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COMPACT HYPERBOLIC 4-MANIFOLDS OF SMALL VOLUME

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Abstract. We prove the existence of a compact non-orientable hyperbolic 4-manifold of volume $32\pi^2/3$ and a compact orientable hyperbolic 4-manifold of volume $64\pi^2/3$, obtainable from torsion-free subgroups of small index in the Coxeter group $[5,3,3,3]$. At the time of writing these are the smallest volumes of any known compact hyperbolic 4-manifolds.

1. Introduction

The smallest volume (area) of a compact orientable hyperbolic 2-manifold is 4π and is achieved by any closed hyperbolic surface of genus 2 and so Euler characteristic -2. For non-compact orientable 2-manifolds, the smallest volume is 2π, achieved by a once-punctured torus which has Euler characteristic -1. For hyperbolic 3-manifolds, the work of Thurston and Jørgensen (see [18, 2]) has shown the existence of a smallest volume for compact orientable hyperbolic 3-manifolds which can be achieved by a finite number of manifolds. The prime candidate is the Weeks-Matveev-Fomenko manifold [21, 14], whose volume can be given by the closed formula $\frac{12(23^{3/2})\,\zeta_k(2)}{(4\pi^2)^2}$, where the Dedekind zeta function ζ_k is over the field k of degree 3 over the rationals with one complex place and discriminant -23. This volume is approximately 0.942707. In the case of non-compact orientable hyperbolic 3-manifolds, the minimal volume is known to be $\frac{12(3^{3/2})\,\zeta_k(2)}{4\pi^2}$ (≈ 2.029883), where $k = \mathbb{Q}(\sqrt{-3})$, and is achieved by the figure 8 knot complement and its sister manifold [5]. These two manifolds and the Weeks manifold are known to be arithmetic, and if one restricts to arithmetic 3-manifolds, then the minimum volume is known to be achieved by the Weeks manifold [6].

In the case of dimension 4, as with all even dimensions, the volume of a hyperbolic manifold is a constant multiple of its Euler characteristic. In dimension 4, this is given by $\mathrm{Vol}(M) = 4\pi^2\chi(M)/3$ (see [17, 11]). Furthermore the Euler characteristic of a compact orientable hyperbolic 4-manifold is always even. It is known that there exist *non-compact* orientable hyperbolic 4-manifolds of minimal Euler characteristic 1 [15, 9]. A well-studied example of a *compact* orientable hyperbolic 4-manifold is the Davis manifold [8, 10, 16], which has Euler characteristic 26.

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In this paper, we establish the existence of a compact non-orientable hyperbolic 4-manifold of Euler characteristic 8, and an orientable double cover of this manifold, of Euler characteristic 16. These examples give the smallest known volumes so far in the compact case. Furthermore, these two 4-manifolds and the Davis manifold are all arithmetic, and have the same arithmetic structure, and hence are commensurable in that they then have a common finite cover (see comments in §2). We will not make use of this fact here, but remark that, following the identification of the compact arithmetic hyperbolic 4-orbifold of smallest volume in [1] as the quotient $\mathbb{H}^4/\Gamma_1$, where Γ_1 is defined in §2, it follows, as is shown in [1], that the as-yet-unknown compact arithmetic orientable hyperbolic 4-manifold of smallest volume has the form $\mathbb{H}^4/\Gamma_0$, where Γ_0 is a torsion-free subgroup of Γ_1 of finite index. This is precisely how our manifolds are obtained.

2. Preliminaries

In hyperbolic 4-space $\mathbb{H}^4$, there are five compact Coxeter simplices — that is, simplices whose faces are geodesic and whose dihedral angles between faces of codimension 1 are submultiples of π (see [13, 19]). If Γ is the group generated by reflections in the 3-dimensional faces of such a simplex, then Γ is a discrete subgroup of $\operatorname{Isom}\mathbb{H}^4$, and the simplex is a fundamental domain for Γ so that its images under Γ tessellate $\mathbb{H}^4$ (see [19]). The Coxeter symbols Δ_i representing these simplices are given in Figure 1, and the corresponding reflection groups, in the same numbering, are denoted by Γ_i $(i = 1, 2 \ldots, 5)$.

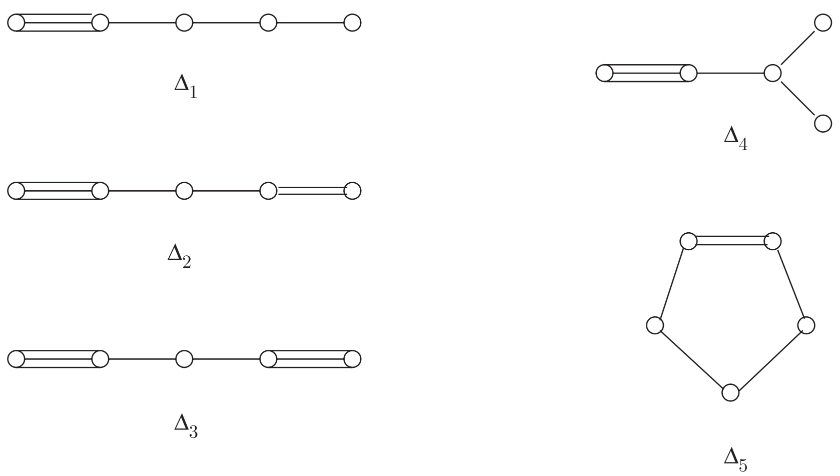

Figure 1. Five compact Coxeter simplices in hyperbolic 4-space

The group Γ_1 is also conveniently denoted as $[5, 3, 3, 3]$. Each of these reflection groups Γ has a torsion-free subgroup Γ_0 of finite index, so that one can define a rational Euler characteristic for Γ by $\chi(\Gamma) = \frac{\chi(\Gamma_0)}{[\Gamma:\Gamma_0]}$ where $\chi(\Gamma_0)$ is the Euler characteristic of the manifold $\mathbb{H}^4/\Gamma_0$. This is well defined, and furthermore, $\chi(\Gamma)$ can be computed directly from the simplex by the formula

$$\chi(\Gamma) = \sum_{\tau} \frac{(-1)^{\dim\tau}}{\circ(\operatorname{Stab}\tau)},$$

where the sum is over all the cells τ of the simplex (see [4] for example). The resulting Euler characteristics in the five compact cases are:

$$\chi(\Gamma_1) = \frac{1}{14400}, \qquad \chi(\Gamma_2) = \frac{17}{28800}, \qquad \chi(\Gamma_3) = \frac{26}{14400},$$
$$\chi(\Gamma_4) = \frac{17}{14400}, \qquad \text{and} \qquad \chi(\Gamma_5) = \frac{11}{5760}.$$

Any torsion-free subgroup inside one of these reflection groups will give rise to a compact hyperbolic manifold, so that, for minimal volume, we endeavour to find torsion-free subgroups of as small an index as possible. The manifold will be orientable if and only if the torsion-free subgroup is contained in the index 2 subgroup generated by products of pairs of the generating reflections (or equivalently, contains no element expressible as a word of odd length in the generating reflections). For a systematic approach to Coxeter groups in dimensions 4 and higher, see [9].

Note that if H is a finite subgroup of the reflection group Γ, then for any torsion-free subgroup Γ_0, the index $[\Gamma : \Gamma_0]$ must be divisible by the order of H (since $H \cap \Gamma_0$ has to be trivial). Thus for each of the five groups concerned, any resulting compact orientable manifold $\mathbb{H}^4/\Gamma_0$ will have Euler characteristic a multiple of $2, 34, 26, 34$ or 22, respectively. For details on Γ_1, see §3. The Davis manifold arises from a normal torsion-free subgroup of Γ_3 of index precisely 14400 — indeed the unique such subgroup — so has Euler characteristic 26 (see [8, 10, 16]). We also note that the groups $\Gamma_1, \dots, \Gamma_5$ are arithmetic, the first four have the same arithmetic structure and are hence commensurable [20, 12]. Thus any subgroups of finite index in these four groups will also be pairwise commensurable.

3. Torsion elements of the group $\Gamma_1 = [5, 3, 3, 3]$

In order to find torsion-free subgroups of finite index in the Coxeter group Γ_1, we first determine representatives of conjugacy classes of torsion elements of prime order in Γ_1. Let a, b, c, d, e represent the reflections in the faces labelled A, B, C, D, E in the symbol Δ_1 in Figure 2.

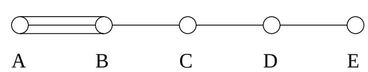

FIGURE 2. Coxeter symbol for the group $\Gamma_1 = [5, 3, 3, 3]$

Note that if a subgroup contains a torsion element x, of order n, say, and p is any prime divisor of n, then $x^{n/p}$ is a torsion element of order p lying in the same subgroup, and hence we may restrict our attention to prime orders.

Every element of finite order in Γ_1 has a fixed point in $\mathbb{H}^4$ and so is conjugate in Γ_1 to an element stabilising a vertex. The vertex stabilisers are the five 4-generator subgroups obtained by deleting one generator, namely $\langle a, b, c, d\rangle$, $\langle a, b, c, e\rangle$, $\langle a, b, d, e\rangle$, $\langle a, c, d, e\rangle$ and $\langle b, c, d, e\rangle$. These five subgroups are all finite, of orders 14400, 240, 60, 48 and 120 respectively, and their structure is well known (and also can be found with the help of the MAGMA system [3]). The conjugacy classes of the elements of prime order in each of these finite subgroups is given in the table below (where in some cases, for information, we have included more than one representative).

(i) Subgroup $\langle a,b,c,d\rangle$, the group of the 120-cell, of order 14400:

Order of element	Class size	Class representative(s)
2	1	$(abcd)^{15}$
2	60	a, b, c, d
2	60	$(abc)^5$
2	450	ac, ad, bd
3	40	$(abcd)^{10}$
3	400	bc, cd
5	24	$(abcd)^6$
5	24	$(abcd)^{12}$
5	144	ab
5	144	$(ab)^2$
5	288	$(abcd)^{10}(abc)^{-2}$

(ii) Subgroup $\langle a,b,c,e\rangle \cong A_5 \times C_2 \times C_2$, of order 240:

Order of element	Class size	Class representative(s)
2	1	e
2	1	$(abc)^5$
2	1	$(abc)^5e$
2	15	a, b, c
2	15	ac
2	15	ae, be, ce
2	15	ace
3	20	bc
5	12	ab
5	12	$(ab)^2$

(iii) Subgroup $\langle a,b,d,e\rangle \cong D_5 \times D_3$, of order 60:

Order of element	Class size	Class representative(s)
2	5	a, b
2	3	d, e
2	15	ad, ae, bd, be
3	2	de
5	2	ab
5	2	$(ab)^2$

(iv) Subgroup $\langle a,c,d,e\rangle \cong S_4 \times C_2$, of order 48:

Order of element	Class size	Class representative(s)
2	1	a
2	3	ce
2	3	ace
2	6	ac, ad, ae
2	6	c, d, e
3	8	cd, de

(v) Subgroup $\langle b,c,d,e\rangle \cong S_5$, of order 120:

Order of element	Class size	Class representative(s)
2	10	b, c, d, e
2	15	bd, be, ce
3	20	bc, cd, de
5	24	$bcde$

From these observations the following theorem follows easily:

Theorem 3.1. *A subgroup of the Coxeter group Γ_1 is torsion-free if and only if it contains no element that is conjugate to one of the elements in the list* $L = [\,a, ab, ac, bc, ace, (abc)^5, (abc)^5e, (abcd)^{15}, (abcd)^{10}, (abcd)^6, bcde, (abcd)^{10}(abc)^{-2}\,]$.

Note that the subgroup $\langle a, b, c, d\rangle$ is isomorphic to the Coxeter group $[5, 3, 3]$, also known as the automorphism group of the 120-cell. Any torsion-free subgroup of Γ_1 must intersect this subgroup trivially, and therefore its index in Γ_1 has to be divisible by 14400. The torsion-free subgroup Σ we find in section 5 has index 8×14400.

4. Subgroups of small index in Γ_1

Using the `LowIndexSubgroups` command in Magma [3], it is easy to find conjugacy classes of subgroups of small index in the group Γ_1. In fact there are 24 classes of subgroups of index up to 240: one of index 1, one of index 2, two of index 85, two of index 120, two of index 136, two of index 156, four of index 170, and ten of index 240.

Of particular interest to us are the two conjugacy classes of index 120.

One of these contains the subgroup H_1 generated by a, b, c, $dcbabacebabcd$, $dedcbabacbabcdcbabcababcded$ and $edcbabacbabdcbaebcdabcababcde$. Now in the permutation representation of Γ_1 on (right) cosets of this subgroup of index 120 by (right) multiplication, the elements of the list L in Theorem 3.1 have cycle structures $1^{32}2^{44}$, $1^{10}5^{22}$, 1^82^{56}, 1^63^{38}, 1^82^{56}, $1^{32}2^{44}$, $1^{32}2^{44}$, 1^{120}, 3^{40}, 5^{24}, 5^{24} and 5^{24}, respectively. (Note: here the notation r^{c_r} indicates c_r cycles of length r.) As an element $g \in \Gamma_1$ fixes the coset H_1x in this permutation representation if and only if $xgx^{-1} \in H_1$, it follows that H_1 contains conjugates of each of the torsion elements $a, ab, ac, bc, ace, (abc)^5, (abc)^5e$ and $(abcd)^{15}$, but contains no conjugates of any of the torsion elements $(abcd)^{10}, (abcd)^6, bcde$ or $(abcd)^{10}(abc)^{-2}$.

The other class of subgroups of index 120 contains the subgroup H_2 generated by ac, bd, $ababcdedcababa$, $adcedcbabcdeba$ and $ababacbaebcdaba$, and in the permutation representation on cosets of this subgroup the elements of the list L have cycle structures 2^{60}, 5^{24}, 1^82^{56}, 3^{40}, 1^82^{56}, 2^{60}, 2^{60}, 1^{120}, 1^63^{38}, $1^{10}5^{22}$, 5^{24} and 5^{24}, respectively.

Now consider the subgroup $H_1 \cap H_2$. This has index $120^2 = 14400$ in Γ_1 and in the permutation representation on its cosets the elements of the list L have cycle structures 2^{7200}, 5^{2880}, $1^{64}2^{7168}$, 3^{4800}, $1^{64}2^{7168}$, 2^{7200}, 2^{7200}, 1^{14400}, 3^{4800}, 5^{2880}, 5^{2880} and 5^{2880}, respectively. Hence this intersection contains conjugates of each of the torsion elements ac, ace and $(abcd)^{15}$, but not of any of the other elements of the list L.

In order to eliminate conjugates of the three elements ac, ace and $(abcd)^{15}$, we need to dig deeper into the subgroup lattice of Γ_1. Another good candidate subgroup is a subgroup H_3 of index 2 in H_2 (and index 240 in Γ_1), generated by ac, bd, $ababcdedcababa$ and $ababcbabadcbabcde$. In the permutation representation of Γ_1 on the 240 cosets of this subgroup, the elements of the list L in Theorem 3.1 have cycle structures 2^{120}, 5^{48}, $1^{16}2^{112}$, 3^{80}, 2^{120}, 2^{120}, 2^{120}, 1^{240}, $1^{12}3^{76}$, $1^{20}5^{44}$, 5^{48} and 5^{48}, respectively.

The subgroup $H_1 \cap H_3$ has index 28800 in Γ_1, and in the permutation representation on its cosets the elements of the list L have cycle structures 2^{14400}, 5^{5760},

$1^{128}2^{14336}$, 3^{9600}, 2^{14400}, 2^{14400}, 2^{14400}, 1^{28800}, 3^{9600}, 5^{5760}, 5^{5760} and 5^{5760}, respectively. Hence this intersection still contains conjugates of each of ac and $(abcd)^{15}$, but none of any of the other elements of the list L.

Similar observations can be made for other subgroups of index 240, although the outcome is best for H_1 and H_3.

5. Construction of the 4-manifold

A further application of the `LowIndexSubgroups` command in Magma [3] to the subgroup H_3 described in section 4 produces a subgroup H_4 of index 4 in H_3 (and index 960 in Γ_1), generated by *ababcbabadcbabcde* and *abacbdcbaedecbabcaba*. In the permutation representation of Γ_1 on the 960 cosets of this subgroup, the elements of the list L in Theorem 3.1 have cycle structures 2^{480}, 5^{192}, 2^{480}, 3^{320}, 2^{480}, 2^{480}, 2^{480}, 2^{480}, $1^{48}3^{304}$, $1^{80}5^{176}$, 5^{192} and 5^{192}, respectively.

The intersection $\Sigma = H_1 \cap H_4$ is a subgroup of index 115200 in Γ_1, and in the permutation representation on its cosets the elements of the list L have cycle structures 2^{57600}, 5^{23040}, 2^{57600}, 3^{38400}, 2^{57600}, 2^{57600}, 2^{57600}, 2^{57600}, 3^{38400}, 3^{9600}, 5^{23040}, 5^{23040} and 5^{23040}, respectively. In particular, none of the elements of the list L fix any points, and hence by Theorem 3.1, this subgroup is torsion free.

Thus we have found a torsion-free subgroup Σ of index 8×14400 in Γ_1, and therefore of Euler characteristic 8, giving a compact hyperbolic 4-manifold of volume $4\pi^2/3 \times 8 = 32\pi^2/3$.

This manifold is non-orientable. In fact the subgroup $\Sigma = H_1 \cap H_4$ is generated by the elements *cbdcbabcdedcbabcda* (of length 18), *babacbabacbabcdcbabcdedcbabcdab* (length 31), *ababacdedcbabacbabcdcbabcababcded* (length 33), and *bedcbabacbabdcba bacbadcbaedcbabcdabcababcdbabcbaba* (length 50). Its unique subgroup of index 2 containing only words of even length is a torsion-free subgroup Σ^o of index 16×14400 in Γ_1, giving a compact hyperbolic 4-manifold of Euler characteristic 16 and volume $64\pi^2/3$.

Our findings can be summed up in the following theorem:

Theorem 5.1. *In the Coxeter group* $\Gamma_1 = [5, 3, 3, 3]$, *the intersection* Σ *of the index* 120 *subgroup generated by the elements a, b, c, dcbabacebabcd, dedcbabacbabcdcbabca babcded and edcbabacbabdcbaebcdabcababcde with the index* 960 *subgroup generated by ababcbabadcbabcde and abacbdcbaedecbabcaba is torsion-free and of index* 115200 *in* Γ_1, *and the quotient space* $\mathbb{H}^4/\Sigma$ *is a compact but non-orientable hyperbolic* 4-*manifold of volume* $32\pi^2/3$ *and Euler characteristic* 8. *Furthermore,* Σ *has a subgroup* Σ^o *of index* 2 *such that the quotient space* $\mathbb{H}^4/\Sigma^o$ *is a compact orientable hyperbolic* 4-*manifold of volume* $64\pi^2/3$ *and Euler characteristic* 16.

6. Closing remarks

There may still be torsion-free subgroups in Γ_1 of smaller index than the subgroups we have described above, however we have not been able to find any using the computational methods available and a number of new approaches to this question. There is certainly no torsion-free subgroup of smaller index that is obtainable as an intersection of two subgroups of index up to 480 in Γ_1, or as an intersection of two subgroups of index up to 4 in any of these, but of course there may be other subgroups, possibly maximal in Γ_1, or otherwise lying inside some maximal subgroup of index larger than 480.

The abelianisations of the subgroups H_1, H_2, H_3 and H_4 are $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}_2 \oplus \mathbb{Z}_4$, $\mathbb{Z}_2 \oplus \mathbb{Z}$ and $\mathbb{Z}$ respectively, while those of the intersections $H_1 \cap H_2$, $H_1 \cap H_3$ and $\Sigma = H_1 \cap H_4$ are $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$, respectively. Finally, the abelianisation of Σ^o is $\mathbb{Z}^2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2^2$. These were found with the help of the `Rewrite` and `AQInvariants` commands in Magma [3].

From this we obtain the integral homology of $M = \mathbb{H}^4/\Sigma^o$ using Poincaré duality and the Euler characteristic. Thus

$$\begin{aligned} H_1(M) &= \mathbb{Z}^2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2^2, \\ H_2(M) &= \mathbb{Z}^{18} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2^2, \quad \text{and} \\ H_3(M) &= \mathbb{Z}^2. \end{aligned}$$

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