

E-ALGEBRAS WHOSE TORSION PART IS NOT CYCLIC

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ABSTRACT. We consider algebras A over a Dedekind domain R with the property $A \cong \text{EndAlg}_R A$ and generalize Schultz' structure theory of the case $R = \mathbb{Z}$ to Dedekind domains. We construct examples of mixed $E(R)$ -algebras, which are non-split extensions of the submodule of elements infinitely divisible by the relevant prime ideals. This is also new in the case $R = \mathbb{Z}$.

1. INTRODUCTION

1.1. Notion of an $E(R)$ -algebra. Let R be a commutative ring and M an R -module. The endomorphism ring $\text{End } M$ of M is an R -algebra acting on M on the right. We will denote the endomorphism ring by $\text{Endo } M$ if we think of it as an R -module.

If M and $\text{Endo } M$ are isomorphic, then M is called an $E(R)$ -module.

If A is an R -algebra, then $\mu: A \rightarrow \text{End } A$ mapping $a \in A$ to right multiplication by a is an algebra homomorphism. If μ is an isomorphism, then A is an $E(R)$ -algebra.

An R -algebra A is a *generalized $E(R)$ -algebra* if it is isomorphic to $\text{End } A$.

Note that $E(R)$ -algebras are precisely the commutative generalized $E(R)$ -algebras. Every generalized $E(R)$ -algebra is an $E(R)$ -module, and every $E(R)$ -module admits a generalized $E(R)$ -algebra structure, which is unique up to isomorphism. Thus $E(R)$ -modules and generalized $E(R)$ -algebras are essentially the same.

The above notions are generalizations of E -rings, i.e. $E(\mathbb{Z})$ -algebras.

These notions go back to Schultz; see [5]. Examples of $E(R)$ -modules are given in Section 2.

1.2. Main results. This subsection summarizes our main results, which will be proved in later sections.

From now on let R be a Dedekind domain. Let Q denote the quotient field of R . If S is a set of prime ideals of R , then an R -module M is *S -divisible* if $M = PM$ for all $P \in S$. The module M is *S -reduced* if it has no non-trivial S -divisible submodule. If $N \subseteq M$, then N is *S -pure* in M (denoted by $N \subseteq_S M$) if $IN = N \cap IM$ for all ideals I of R which are products of prime ideals from S .

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For example, $T := \bigoplus_{P \in S} R/P^{n_P}$ is an S -pure submodule of

$$\Pi := \prod_{P \in S} R/P^{n_P}$$

where the n_P are arbitrary positive integers. Both T and Π are S -reduced, and they are divisible by all prime ideals not in S . Note that Π/T is divisible: if $Q \in S$, then Π/T is a factor of the Q -divisible module $\Pi := \prod_{P \in S \setminus \{Q\}} R/P^{n_P}$ and hence it is Q -divisible. If $Q \notin S$, then Π/T is Q -divisible since Π is already Q -divisible.

Now we can reformulate the structure theory of E -rings from Schultz [5, p. 63, Theorem 6] for Dedekind domains.

Theorem 1.1. *Let A be a generalized $E(R)$ -algebra. Then*

(i) *The primary components of the torsion part T of A are cyclic:*

$$(1) \quad T = \bigoplus_{P \in S} R/P^{n_P}$$

where S is a set of prime ideals of R and the n_P are positive integers. After Schultz, the elements of S are called the relevant prime ideals.

(ii) *Using the above notation, let D be the largest S -divisible submodule of A . Then D is an ideal, A/D is an S -reduced subalgebra of $\prod_{P \in S} R/P^{n_P}$ and*

$$(2) \quad T = \bigoplus_{P \in S} R/P^{n_P} \subseteq_S A/D \subseteq_S \prod_{P \in S} R/P^{n_P} =: \Pi.$$

Using the notation and assumptions of the previous theorem, we have a pull-back diagram:

$$(3) \quad \begin{array}{ccc} A & \longrightarrow & A/T \\ \downarrow & & \downarrow \\ A/D & \longrightarrow & A/(D+T) \end{array}$$

This suggests that we can construct mixed $E(R)$ -algebras as pull-backs; see Lemma 4.1. It is a fruitful method, as it gives the first example of an $E(R)$ -module A for which D is not a direct summand as stated in the next theorem. Let R_S be the localization of R at the set S of prime ideals.

Theorem 1.2. *Let S be an infinite set of prime ideals of R . Suppose that S does not contain all the prime ideals and R_S is not a complete discrete valuation ring. Then for any family $\{n_P : P \in S\}$ of positive integers there exists an arbitrarily large $E(R)$ -algebra A whose torsion submodule is given by (1) and whose S -divisible part D is not a direct summand.*

This indicates that the classification of mixed $E(R)$ -modules is a difficult task. However, the classification of non-reduced E -rings generalizes to $E(R)$ -algebras:

Theorem 1.3. *The non-reduced $E(R)$ -modules over a Dedekind domain R are exactly the R -modules of the form $Q \oplus C$ where C is a torsion cyclic R -module. In particular they are all $E(R)$ -algebras.*

2. EXAMPLES OF $E(R)$ -MODULES

2.1. Torsion-free $E(R)$ -modules. Though most of the references below deal only with the case $R = \mathbb{Z}$, their methods generalize without problem to Dedekind domains R which are neither fields nor complete discrete valuation domains. Therefore we state the results for Dedekind domains.

If R is a field, then R is also the only $E(R)$ -algebra. Let us note that all $E(R)$ -algebras over a complete discrete valuation domain are $R, Q(R)$ and the direct sums $Q(R) \oplus R/p^n R$ for all $n \in \mathbb{N}$, so the exclusion of these rings is necessary. We also note that the generalized $E(R)$ -algebras over a complete discrete valuation domain are $E(R)$ -algebras; see [3]. Thus it is also necessary to exclude these rings.

Classical examples of torsion-free $E(R)$ -algebras are the subalgebras of Q and the pure subalgebras of J_P , the algebra of P -adic integers over R where P is a prime ideal.

Arbitrarily large $E(R)$ -algebras were constructed using the Black Box in Dugas, Mader, Vinsonhaler [1]. The proof can be simplified by use of the Strong Black Box; see [4] and also [3]. These $E(R)$ -algebras are distorted polynomial algebras over R .

Finally, we mention some more constructions of E -rings. Faticoni [2] showed that every countable reduced torsion-free commutative ring is a pure subring of an E -ring. The same is true for rings of cardinality $\leq 2^{\aleph_0}$; see [3].

2.2. $E(R)$ -modules whose torsion part is a direct summand. An R -module is an $E(R)$ -module whose torsion part is a direct summand iff, for some non-zero ideal I of R , it is of the form $R/I \oplus N$ where N is a torsion-free I -divisible $E(R)$ -module. (N is I -divisible means $N = IN$.) This follows directly from Theorem 1.1.

Thus this class of $E(R)$ -modules is classified modulo torsion-free $E(R)$ -modules. In particular, the torsion $E(R)$ -modules are precisely the cyclic modules.

2.3. Other mixed $E(R)$ -modules. The following example is due to Schultz. Let S be a set of prime ideals of R . Let A be a subalgebra of $\prod_{P \in S} R/P^{n_P}$ such that

$$(4) \quad \bigoplus_{P \in S} R/P^{n_P} \subseteq_S A \subseteq_S \prod_{P \in S} R/P^{n_P}.$$

Then A is an $E(R)$ -algebra and S is the set of relevant prime ideals.

So far we have only seen $E(R)$ -modules whose S -divisible part is a direct summand. But this is not true for our examples in Section 4:

Mixed $E(R)$ -modules can be constructed as a pull-back of the last example and a torsion-free $E(R)$ -module. See Lemma 4.1 for the precise construction.

3. THE STRUCTURE THEORY OF $E(R)$ -ALGEBRAS

In this section we prove Theorems 1.1 and 1.3, which describe the structure of $E(R)$ -algebras.

We will use $N \sqsubseteq M$ to denote that N is a direct summand of M as an R -module.

We recall some well-known facts about torsion submodules over Dedekind domains used below. Torsion modules decompose into primary components. Each torsion module M has a basic submodule i.e. a pure submodule B which is a direct sum of cyclic modules and M/B is divisible.

Basic submodules of M are not unique but the number of cyclic summands R/P^k is unique for all k . For example, this number is the largest cardinal κ for which $(R/P^k)^{(\kappa)}$ is a direct summand of M .

Divisible submodules and pure bounded submodules are direct summands in any module. Divisible modules are direct sums of modules R_{P^∞} 's and Q . The number of the summands R_{P^∞} , respectively Q is independent of the decomposition. Namely, in every decomposition of a divisible module M , the number of the summands Q is the largest cardinal κ for which $Q^{(\kappa)}$ is a direct summand of M . The same result holds for R_{P^∞} .

Recall that $J_P \cong \text{End } R_{P^\infty}$ is the algebra of P -adic integers over R . This gives an embedding of R_{P^∞} in $\text{Hom}(J_P, R_{P^\infty})$. As we already mentioned, $J_P \cong \text{End } J_P$.

Recall that $\text{Hom}(A, B)$ is torsion-free if B is divisible, and $\text{Hom}(A, B)$ is divisible if A is torsion-free and B is divisible.

The proofs of the next two lemmas are obvious.

Lemma 3.1. *Let M be a torsion cyclic module or the quotient field Q . Then for every cardinal κ ,*

$$(5) \quad \text{Endo } M^{(\kappa)} \cong M^{(\lambda)}$$

for some λ . If $\kappa > 1$, then $\lambda > \kappa$.

Lemma 3.2. *Let A be an $E(R)$ -module. Suppose that $A = B \oplus C$ is a direct sum of R -modules. Then $\text{Endo } B \oplus \text{Endo } C$ is a direct summand of A as an R -module. Moreover,*

$$(6) \quad A \cong \text{Endo } A = \text{Endo } B \oplus \text{Hom}(B, C) \oplus \text{Hom}(C, B) \oplus \text{Endo } C.$$

3.1. The main structure theorem. We are ready to prove Theorem 1.1. The two parts of the theorem are proven separately.

Proof of Theorem 1.1(i). By the preceding remarks, a basic submodule of the P -component A_P of A is a direct sum of cyclic P -modules: $\bigoplus_{k=1}^\infty (R/P^k)^{(\kappa(k))}$ where $\kappa(k)$ is the largest cardinal κ for which $R_{P^k}^{(\kappa)}$ is a direct summand of A . First we prove that $\kappa(k)$ is at most 1.

From Lemmas 3.1 and 3.2 we have a direct summand $R_{P^k}^{(\lambda)}$ of A where $\lambda > \kappa(k)$ if $\kappa(k) > 1$. Hence $\kappa(k) \leq 1$.

Now we prove that the basic subgroup of A_P is cyclic. Otherwise $R_{P^k} \oplus R_{P^l}$ would be a direct summand of A for some $k < l$. Then

$$(7) \quad \text{Endo}(R_{P^k} \oplus R_{P^l}) = \text{Endo } R_{P^k} \oplus \text{Hom}(R_{P^k}, R_{P^l}) \oplus \text{Hom}(R_{P^l}, R_{P^k}) \oplus \text{Endo } R_{P^l}$$

is a direct summand of A isomorphic to $R_{P^k}^3 \oplus R_{P^l}$, contradicting the previous paragraph. Hence the basic submodule of the P -component is cyclic.

It follows that A_P is a direct sum of a cyclic P -module (a basic submodule) and a divisible one $R_{P^\infty}^{(\kappa)}$, where κ is maximal with $R_{P^\infty}^{(\kappa)}$ being a direct summand of A . We show that $\kappa = 0$ and hence A_P is cyclic.

By a double application of Lemma 3.2,

$$(8) \quad \text{Hom}(\text{Endo } R_{P^\infty}^{(\kappa)}, R_{P^\infty}^{(\kappa)}) \subseteq A.$$

Now $\text{Endo } R_{P^\infty}^{(\kappa)}$ is torsion-free since $R_{P^\infty}^{(\kappa)}$ is divisible. Hence

$$\text{Hom}(\text{Endo } R_{P^\infty}^{(\kappa)}, R_{P^\infty}^{(\kappa)})$$

is divisible. Its P -divisible part is $R_{P^\infty}^{(\lambda)}$, where λ is the dimension of the P -socle:

$$(9) \quad \text{Hom}(\text{Endo } R_{P^\infty}^{(\kappa)}, R_{P^\infty}^{(\kappa)})[P] \cong \text{Hom}((\text{Endo } R_{P^\infty}^{(\kappa)}) \otimes R/P, R_P^{(\kappa)}) \sqsupseteq \text{Endo } R_P^{(\kappa)}.$$

The dimension of $(\text{Endo } R_{P^\infty}^{(\kappa)}) \otimes R/P$ over R/P is obviously at least κ (e.g. since the canonical projections to the coordinates are linearly independent), which justifies the last relation in (9). If $\kappa > 1$, then we have $\lambda > \kappa$ since the dimension of $\text{Endo } R_P^{(\kappa)}$ is larger than κ by Lemma 3.1. But $\lambda > \kappa$ contradicts the maximality of κ . Thus $\kappa \leq 1$.

We also have to exclude the case $\kappa = 1$. So suppose that $R_{P^\infty} \sqsubseteq A$. Then $J_P \cong \text{Endo } R_{P^\infty} \sqsubseteq A$. So $R_{P^\infty} \oplus J_P \sqsubseteq A$. By Lemma 3.2,

$$(10) \quad J_P \oplus J_P \cong \text{Endo } R_{P^\infty} \oplus \text{Endo } J_P \sqsubseteq A.$$

Hence $R_{P^\infty}^2 \sqsubseteq \text{Hom}(J_P^2, R_{P^\infty}) \sqsubseteq A$, which is impossible.

So every P -component is reduced with cyclic basic submodule; hence all of them are cyclic. □

Now we are ready to prove the second part of the main structure theorem.

Proof of Theorem 1.1(ii). If $P \in S$, then the P -component R/P^{n_P} of A is cyclic and pure in A . Hence it is a direct summand:

$$(11) \quad A = R/P^{n_P} \oplus B_P.$$

Thus we have

$$(12) \quad A \cong \text{Endo } A = \text{Endo } R/P^{n_P} \oplus \text{Hom}(R/P^{n_P}, B_P) \oplus \text{Hom}(B_P, R/P^{n_P}) \oplus \text{Endo } B_P.$$

The first three summands are all P -modules. But $\text{Endo } R/P^{n_P} \cong R/P^{n_P}$, so this is the whole P -component of A .

In particular, we must have $\text{Hom}(B_P, R/P^{n_P}) = 0$ so B_P is P -divisible.

It follows that B_P is the largest P -divisible submodule of A and hence is an ideal of A . Also, R/P^{n_P} is an ideal of A , being the P -torsion part of A . So $A = R/P^{n_P} \times B_P$ as R -algebras and this gives rise to an algebra epimorphism $\varphi_P: A \rightarrow R/P^{n_P}$.

Combining the algebra homomorphisms φ_P together for all $P \in S$, we obtain a homomorphism

$$(13) \quad \varphi: A \rightarrow \prod_{P \in S} R/P^{n_P}.$$

Obviously, φ restricted to the torsion part T of A is just the canonical embedding of $T = \bigoplus_{P \in S} R/P^{n_P}$ into $\prod_{P \in S} R/P^{n_P}$.

Let $D := \bigcap_{P \in S} B_P$. This is the kernel of φ . Since $\prod_{P \in S} R/P^{n_P}$ is S -reduced, every S -divisible submodule of A must be contained in D . We prove that D is S -divisible and so it is the largest S -divisible submodule of A .

If $P \in S$, then $B_P/D = \prod_{Q \in S \setminus \{P\}} R/Q^{n_Q}$ is P -divisible. In other words, $B_P = D + PB_P$ and hence the localization of P at D is B_P , and so P -divisible. Hence D itself is P -divisible. All in all, D is P -divisible for all $P \in S$ and so D is S -divisible.

So far we have proved (2) except that A/D is S -pure in $\prod_{P \in S} R/P^{n_P} = \Pi$. Since T is S -pure in Π and Π/T is S -divisible, this is equivalent to saying that $(A/D)/T$ is S -divisible. But already A/T is S -divisible as can be easily seen: for $P \in S$, the factor A/T is P -divisible, being a factor of the P -divisible B_P by (11). □

3.2. Non-reduced $E(R)$ -algebras. We prove Theorem 1.3. The methods are similar to the proof of Theorem 1.1(i).

Proof of Theorem 1.3. Let A be a non-reduced $E(R)$ -module. Its torsion part must be reduced by Theorem 1.1(i). It follows that the divisible part must be $Q^{(\kappa)}$ for some $\kappa \geq 1$. By Lemma 3.1, if $\kappa > 1$, then $Q^{(\lambda)}$ is a direct summand of A for some $\lambda > \kappa$. This is impossible, and therefore the divisible part of A is Q . So A splits as $A = Q \oplus C$ for some reduced module C . We have

$$(14) \quad A \cong \text{Endo } A = \text{Endo } Q \oplus \text{Hom}(Q, C) \oplus \text{Hom}(C, Q) \oplus \text{Endo } C.$$

The first three summands are divisible and $\text{Endo } Q \cong Q$. Since the divisible part of A is only Q , the second and third summands must be zero. In particular, $\text{Hom}(C, Q) = 0$ means C is torsion, so C is the torsion part of A . Hence the torsion part C of A is a direct summand, so C is cyclic as shown in Subsection 2.2. \square

4. CONSTRUCTION OF MIXED E -RINGS

The next lemma is our main tool for constructing mixed (generalized) $E(R)$ -algebras out of torsion-free ones:

Lemma 4.1. *Let S be an infinite set of prime ideals and B an algebra such that*

$$(15) \quad T := \bigoplus_{P \in S} R/P^{n_P} \subseteq_S B \subseteq_S \prod_{P \in S} R/P^{n_P} =: \Pi.$$

Moreover, let C be an S -divisible, torsion-free generalized $E(R)$ -algebra. Fix an R -algebra isomorphism $\phi: C \cong \text{End } C$. Let D be an ideal of C such that $C/D \cong B/T$. Suppose that D is invariant under endomorphisms and that the induced map $C \cong \text{End } C \rightarrow \text{End } C/D$ maps every $c \in C$ to the multiplication by the residue class of c . (Note that $C/D \cong B/T$ is commutative.)

Then the pull-back algebra A defined by the next diagram is a generalized $E(R)$ -algebra.

$$(16) \quad \begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & B/T \cong C/D \end{array}$$

A is commutative exactly if C is commutative.

Proof. Note that the pull-back diagram (16) extends to a diagram of exact sequences of R -modules:

$$(17) \quad \begin{array}{ccccccc} & & 0 & \longrightarrow & D & \xlongequal{\quad} & D \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T & \longrightarrow & A & \longrightarrow & C \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T & \longrightarrow & B & \longrightarrow & B/T \cong C/D \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

Therefore we can regard T and D as R -submodules of A ; they are even ideals. Since $B \cong A/D$ is S -reduced and D is S -divisible (as an ideal of the S -divisible algebra C), D is the largest S -divisible submodule of A . Similarly, since T is torsion and $C \cong A/T$ is torsion-free, T is the torsion part of A .

We can look at diagram (16) as a pull-back diagram of algebras of endomorphisms: B resp. B/T act on themselves via multiplication and C acts on itself via ϕ . The conditions of the lemma guarantee the compatibility of these actions with the maps of (16). Hence, by pull-back, A is isomorphic to the subalgebra of $\text{End } A$ consisting of those endomorphisms which induce an endomorphism on both B and C i.e. under which D and T are invariant. But this subalgebra is clearly the whole $\text{End } A$ since every endomorphism of A must leave the torsion part T and the largest S -divisible submodule D invariant. Thus A is a generalized $E(R)$ -algebra. It is obvious from the pull-back diagram (16) that A is commutative exactly if C is commutative. \square

Proof of Theorem 1.2. We are going to apply Lemma 4.1 to obtain the required algebras.

By the *localization* of an R -module M at a set W of some prime ideals we mean the R_W -module $M_W := M \otimes_R R_W$. Note that M_W is an R_W -algebra if M is an R -algebra.

Let P be any prime ideal not in S . Choose any algebra B_0 as in (15), which is P -reduced and $E := B_0/T$ is P -cotorsion-free. (Take e.g. the B_0 determined by $B_0/T = R_S$.)

Since E is P -cotorsion-free, there is a set X , larger than any given cardinal, and an $E(R)$ -algebra C such that

$$(18) \quad E[X] \subseteq C \subseteq E[X]_P = E_P[X],$$

where $E[X]$ denotes the polynomial R -algebra over E in the set of variables X . (Such a C can be constructed in the following way. Let us start with a polynomial algebra $C_0 := E[X_0]$. In the P -adic completion of C_0 , we choose an appropriate set X_1 of independent elements over C_0 . Let $X := X_0 \cup X_1$ and C be the purification of $C_0[X_1] = E[X]$ in the P -adic completion of C_0 . See [1, 3, 4] for more details.)

We view the elements of C as polynomials in $E_P[X]$. Let D be the ideal of C consisting of polynomials with zero constant term, i.e. the kernel of the algebra homomorphism which substitutes 0 for every variable. Thus $E \subseteq C/D \subseteq E_P$. Since Π/T is a divisible torsion-free algebra, we can identify E_P , and hence C/D , as a subalgebra B/T of Π/T for some subalgebra B of Π .

Now let us apply Lemma 4.1 for the data B, T, C and D . The result is an $E(R)$ -algebra A such that $A/D \cong B$ and $A/T \cong C$. We prove that D is not a direct summand of A . If it were a direct summand, then the projection onto D would be a multiplication by some $p \in A$ with $p^2 = p$ (since A is an $E(R)$ -algebra). Since the image of multiplication by p is D , we have $p = p \cdot 1 \in D$. So, by $D \subseteq E[X]_P$, the element p would be a non-zero idempotent polynomial with zero constant term. But no such polynomial exists. \square

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