $w^*$-dense in $B_\rho(G)$.

**Lemma 1.2.** Let $G$ be a locally compact group and let $f \in A_c(G)$. Then $Z_f$ has empty interior if and only if $f$ generates $B_\rho(G)$ in the $w^*$-topology.

**Proof.** Let $I_f$ denote the closed ideal of $C^*_r(G)$ generated by $\rho(f)$. If $\hat{Z}_f \neq \emptyset$, then the non-zero ideal $J$ of $C^*_r(G)$ defined by $\hat{J} = \hat{Z}_f$ satisfies $J \cap I_f = \{0\}$ since $\hat{I}_f = \hat{G}_r \setminus \hat{Z}_f$. Conversely, if there exists a non-zero ideal $J$ of $C^*_r(G)$ such that $J \cap I_f = \{0\}$, then $\hat{J} \cap (\hat{G}_r \setminus \hat{Z}_f) = \hat{J} \cap \hat{I}_f = (J \cap I_f)^\wedge = \emptyset$ and hence $\emptyset \neq \hat{J} \subseteq Z_f$, so
Given \( J \) zero ideal \( \mathcal{I} \). Therefore, to prove the lemma it suffices to show that there exists a non-zero ideal \( J \) of \( C^*_r(G) \) with \( J \cap I_f = \{0\} \) if and only if \( E_f \), the \( w^* \)-closed subspace of \( B_\rho(G) \) generated by all two-sided translates of \( f \), does not coincide with \( B_\rho(G) \).

Let \( \Delta \) denote the modular function of \( G \) and for any function \( h \) on \( G \), define \( \tilde{h}(y) = \Delta(y^{-1})h(y^{-1}) \). Then note that \( \langle \rho(h)\rho(g), u \rangle = \langle \rho(g), \tilde{h} * u \rangle \) for all \( g \in C_c(G) \) and \( h, u \in A_c(G) \), and hence \( \langle \rho(h)T, u \rangle = \langle T, \tilde{h} * u \rangle \) for all \( T \in C^*_r(G) \) and \( h, u \in A_c(G) \).

Now let \( J \) be an ideal as above. Then \( (T, \tilde{f} * u) = 0 \) for all \( T \in J \) and \( u \in A_c(G) \) since \( \rho(f)T = 0 \). Writing \( f \) as \( f = \langle \pi(\cdot)\xi, \eta \rangle \) for some representation \( \pi \) of \( G \) and \( \xi, \eta \in \mathcal{H}_x \) and choosing \( u \) such that \( u \geq 0 \) and \( \|u\|_1 = 1 \), we have

\[
(f * u)(x) - f(x) = \int_G u(y^{-1})\langle \pi(x)\pi(y)\xi, \eta \rangle dy - \langle \pi(x)\xi, \eta \rangle = \left\langle \pi(x) \left( \int_G u(y^{-1})\langle \pi(y)\xi - \xi \rangle dy \right), \eta \right\rangle.
\]

This implies that

\[
\|f * u - f\|_{B_\rho(G)} \leq \|\eta\| \sup \{\|\pi(y)\xi - \xi\| : y^{-1} \in \text{supp} u\}.
\]

Since \( \langle T, \tilde{f} * u \rangle = 0 \), taking for \( u \in A_c(G) \) the usual approximate identity for \( L^1(G) \), it follows that

\[
|\langle T, f \rangle| \leq \|T\| \cdot \|f * u - f\|_{B_\rho(G)} \to 0
\]

for each \( T \in J \). Since \( J^\perp \) is a \( w^* \)-closed two-sided translation invariant subspace of \( B_\rho(G) \), we conclude that \( E_f \subseteq J^\perp \), a proper subspace of \( B_\rho(G) \).

Conversely, suppose that \( E_f \neq B_\rho(G) \) and let

\[
J = \{T \in C^*_r(G) : (T, u) = 0 \text{ for all } u \in E_f \}.
\]

Then \( J \) is a non-zero closed ideal \( C^*_r(G) \). We show that \( J \cap I_f = \{0\} \). For \( T \in C^*_r(G) \), \( s, t \in G \) and \( u \in A_c(G) \), we have

\[
\langle \rho(L_{s^{-1}}R_{t^{-1}}f), u \rangle = \Delta(ts)(T, (L_s R_t f) * u).
\]

Therefore, it will follow that \( J \cap I_f = \{0\} \) once we have verified that if \( E \) is a \( w^* \)-closed translation invariant subspace of \( B_\rho(G) \), then \( E * u \subseteq E \) for every \( u \in A_c(G) \). This can be seen as follows. If \( v(\cdot) = \langle \pi(\cdot)\xi, \eta \rangle \in E \), then

\[
(v * u)(\cdot) = \left\langle \pi(\cdot) \left( \int_G u(y^{-1})\pi(y)\xi dy \right), \eta \right\rangle.
\]

Given \( \epsilon > 0 \), there exist \( y_1, \ldots, y_n \in G \) and \( c_1, \ldots, c_n \geq 0 \) such that

\[
\left\| \int_G u(y^{-1})\pi(y)\xi dy - \sum_{j=1}^n c_j \pi(y_j)\xi \right\| \leq \epsilon.
\]

Then \( w(\cdot) = \sum_{j=1}^n c_j \langle \pi(\cdot)\pi(y_j)\xi, \eta \rangle \in E \) and

\[
\|v * u - w\|_{B_\rho(G)} \leq \|\eta\| \cdot \left\| \int_G u(y^{-1})\pi(y)\xi dy - \sum_{j=1}^n c_j \pi(y_j)\xi \right\| \leq \epsilon \|\eta\|.
\]

This finishes the proof. \( \Box \)
It is worth pointing out that, according to the following lemma, the topological Paley-Wiener property is equivalent to a dichotomy for the subspaces $B_{\pi}(G) \cap A_c(G)$ of $A(G)$, which has been investigated in Section 5 of [13].

**Lemma 1.3.** For any locally compact group $G$, the following two conditions are equivalent:

(i) $G$ has the topological Paley-Wiener property.

(ii) For any unitary representation $\pi$ of $G$, either $B_{\pi}(G) \cap A_c(G) = \{0\}$ or $B_{\pi}(G) \cap A_c(G) = A(G)$.

**Proof.** (i) $\Rightarrow$ (ii) Let $\pi$ be any representation of $G$, and suppose there exists $f \in B_{\pi}(G) \cap A_c(G), f \neq 0$. By (i), $Z_f$ has empty interior and hence $f$ generates $B_{\rho}(G)$ in the $w^*$-topology (Lemma 1.2). Thus

$$B_{\rho}(G) \subseteq \overline{B_{\pi}(G) \cap A_c(G)}^{w^*} \subseteq B_{\pi}(G),$$

and this in turn implies that

$$A(G) = \overline{B_{\rho}(G) \cap A_c(G)}^{w^*} \subseteq \overline{B_{\pi}(G) \cap A_c(G)}^{w^*},$$

as required.

(ii) $\Rightarrow$ (i) Let $f$ be a non-zero function in $A_c(G)$, and let $E$ denote the closed two-sided invariant subspace of $A(G)$ generated by $f$. There exists a representation $\pi$ of $G$ such that $E = A_{\pi}(G)$ [2 - Théorème 3.17]. By hypothesis, either $B_{\pi}(G) \cap A_c(G) = \{0\}$ or $B_{\pi}(G) \cap A_c(G) = A(G)$. However, $f \in B_{\pi}(G) \cap A_c(G)$. Therefore

$$\overline{E}^{w^*} \subseteq B_{\rho}(G) = \overline{A(G)}^{w^*} = \overline{B_{\pi}(G) \cap A_c(G)}^{w^*} \subseteq \overline{E}^{w^*},$$

so that $f$ generates $B_{\rho}(G)$ in the $w^*$-topology. By Lemma 1.2, $\mathbb{Z}_f \neq \emptyset$. \hfill $\square$

An interesting consequence of Lemma 1.3 is that the topological Paley-Wiener property can be interpreted as an approximation property for $A(G)$.

A locally compact group $G$ is called a SIN-group (a group with small invariant neighbourhoods) if there exists a neighbourhood basis of the identity consisting of sets $V$ such that $x^{-1}Vx = V$ for all $x \in G$. SIN-groups provide a large class of locally compact groups comprising all abelian groups, compact groups and discrete groups. For the structure theory of SIN-groups, see [11].

**Theorem 1.4.** For a SIN-group $G$, the following two conditions are equivalent:

(i) $G$ contains no non-trivial compact normal subgroup.

(ii) $G$ has the topological Paley-Wiener property.

**Proof.** This is an immediate consequence of Lemma 1.3 and Theorem 5.6 of [13]. \hfill $\square$

We conclude this section with the observation that the topological Paley-Wiener property is in general not inherited by normal subgroups. Indeed, an easy example showing this can be constructed as follows. Let $F$ be a finite abelian group and let $N = \sum_{j=1}^\infty F_j$ be the direct sum of copies $F_j$ of $F$. Let $G = S_\infty \ltimes N$ be the semidirect product, where the infinite symmetric group $S_\infty$ acts on $N$ by permuting the index set $\mathbb{N}$. Obviously, $G$ is a discrete ICC group and hence has the weak Paley-Wiener property (Theorem 1.4), whereas the normal subgroup $N$ is an abelian torsion-group.
2. Extensions and applications

In this section we are going to prove two theorems of the nature that if a locally compact group $G$ contains a closed normal subgroup that has the topological Paley-Wiener property, then under certain additional hypotheses, $G$ also has the topological Paley-Wiener property. Both of these results apply to various (classes of) locally compact groups. A similar, but much less applicable, result has been shown in [13, Proposition 5.3].

Let $H$ be a closed subgroup of $G$ and let $\tau$ be a representation of $H$. The representation of $G$ induced by $\tau$ is denoted $\text{ind}_H^G \tau$. In the sequel we shall use that $\pi \otimes \text{ind}_H^G \tau = \text{ind}_H^G (\pi|_H \otimes \tau)$ and that $\pi \prec \text{ind}_H^G (\pi|_H)$ when $G$ is amenable [10, Theorem 5.1].

Lemma 2.1. Let $H$ be a closed subgroup of the locally compact group $G$. Let $U$ be a non-empty open subset of $\hat{G}_r$, and let

$$V = \{ \tau \in \hat{H}_r : \text{supp}(\text{ind}_H^G \tau) \nsubseteq \hat{G}_r \setminus U \}.$$ 

Then $V$ is non-empty and open in $\hat{H}_r$.

Proof. To show that $\hat{H}_r \setminus V$ is closed in $\hat{H}_r$, let $(\tau_\alpha)_\alpha$ be a net in $\hat{H}_r \setminus V$ such that $\tau_\alpha \to \tau$ for some $\tau \in \hat{H}_r$. Since inducing is continuous in Fell’s subgroup representation topology [13, Theorem 4.2], $\text{ind}_H^G \tau_\alpha \to \text{ind}_H^G \tau$. Since $\text{ind}_H^G \tau_\alpha \nsubseteq \hat{G}_r \setminus U$ for all $\alpha$ and $\hat{G}_r \setminus U$ is closed in $\hat{G}_r$, it follows that $\text{supp}(\text{ind}_H^G \tau) \subseteq \hat{G}_r \setminus U$ and hence $\tau \in \hat{H}_r \setminus V$.

Assume that $V = \emptyset$, that is, $\text{supp}(\text{ind}_H^G \tau) \subseteq \hat{G}_r \setminus U$ for all $\tau \in \hat{H}_r$. Then, since $\rho_G = \text{ind}_H^G \rho_H$, we obtain that

$$\hat{G}_r \sim \{ \text{ind}_H^G \tau : \tau \in \hat{H}_r \} \sim \bigcup_{\tau \in \hat{H}_r} \text{supp}(\text{ind}_H^G \tau) \subseteq \hat{G}_r \setminus U.$$ 

This contradiction shows that $V$ is non-empty. \hfill \Box

Recall that a locally compact group $G$ is said to be compact-free if the identity is the only element of $G$ generating a relatively compact subgroup.

Theorem 2.2. Let $G$ be a locally compact group containing a closed normal subgroup $N$ such that $G/N$ is abelian and compact-free. If $N$ has the topological Paley-Wiener property, then so does $G$.

Proof. Let $f \in C_c(G)$ be such that $\hat{Z}_f \neq \emptyset$. We have to show that $L_x f|_N = 0$ for all $x \in G$. Temporarily, fix $\pi \in \hat{Z}_f$ and $\xi, \eta \in \mathcal{H}_\pi$, and consider on $\hat{G}/N$ the function

$$\chi \to ((\pi \otimes \chi)(f)(\xi \otimes 1), \eta \otimes 1).$$

Define $h \in C_c(G/N)$ by

$$h(xN) = \int_N f(xn) \langle \pi(xn)\xi, \eta \rangle dn,$$

$x \in G$. Then, for $\chi \in \hat{G}/N$,

$$\hat{h}(\chi) = \int_{G/N} \chi(xN) \left( \int_N f(xn) \langle \pi(xn)\xi, \eta \rangle \right) d(xN) = \langle ((\pi \otimes \chi)(f)(\xi \otimes 1), \eta \otimes 1).$$
Since $G/N$ is abelian and compact-free, $\hat{h}$ can vanish only on a set of Haar measure zero in $\hat{G}/\hat{N}$ [13, Lemma 5.1]. However, $\hat{h}$ vanishes on the set
\[ X = \{ \chi \in \hat{G}/\hat{N} : \pi \otimes \chi \in \hat{Z}_f \}, \]
and $X$ is a non-empty open subset of $\hat{G}/\hat{N}$ since $\pi \in X$ and the mapping $\chi \rightarrow \pi \otimes \chi$ from $\hat{G}/\hat{N}$ into $\hat{G}$ is continuous. We conclude that $h = 0$. Since $\xi, \eta \in \mathcal{H}_\pi$ are arbitrary, it follows that $(\pi \otimes \chi)(f) = 0$ for all $\pi \in \hat{Z}_f$ and $\chi \in \hat{G}/\hat{N}$. Now,
\[ \pi \otimes \hat{G}/\hat{N} = \{ \pi \otimes \chi : \chi \in \hat{G}/\hat{N} \} \sim \pi \otimes \text{ind}^G_N 1_N = \text{ind}^G_N (\pi|_N), \]
and since $\pi \prec \rho_G$ implies that $\pi|_N \prec \rho_N$, we get that
\[ \text{ind}^G_N (\pi|_N) \prec \text{ind}^G_N \rho_N = \rho_G \]
and therefore $\pi \otimes \hat{G}/\hat{N} \subseteq \hat{G}_r$. Thus we have seen that $\hat{Z}_f \otimes \hat{G}/\hat{N} \subseteq \hat{Z}_f$. Since $\sigma \rightarrow \sigma \otimes \chi$ is a homeomorphism of $\hat{G}_r$,
\[ \hat{Z}_f \otimes \hat{G}/\hat{N} = \bigcup_{\chi \in \hat{G}/\hat{N}} \hat{Z}_f \otimes \{ \chi \} \]
is open in $\hat{G}_r$, whence $\hat{Z}_f \otimes \hat{G}/\hat{N} = \hat{Z}_f$.

Now, let $x \in G$ and define $g \in C_c(\hat{N})$ by $g(n) = f(xn)$. We have to show that $\hat{Z}_f \neq \emptyset$ for all such functions $g$. To that end, define a subset $V$ of $\hat{N}_r$ by
\[ V = \{ \tau \in \hat{N}_r : \text{supp}(\text{ind}^G_N \tau) \not\subseteq \hat{G}_r \setminus \hat{Z}_f \}. \]
By Lemma 2.1, $V$ is non-empty and open in $\hat{N}_r$. It remains to verify that $V \subseteq \hat{Z}_f$. Fix $\tau \in V$ and choose a net $(\pi_\alpha)$ in $\hat{G}_r$ such that $\pi_\alpha|_N \rightarrow \tau$ in Fell’s topology. We claim that $\pi_\alpha \in \hat{Z}_f$ eventually. Towards a contradiction, after passing to a subnet if necessary, let us assume that $\pi_\alpha \not\in \hat{Z}_f$ for all $\alpha$. Since $\hat{Z}_f = \hat{Z}_f \otimes \hat{G}/\hat{N}$, it follows that
\[ \text{ind}(\pi_\alpha|_N) \sim \pi_\alpha \otimes \hat{G}/\hat{N} \subseteq \hat{G}_r \setminus \hat{Z}_f \]
for all $\alpha$. Therefore, for each $\sigma \in \text{supp}(\pi_\alpha|_N)$,
\[ \text{supp}(\text{ind}_N^G \sigma) \subseteq \text{supp}(\text{ind}_N^G (\pi_\alpha|_N)) \subseteq \hat{G}_r \setminus \hat{Z}_f, \]
and hence $\sigma \in \hat{N}_r \setminus V$. Thus $\pi_\alpha|_N \prec \hat{N}_r \setminus V$ and so $\tau \in \hat{N}_r \setminus V$. This contradiction shows that $\pi_\alpha \in \hat{Z}_f$ eventually. Since
\[ \text{ind}_N^G (\pi_\alpha|_N) \sim \pi_\alpha \otimes \hat{G}/\hat{N} \subseteq \hat{Z}_f \otimes \hat{G}/\hat{N} \subseteq \hat{Z}_f, \]
and $\text{ind}(\pi_\alpha|_N) \rightarrow \text{ind}_N^G \tau$, it follows that $\text{supp}(\text{ind}_N^G \tau) \subseteq \hat{Z}_f$. But $\text{ind}_N^G \tau(f) = 0$ implies that $\tau(L_x f|_N) = 0$ (compare [16, Lemma 2.1]), that is, $\tau \in \hat{Z}_g$. This finishes the proof.

Corollary 2.3. Suppose that $G$ possesses a sequence $H_0 = \{ e \} \subseteq H_1 \subseteq \ldots \subseteq H_m = G$ of closed subgroups such that $H_{j-1}$ is normal in $H_j$ and $H_j/H_{j-1}$ is compact-free abelian ($1 \leq j \leq m$). Then $G$ has the topological Paley-Wiener property.

Proof. Using Theorem 2.2, the statement follows immediately by induction on $m$. □
Corollary 2.4. Let $G$ be a nilpotent locally compact group. Then $G$ has the topological Paley-Wiener property if and only if $G$ is compact-free.

Proof. Suppose that $G$ has the topological Paley-Wiener property. Then, by Lemma 1.1, $G$ cannot have a non-trivial compact normal subgroup. It is easy to see that this implies that $G^c = \{ e \}$. Indeed, let $\{ e \} = Z_0(G) \subseteq Z_1(G) \subseteq \ldots$ denote the ascending central series of $G$ and, assuming that $G^c \neq \{ e \}$, let $m \geq 0$ be maximal with the property that $G^c \cap Z_m(G) = \{ e \}$. Then pick $a \in G^c \cap Z_{m+1}(G)$, $a \neq e$, and let $H$ be the closed subgroup of $G$ generated by $Z_m(G)$ and $a$. Then $H$ is normal in $G$ and hence $H^c$ is a closed normal subgroup of $G$. However, $H^c$ is the closed subgroup generated by $a$, which is compact.

Conversely, if $G$ is compact-free, then $G_0$, the connected component of the identity, is simply connected and $D = G/G_0$ is discrete and torsion-free [9, Theorem 8.3]. We now apply Corollary 2.3 as follows. If $G = G_0$, then $Z_m(G)$ and the quotient group $G/Z_m(G)$ are simply connected for all $m$. Then induction yields that $G$ has the topological Paley-Wiener property. In the general case, note that $D/Z_m(D)$ is torsion-free for all $m$ [3, Corollary 2.11]. Thus, a further induction on the length of nilpotency of $D$ shows that $G$ has the topological Paley-Wiener property.

Corollary 2.5. Let $G$ be a simply connected solvable Lie group. Then $G$ has the topological Paley-Wiener property.

Proof. By the structure theory of simply connected solvable Lie groups, there exists a sequence $H_0 = \{ e \} \subseteq H_1 \subseteq \ldots \subseteq H_m = G$ of closed subgroups such that $H_{j-1}$ is normal in $H_j$ and $H_j/H_{j-1} = \mathbb{R}$ (see [12, Satz III.3.30]). Thus $G$ has the topological Paley-Wiener property by Corollary 2.3.

Theorem 2.6. Let $G$ be a locally compact group and let $N$ be a closed normal subgroup of $G$ such that $G/N$ is amenable. Suppose there exists a dense subset $T$ of $\hat{N}_r$ such that for each $\tau \in T$, $\text{ind}_N^G\tau$ is weakly equivalent to some irreducible representation. If $N$ has the topological Paley-Wiener property, then so does $G$.

Proof. Let $f \in C_c(G)$ such that $\mathring{Z}_f \neq \emptyset$. By Lemma 2.1, the set

$$V = \{ \tau \in \hat{N}_r : \text{supp}(\text{ind}_N^G\tau) \not\subseteq \mathring{G}_r \cup \mathring{Z}_f \}$$

is non-empty and open in $\hat{N}_r$. Since $T$ is dense in $\hat{N}_r$, $T \cap V$ is dense in $V$. By hypothesis, for every $\tau \in T$ there exists $\pi_\tau \in \mathring{G}$ such that $\pi_\tau \sim \text{ind}_N^G\tau$. Since $G/N$ is amenable, $\pi_\tau \in \mathring{G}_r$. Then $\pi_\tau \in \mathring{Z}_f$ for each $\tau \in T \cap V$. Indeed, for such $\tau$,

$$\pi_\tau \sim \text{supp}(\text{ind}_N^G\tau) \not\subseteq \mathring{G}_r \cup \mathring{Z}_f$$

and hence $\pi_\tau \in \mathring{Z}_f$. Now, for any $\tau \in V$ there exists a net $(\tau_\alpha)_\alpha$ in $T \cap V$ converging to $\tau$. Then

$$\pi_{\tau_\alpha} \sim \text{ind}_N^G\tau_\alpha \to \text{ind}_N^G\tau,$$

and since $\pi_{\tau_\alpha}(f) = 0$ for all $\alpha$, it follows that $\text{ind}_N^G\tau(f) = 0$. This in turn implies that $\tau(L_xf|_N) = 0$ for all $x \in G$. Since $N$ has the topological Paley-Wiener property, we conclude that $f = 0$.

We finish the paper with an example to which Theorem 2.6 applies, but neither Theorem 2.2 nor Proposition 5.3 of [13] applies.
Example 2.7. Let $Z_2 = \{1, -1\}$, $K = \prod_{n=1}^{\infty} Z_2$ and $N = \sum_{n=1}^{\infty} Z$, the restricted direct sum of copies of $Z$. Let $G$ be the semidirect product $G = K \rtimes N$, where $K$ acts on $N$ by $(\epsilon \cdot x)_n = \epsilon_n x_n (\epsilon = (\epsilon_n)_n \in K, x = (x_n)_n \in N)$. Identifying $\hat{Z}$ with $T$, we have $\hat{N} = \prod_{n=1}^{\infty} T$ and, for $z = (z_n)_n \in \hat{N}$ and $\epsilon = (\epsilon_n)_n \in K$, $\epsilon \cdot z = z$ if and only if $z_n = \overline{\epsilon_n}$ for all $n \in \mathbb{N}$ such that $\epsilon_n = -1$. Now, $T = \bigcap_{n=1}^{\infty} \{ z \in \hat{N} : \overline{\epsilon_n} \neq z_n \}$ is dense in $\hat{N}$, and each $z \in T$ has a trivial stability group in $K$ and hence the associated induced representation of $G$ is irreducible. By Theorem 2.6, $G$ has the topological Paley-Wiener property.

In [13, Example 5.4] we have given several other examples of locally compact groups having the topological Paley-Wiener property.

References


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