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UNIQUE CONTINUATION ALONG CURVES AND HYPERSURFACES FOR SECOND ORDER ANISOTROPIC HYPERBOLIC SYSTEMS WITH REAL ANALYTIC COEFFICIENTS

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Abstract. In this paper we prove the following kind of unique continuation property. That is, the zero on each geodesic of the solution in a real analytic hypersurface for second order anisotropic hyperbolic systems with real analytic coefficients can be continued along this curve.

1. Introduction

Let Ω be an open connected set in $\mathbb{R}^n$ with $n \geq 2$ and let $C^\omega(\Omega)$ be the set of all analytic functions defined in Ω. Let $\partial_j = \partial/\partial x_j$ and let $u(t,x) = (u_1, \cdots, u_N)^\top$ denote a vector-valued function of size $N \geq 1$. We consider the operator

$$Pu(t,x) := \partial_t^2 u(t,x) - Lu(t,x), \quad (t,x) \in \mathbb{R}^1_t \times \mathbb{R}^n_x, \tag{1.1}$$

where the αth component $Lu_\alpha(t,x)$ is given by

$$(Lu)_\alpha(t,x) = \rho(x)^{-1} \sum_{j,l=1}^{n} \sum_{\beta=1}^{N} C^{jl}_{\alpha\beta}(x) \partial_j \partial_l u_\beta + R_\alpha u(t,x) \quad (1 \leq \alpha \leq N)$$

with a first order linear partial differential operator R_α, where $0 < \rho(x) \in C^\omega(\Omega)$ corresponds to a density in many practical applications.

Throughout the paper we assume that all the coefficients of L are in $C^\omega(\Omega)$. For the principal part of L, we assume the following symmetry and strong ellipticity:

$$\text{(symmetry)} \quad C^{jl}_{\alpha\beta}(x) = C^{lj}_{\beta\alpha}(x) \quad (x \in \Omega, \quad 1 \leq \alpha, \beta \leq N, \quad 1 \leq j, l \leq n); \tag{1.2}$$

$$\text{(strong ellipticity)} \quad \sum_{1 \leq \alpha, \beta \leq N} \sum_{1 \leq j, l \leq n} C^{jl}_{\alpha\beta}(x) a_\alpha \xi_j a_\beta \xi_l \geq \delta |a|^2 |\xi|^2 \tag{1.3}$$

$$(x \in \Omega,\ a = (a_1, \cdots, a_N) \in \mathbf{R}^N,\ \xi = (\xi_1, \cdots, \xi_n) \in \mathbf{R}^n).$$

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In this paper we consider the localized unique continuation property (abbreviated as l-ucp) for (1.1). That is, if a solution of (1.1) is zero in a set $\subset \mathbb{R}^n_x$ with codimension higher than 1, then it must vanish on the connected component containing this set with the same codimension. Here $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ are the space variables. When the codimension of the set of points for the solution of (1.1) is zero, then l-ucp becomes the usual unique continuation property.

The study of l-ucp was initiated by Cheng, Yamamoto and Zhou [3] for the wave equation, and they showed the l-ucp along each line in the hyperplane. They combined the localized Fourier-Gauss transformation to transform (1.1) to the Laplace equation with a small inhomogeneous term and the conditional stability estimate for the unique continuation of the solution of the Laplace equation along lines.

The method of the localized Fourier-Gauss transformation was introduced by Lerner [6] for proving some uniqueness result for an ill-posed problem, and it was also used by Robbiano [9] to prove some kind of unique continuation property. The purpose of this paper is to generalize the result in [3] to second order anisotropic hyperbolic systems with real analytic coefficients. For related results, the readers should see the references given in [3].

We will focus on a neighborhood of the origin in an open analytic hypersurface S such that $\bar{S} \subset \Omega$. Let $d(x, y)$ be the distance between two points $x, y \in S$ with respect to the metric on S induced from the Euclidean metric of $\mathbb{R}^n$. For $r > 0$, define $B_S(x, r) := \{y \in S : \ d(x, y) < r\}$.

Theorem 1.1. *Let $R_0 > 0$ be such that $B_S(0, R_0) \subset \Omega$ in which the exponential map $\exp_0 : T_0S \to S$ gives a local coordinate. There exists R $(0 < R < R_0)$ with the following property: Fix a constant r such that $0 < r < R$. Suppose $u \in C^2((-T, T) \times \Omega)$ satisfies $Pu = 0$ in $(-T, T) \times \Omega$ and*

$$u(t, x) = 0 \quad ((t, x) \in (-T, T) \times B_S(0, r) \cap \gamma)$$

for a geodesic γ in $B_S(0, R)$ starting from 0 with distance d $(r < d < R)$. Then there exists a constant $K = K(r, R, s_0) > 0$ for a given small $s_0 \in (0, T)$, and we have

$$u(t, x) = 0 \quad \textit{for} \quad (t, x) \in (-T + s_0, T - s_0) \times (B_S(0, R) \cap \gamma) \tag{1.4}$$

such that

$$|t| + K(R - d)^{-\frac{1}{2}}(d - r)^{\frac{1}{2}} < T - s_0.$$

Corollary 1.2. *Let R, r, d, K and s_0 be as given in Theorem 1.1. Then if $u \in C^2((-T, T) \times \Omega)$ satisfies $Pu = 0$ in $(-T, T) \times \Omega$ and*

$$u(t, x) = 0 \quad ((t, x) \in (-T, T) \times B_S(0, r)),$$

we have

$$u(t, x) = 0 \quad \textit{for} \quad (t, x) \in (-T + s_0, T - s_0) \times B_S(0, d)$$

such that

$$|t| + K(R - d)^{-\frac{1}{2}}(d - r)^{\frac{1}{2}} < T - s_0.$$

Proof. This easily follows from the proof of Theorem 1.1 given later which shows that $K = K(r, R, s_0)$ of the theorem does not depend on the geodesic γ. □

Remark 1.3. (1) We remark on a byproduct of Corollary 1.2. Let the assumptions in Corollary 1.2 be satisfied. From the proof of Theorem 1.1 given later, K in Theorem 1.1 is determined from the harmonic measure associated with a solution

to the elliptic equation $Q_{s,x}\phi_{a,\lambda} = 0$ obtained by applying the localized Fourier-Gauss transform (defined later) to equation (1.1), and it can be controlled by the domain of the analyticity of the coefficients in equation (1.1). Also, R and s_0 in the same theorem depend on the size of the domain in which we can construct the fundamental solution for $Q_{s,x}$. This size can be controlled by the domain of analyticity of the coefficients, the symmetry and strong ellipticity for $Q_{s,x}$ being inherited from that of L. Therefore, by a repeated use of Corollary 1.2, we have $u(t,x) = 0$ $((t,x) \in (-\frac{T}{2}, \frac{T}{2}) \times S)$ if T is large enough and $u(t,x) = 0$ $((t,x) \in (-T,T) \times B_S(0,r))$.

(2) L can be an anisotropic elasticity operator:

$$(Lu)_\alpha(t,x) = \rho(x)^{-1} \sum_{\beta,j,l=1}^{n} \partial_j (C^{jl}_{\alpha\beta}(x)\partial_l u_\beta) \quad (1 \le \alpha \le N) \tag{1.5}$$

with the elasticity tensor $C^{jl}_{\alpha\beta}(x)$, where $1 \le \alpha, \beta, j, l \le N = n$ ($n = 2$ or 3). We assume $C^{jl}_{\alpha\beta}(x) \in C^\omega(\Omega)$ $(1 \le \alpha, \beta, j, l \le n)$, symmetry and strong ellipticity. If we replace the symmetry by the full symmetry:

$$C^{jl}_{\alpha\beta}(x) = C^{lj}_{\beta\alpha}(x) = C^{jl}_{\beta\alpha}(x) \quad (x \in \Omega,\ 1 \le \alpha, \beta, j, l \le n),$$

which is a condition stronger than the symmetry condition, we can replace $\partial_l u_\beta(t,x)$ by the strain tensor $\varepsilon_{\beta l}(t,x) := \frac{1}{2}(\partial_l u_\beta(t,x) + \partial_\beta u_l(t,x))$. This is the usual form of the anisotropic elasticity operator.

(3) By the previous remark, we have all the results given in Theorem 1.1 and Corollary 1.2 for the anisotropic elasticity operator.

From Remark 1.3, we have a very interesting result concerning the observation data for an inverse source problem, which includes the nondestructive testing for the acoustic emission sources. That is, consider a solution $u(t,x) \in C^2((-T,T)\times\Omega)$ to $Pu = f$ with an unknown source $f(t,\cdot) \in C^0((-T,T) \times \Omega)$, where $P = \partial_t^2 - L$ with L given by (1.5). Then, we have the following.

Corollary 1.4. *Let the assumptions in Corollary* 1.2 *be satisfied. Assume the support of the unknown source f is away from S. Then, the observation data $u(t,x)$ given for $(t,x) \in (-T,T) \times B_S(0,r)$ determines $u(t,x)$ $((t,x) \in (-T/2, T/2) \times S)$. More precisely, if $u_j(t,x) \in C^2((-T,T) \times \Omega)$ $(j = 1,2)$ are the respective solutions of $Pu_j = f_j$ with unknown sources $f_j(t,x) \in C^0((-T,T) \times \Omega)$ such that $\mathrm{supp} f_j \cap ((-T,T) \times S) = \emptyset$ $(j = 1,2)$ and $u_1(t,x) = u_2(t,x)$ $((t,x) \in (-T,T) \times B_S(0,r))$, then we have $u_1(t,x) = u_2(t,x)$ $((t,x) \in (-T/2, T/2) \times S)$.*

Proof. This is an immediate consequence of Remark 1.3. □

Remark 1.5. Let $n = 2$ or 3. By the standard regularity argument used in the mixed problem for hyperbolic equations, the regularity of the source f can be replaced by $f(t,\cdot) \in C_0^4((0,T], H^{-1}(\Omega))$ if $u(t,\cdot) \in \bigcap_{k=0}^1 C^{5-k}([0,T], H^{k-1}(\Omega))$ is the solution to a mixed problem in $[0,T] \times \Omega$ with homogeneous initial and boundary conditions at $t = 0$ and $\partial\Omega$, respectively. Note that $u(t,x) \in C^2$ for $t \in [0,T]$ and x near S. In particular, if $f(t,x) = \delta_\Sigma(x) b(t,x)$ with $b(t,\cdot) \in C^4([0,T], C^\infty(\Omega))$ and Σ is an open surface inside Ω representing a crack, this $f(t,x)$ is the kind of source considered for the acoustic emission sources (see [8]).

2. Proof of Theorem 1.1

We now perform a change of coordinates near 0 to flatten S. Since the exponential map $\exp_0 : T_0S \to S$ gives local coordinates in $B_S(0, R_0) \subset \Omega$ and the dimension of the tangent space at 0 is $n-1$, using the normal vector field of S, we can construct new coordinates $(y_1, y_2, \cdots, y_n)$ such that the geodesic γ in $B_S(0, R_0)$ is just on the y_1-axis directed to its positive direction.

In the new coordinates $y = (y_1, y_2, \cdots, y_n)$, (1.1) becomes

$$Pu(t, y) := \partial_t^2 u(t, x(y)) - L_y u(t, x(y)),$$

where the principal part of the partial differential operator L_y is defined by

$$(L_y u)_\alpha = \sum_{j,l=1}^{n} \sum_{\beta=1}^{N} \widetilde{C}^{pq}_{\alpha\beta}(y) \partial_p \partial_q u_\beta$$

with

$$\widetilde{C}^{pq}_{\alpha\beta}(y) = |J(x(y))| \sum_{j,l=1}^{n} C^{jl}_{\alpha\beta}(x(y)) \partial_j y_p \partial_l y_q$$

and

$$J(x(y)) := \text{Jacobian of the change of variables.}$$

It is not hard to check that $\widetilde{C}^{pq}_{\alpha\beta}(y)$ satisfies (1.2) and (1.3) with possibly different constants.

To avoid heavy notation, the coordinates $x = (x_1, x_2, \cdots, x_n)$ represent the Cartesian coordinates in the Euclidean space $\mathbb{R}^n$, the geodesic γ is the segment on the x_1-axis starting from the origin directed to its positive direction and $\widetilde{C}^{pq}_{\alpha\beta}(y) = C^{pq}_{\alpha\beta}(x)$ etc. from now on. Also, for $r > 0$, define $B(x, r) := \{y \in \mathbb{R}^n : \ |x - y| < r\}$.

2.1. Preliminary facts. We will use the method given in [3] to prove Theorem 1.1. We define the localized Fourier-Gauss transformation (LFGT for its abbreviation) $v_{a,\lambda}(s, x)$ of $u(t, x)$ by

$$v_{a,\lambda}(s, x) := \sqrt{\frac{\lambda}{2\pi}} \int_{-T}^{T} e^{-\frac{\lambda}{2}(is+a-t)^2} u(t, x) dt, \tag{2.1}$$

where $\lambda > 0$, $a \in \mathbb{R}$ and $i = \sqrt{-1}$. In connection with the operator $P = \partial_t^2 - L$, we define an elliptic operator by

$$Q_{s,x} := \partial_s^2 + L \quad ((s, x) \in \mathbb{R}^1_s \times \mathbb{R}^n_x)$$

and set

$$\chi_{a,\lambda} := Q_{s,x} v_{a,\lambda}.$$

The following properties of LFGT are given in [3].

Lemma 2.1. *Let $u \in C^2([-T, T] \times \overline{B(0, R)})$ and $s_0 \in (-T, T)$ be fixed. If u satisfies $Pu = 0$ in $[-T, T] \times \overline{B(0, R)}$, then*

$$v_{a,\lambda}(0, x) \to u(a, x) \quad \text{as} \quad \lambda \to \infty, \quad |a| < T, \tag{2.2}$$

$$\begin{aligned} &|v_{a,\lambda}(s, x)|, \quad |(\partial_{x_j} v_{a,\lambda})(s, x)|, \quad |(\partial_s v_{a,\lambda})(s, x)| \\ &\le C_1 \lambda^{\frac{1}{2}} e^{\frac{\lambda}{2} s_0^2} \quad (1 \le j \le n, \ (s, x) \in (-s_0, s_0) \times B(0, R)), \end{aligned} \tag{2.3}$$

$$|\chi_{a,\lambda}(s, x)| \le C_2 \lambda^{\frac{3}{2}} e^{-\frac{\lambda}{2}[(T-|a|)^2 - s_0^2]} \quad ((s, x) \in (-s_0, s_0) \times B(0, R)), \tag{2.4}$$

where $C_1 > 0$ *depends on* $\|u\|_{C^1([-T,T]\times\overline{B(0,R)})}$ *and* $C_2 > 0$ *depends on* s_0, T, a *and* $\|u\|_{C^2([-T,T]\times\overline{B(0,R)})}$.

Remark 2.2. In [3], (2.4) is shown for $\partial_s^2 + \Delta$. The proof for $Q_{s,x}$ is almost the same as that given in [3].

The proof of Theorem 1.1 is done by using Lemma 2.1 and the following conditional stability for the unique continuation along the curve for $Q_{s,x}$.

Theorem 2.3. *Let* $\varphi \in C^2((-s_0, s_0) \times B(0,R))$ *satisfy* $Q_{s,x}\varphi = 0$ *in* $(-s_0, s_0) \times B(0,R)$ *and*

$$\|\varphi\|_{C([-s_0,s_0]\times\overline{B(0,R)})} \le M$$

with some constant $M > 0$. *Then, for* $\rho \in (r, R)$, *there exist positive constants* $C_3 = C_3(r, R, s_0)$ *and* $\alpha = \alpha(\rho, r, R, S_0) \in (0,1)$ *such that*

$$\|\varphi(0,\cdot,0)\|_{L^\infty(0,\rho)} \le C_3(M + C_\rho)^{1-\alpha}\|\varphi(0,\cdot,0)\|^{\alpha}_{L^\infty(0,r)}, \tag{2.5}$$

where the positive number $C_\rho > 0$ *depends on* ρ. *Moreover,*

$$\lim_{\rho\uparrow R}\alpha(\rho, r, R, S_0) = 0, \quad \lim_{\rho\downarrow r}\alpha(\rho, r, R, S_0) = 1, \tag{2.6}$$

and

$$\alpha(\rho, r, R, S_0) \ge C_4(R - \rho), \quad 1 - \alpha(\rho, r, R, S_0) \le C_5(\rho - r)^{1/2}, \tag{2.7}$$

where $C_4 = C_4(r, R, s_0) > 0$ *and* $C_5 = C_5(r, R, s_0) > 0$ *are constants.*

We first apply this theorem to finish proving Theorem 1.1, and after that we will give the proof of Theorem 2.3.

Proof of Theorem 1.1. First we have

$$\chi_{a,\lambda} = Q_{s,x}v(a,\lambda) \quad ((s,x) \in (-s_0, s_0) \times B(0,R)). \tag{2.8}$$

The definition of $v_{a,\lambda}(s,x)$ implies that

$$v_{a,\lambda}(s,x) = 0 \quad ((s,x) \in (-s_0, s_0) \times B^{'}(0,r)),$$

where $B^{'}(0,r) = B(0,r) \cap \{x = (x_1, x^{'}) : x_1 \ge 0,\ x^{'} = 0\}$.

Now we need to use the following Lemma 2.4 to estimate $v_{a,\lambda}$. □

Lemma 2.4. *There exist constants* R $(0 < R < R_0)$ *and* $C_6 > 0$ *depending on* $\delta, S, C^{jl}_{\alpha\beta}$ *such that for any* $f \in L^\infty((-s_0, s_0)\times B(0,R))$, *there exists* $w \in H^2(\mathbb{R}^{n+1})$ *satisfying* $Q_{s,x}w = 0 \quad ((s,x) \in (-s_0, s_0) \times B(0,R))$ *and*

$$|w(s,x)| \le C_6\|f\|_{L^\infty((-s_0,s_0)\times B(0,R))} \quad ((s,x) \in (-s_0, s_0) \times B(0,R)). \tag{2.9}$$

The proof of this lemma is given in the appendix.

By the lemma, there exist $\psi_{a,\lambda} \in H^2(\mathbb{R}^{n+1})$ and $\varphi_{a,\lambda} \in C^\infty((-s_0, s_0)\times B(0,R))$ such that

$$v_{a,\lambda} = \varphi_{a,\lambda} + \psi_{a,\lambda} \tag{2.10}$$

and

$$\begin{aligned} &Q_{s,x}\varphi_{a,\lambda} = 0 \quad ((s,x) \in (-s_0, s_0) \times B(0,R)), \\ &|\psi_{a,\lambda}(\xi)| \le C_6\|\chi_{a,\lambda}\|_{L^\infty((-s_0,s_0)\times B(0,R))} \quad (\xi = (s,x) \in (-s_0, s_0) \times B(0,R)) \end{aligned} \tag{2.11}$$

for some positive constant C_6.

Since $v_{a,\lambda}(s,x)=0$ for $(s,x)\in(-s_0,s_0)\times B^{'}(0,r)$, we obtain from (2.10), (2.11) and (2.4) that

$$\begin{aligned}|\varphi_{a,\lambda}(0,x)| = |\psi_{a,\lambda}(0,x)| &\le C\|\chi_{a,\lambda}\|_{L^\infty((-s_0,s_0)\times B(0,R))}\\ &\le C\lambda^{\frac{3}{2}}e^{-\frac{\lambda}{2}[(T-|a|)^2-s_0^2]} \qquad (x\in B^{'}(0,r));\end{aligned} \tag{2.12}$$

here and hereafter in the proof the constant C varies from line to line. Similarly, the equations (2.3), (2.4), (2.10) and (2.11) imply

$$|\varphi_{a,\lambda}(s,x)|\le C\lambda^{\frac{3}{2}}e^{\frac{\lambda}{2}s_0^2} \quad ((s,x)\in(-s_0,s_0)\times B(0,R),\ \lambda\ge C_\rho), \tag{2.13}$$

where $C_\rho>0$ is the constant in (2.5).

Now, we apply Theorem 2.3 to $\varphi_{a,\lambda}$. Then, in terms of (2.12) and (2.13), we have

$$|\varphi_{a,\lambda}(0,x)|\le C\lambda^{\frac{3}{2}}e^{-\frac{\lambda}{2}[(T-|a|)^2\alpha-s_0^2]} \quad (x\in B^{'}(0,\rho));$$

here the constant C is independent of $\lambda\ge C_\rho$.

Consequently, by (2.10), we get

$$|v_{a,\lambda}(0,x)|\le C\lambda^{\frac{3}{2}}e^{-\frac{\lambda}{2}[(T-|a|)^2\alpha-s_0^2]} \quad (x\in B^{'}(0,\rho)). \tag{2.14}$$

Recall that $\alpha=\alpha(\rho,r,R,s_0)$ in Theorem 2.3. It is convenient to write $\alpha=\alpha(\rho)$. By (2.6) and (2.7), we can estimate

$$\frac{1}{\sqrt{\alpha(\rho)}}-1\le\frac{1-\sqrt{\alpha(\rho)}}{\sqrt{\alpha(\rho)}}\le\frac{C_5(\rho-r)^{1/2}}{\sqrt{C_4(R-\rho)}}. \tag{2.15}$$

Therefore, define $K=K(r,R,s_0):=\frac{s_0C_5}{\sqrt{C_4}}$ and let a satisfy the following inequality:

$$|a|+K(R-\rho)^{-\frac{1}{2}}(\rho-r)^{\frac{1}{2}}<T-s_0.$$

By (2.15), it is easy to see that

$$s_0(\frac{1}{\sqrt{\alpha(\rho)}}-1)<T-s_0-|a|,$$

namely,

$$0<(T-|a|)^2\alpha-s_0^2.$$

Due to (2.14), we obtain

$$\lim_{\lambda\to\infty}|v_{a,\lambda}(0,x)|=0,$$

which implies $u(a,x)=0$ by (2.2). Thus, the proof is done.

3. Proof of Theorem 2.3

In this section, we give the proof of Theorem 2.3. First of all, we extend φ satisfying $Q_{s,x}\varphi=0$ $((s,x)\in(-s_0,s_0)\times B(0,R))$ to an analytic function with respect to the x_1 variable as follows.

Lemma 3.1. *Let $\overline{B(x_0,r)}\subset\omega$ for $x_0=(x_1^0,x_2^0,\cdots,x_n^0)$ and let* Imz *denote the imaginary part of $z\in\mathcal{C}$. Then for any given function $\varphi=\varphi(x_1,x_2,\cdots,x_n)$ satisfying $Q_{s,x}\varphi=0$ in ω, we can construct an analytic function $v=v(z)$ in $U_{\frac{r}{2}}(x_1^0,0)=\{z=\xi+i\eta:(\xi-x_1^0)^2+\eta^2<(\frac{r}{2})^2\}$ such that*

$$v(z)=\varphi(z,x_2^0,\cdots,x_n^0) \quad \textit{for} \quad \mathrm{Im}z=0,\ |z-x_1^0|<\frac{r}{2}$$

and there exists a constant $C_7 = C_7(r, x_0, \omega, n) > 0$ *such that*

$$\|v\|_{C(\overline{U_{\frac{r}{2}}(x_1^0,0)})} \leq C_7\|\varphi\|_{C(\omega)} + 1. \tag{3.1}$$

Remark 3.2. We change the form of the estimate in Lemma 3.1 of [1] so that there is no need to use a fundamental solution with a particular form. The change of the estimate will not influence the result because the right-hand side $C\lambda^{\frac{3}{2}}e^{\frac{\lambda}{2}s_0^2}$ of the inequality (2.13) in the proof of Theorem 1.1 increases to ∞ as λ tends to ∞.

Proof. By the analytic hypoellipticity for $Q_{s,x}\varphi = 0$ (see Theorem 9.5.1 in [4]), the function φ is real analytic in ω. Therefore, $\varphi(x_1, x_0^{'}) = \varphi(x_1, x_2^0, \cdots, x_n^0)$ can be represented as a power series

$$\varphi(x_1, x_0^{'}) = \sum_{k=0}^{\infty} \varphi_k (x_1 - x_1^0)^k \quad (|x_1 - x_1^0| < r),$$

where $\varphi_k = \partial_{x_1}^k \varphi(x_0)$. For the region $U_{\frac{r}{2}}(x_1^0, 0) \subset \mathcal{C}$, there exists a unique analytic function

$$v(z) = \sum_{k=0}^{\infty} \varphi_k (z - x_1^0)^k$$

such that

$$v(z) = \varphi(z, x_2^0, \cdots, x_n^0) \quad \text{for} \quad \mathrm{Im} z = 0,\ |z - x_1^0| < \frac{r}{2}.$$

We can find a positive number N to obtain

$$\sum_{k=N}^{\infty} |\varphi_k (z - x_1^0)^k| \leq 1 \quad (z \in U_{\frac{r}{2}}(x_1^0, 0)).$$

On the other hand, we apply some interior estimates of derivatives of $Q_{s,x}\varphi = 0$ to estimate φ_k so that

$$\sum_{k=0}^{N-1} |\varphi_k (z - x_1^0)^k| \leq C_7\|\varphi\|_{C(\omega)} \quad (z \in U_{\frac{r}{2}}(x_1^0, 0)).$$

Since

$$v(z) = \sum_{k=0}^{\infty} \varphi_k (z - x_1^0)^k = \sum_{k=0}^{N-1} \varphi_k (z - x_1^0)^k + \sum_{k=N}^{\infty} \varphi_k (z - x_1^0)^k,$$

the triangle inequality implies the proof. □

Next we define a harmonic measure in the sector D:

$$D = \{z \in \mathcal{C} : 0 < |z - z_0| < r, \quad |arg(z - z_0)| < \theta\}$$

with $z_0 \in \mathbb{R}$ and $0 < \theta < \frac{\pi}{2}$.

Definition 3.3. Let $0 < r_1 < r_2 < r$. A function $\alpha(z)$ is called the harmonic measure for D and $[r_1, r_2]$ if α satisfies

$$\begin{aligned} &\Delta\alpha(z) = 0, \quad z \in D \setminus [z_0 + r_1, z_0 + r_2], \\ &\alpha(z) = 0, \quad z \in \partial D, \\ &\alpha(z) = 1, \quad z \in [z_0 + r_1, z_0 + r_2]. \end{aligned}$$

The harmonic measure satisfies the following properties and their proof can be found in Lemma 4.2 of [2].

Lemma 3.4. *There exists a positive constant $C_8 = C_8(r_1, r_2, r)$ such that*

$$\alpha(x) \geq C_8(z_0 + r - x), \quad z \in [z_0 + r_1, z_0 + r_2],$$

and

$$0 < \alpha(z) < 1, \quad z \in D \setminus [z_0 + r_1, z_0 + r_2].$$

Estimating the harmonic measure, we can obtain a conditional estimate for analytic functions in D. The proof is in Lemma 4.3 of [2].

Lemma 3.5. *Suppose $v = v(z)$ is an analytic function in D and $\epsilon = \max |v(x)|$ for $x \in [z_0 + r_1, z_0 + r_2]$. If $|v(x)| \leq M_1$ for $z \in D$, then we have*

$$|v(x)| \leq M_1^{1-\alpha(x)} \epsilon^{\alpha(x)}, \quad x \in [z_0 + r_1, z_0 + r_2]. \tag{3.2}$$

Now, we proceed to the proof of Theorem 2.3. Using Lemma 3.1 several times, we can find an analytic extension v of φ to a D in $\mathcal{C}$ containing the segment $B^{'}(0, \rho)$ with the estimate (3.1) (see the proof of Theorem 3.1 in [2] for the details). Then, from (3.2), we immediately get the desired stability estimate.

4. Appendix

This appendix is devoted to the proof of Lemma 2.4.

Proof. This can be proven by a quite standard argument for any second order system of elliptic operators except the estimate (2.9). The details of the proof are as follows.

Since $Q = Q_{s,x}$ is a second order system of elliptic operators, there is a parametrix H of Q, which can be given as a properly supported classical pseudodifferential operator of order -2 such that its principal symbol $\sigma(H)(s, x, \sigma, \xi)$ is given by

$$\sigma(H)(s, x, \sigma, \xi) = \sigma(Q)(s, x, \sigma, \xi)^{-1} \quad (\sigma^2 + |\xi|^2 \geq 1), \tag{4.1}$$

where (σ, ξ) is the covector of (s, x) (see page 83 in [5]). We note that the argument given there can be applied for the system of pseudodifferential operators with an obvious change.

Let $\chi_0 \in C^\infty(\mathbb{R}^n)$ be a cutoff function supported near 0 such that $\chi_0 = 1$ in a neighborhood U_0 of 0 and $0 \leq \chi_0 \leq 1$. Define the smoothing operator K by $K := \chi_0(I - QH)$. Let U be a neighborhood of the origin such that $\bar{U} \subset U_0$. Also, let $\chi_j \in C_0^\infty(U_0)\,(j = 1, 2, 3)$ be cutoff functions such that $\chi_1 = 1$ in U, $\chi_2 = 1$ in a neighborhood of suppχ_1 and $\chi_3 = 1$ in a neighborhood of suppχ_2. Define K_0 by $K_0 = \chi_2 K \chi_1$. We take U small enough such that

$$\|K_0 g\|_{L^2(\mathbb{R}^{n+1})} \leq \frac{1}{2} \|g\|_{L^2(\mathbb{R}^{n+1})}.$$

The size of R and s_0 are taken to satisfy $(-s_0, s_0) \times B(0, R) \subset U$. Then, by the argument given in Kumano-go's book (see page 127 in [5]), we have that

$$\sum_{i=1}^{\infty} K_0^i \chi_2 f \quad \text{converges} \quad \text{in} \quad L^2(\mathbb{R}^{n+1})$$

and

$$u := H(\chi_2 f) + HK_0(\sum_{i=1}^{\infty} K_0^{i-1} \chi_2 f) \in H^2(\mathbb{R}^{n+1})$$

satisfies $Qu = f$ in $(-s_0, s_0) \times B(0, R)$, where we have extended f to be zero outside $(-s_0, s_0) \times B(0, R)$. Since K_0 is a smoothing operator, we have from the Sobolev embedding theorem,

$$\|HK_0(\sum_{i=1}^{\infty} K_0^{i-1}\chi_2 f)\|_{L^\infty((-s_0,s_0)\times B(0,R))} \leq C\|f\|_{L^\infty((-s_0,s_0)\times B(0,R))}$$

for some constant $C > 0$. By (4.1) and Lemma 2.17 in [7], the Schwartz kernel $\Gamma = \Gamma(\xi, \eta)$ of H with $\xi = (s, x)$ satisfies the estimate

$$\Gamma(\xi, \eta) = \begin{cases} O(|\xi - \eta|^{1-N}) & (n \geq 2), \\ O(\log|\xi - \eta|) & (n = 1). \end{cases}$$

Hence, we can easily see

$$\|H(\chi_2 f)\|_{L^\infty((-s_0,s_0)\times B(0,R))} \leq C\|f\|_{L^\infty((-s_0,s_0)\times B(0,R))}.$$

Altogether, we have the desired estimate. □

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