PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 8, Pages 2395–2400
S 0002-9939(05)07943-8
Article electronically published on March 15, 2005

EMBEDDING ℓ_1 AS LIPSCHITZ FUNCTIONS

M. RAJA

(Communicated by N. Tomczak-Jaegermann)

Abstract. Let K be a compact Hausdorff space and let d be a lower semi-continuous metric on it. We prove that K is fragmented by d if, and only if, $C(K)$ contains no copy of ℓ_1 made up of Lipschitz functions with respect to d. As applications we obtain a characterization of Asplund Banach spaces and Radon-Nikodým compacta.

1. Introduction

R. C. James [5] and, independently, J. Lindenstrauss and C. Stegall [7] solved negatively a classical problem of S. Banach: *For X a separable Banach space such that its dual X^* is not separable, does X contain a subspace isomorphic to ℓ_1?* These counterexamples seems to be the split between two branches of Banach space theory that have had a great deal of development in the last decades: Asplund spaces and Banach spaces not containing copy of ℓ_1.

Our aim is to present a link between the Asplund property and the existence of subspaces isomorphic to ℓ_1. We shall show that if X is separable and X^* is not, then it is possible to find a copy of ℓ_1 in certain linear space where X imbeds in a very natural way. Consider the unit ball of the dual B_{X^*} endowed with the weak* topology. Then X embeds isometrically into $C(B_{X^*})$ as a subset of norm Lipschitz functions. Of course, $C(B_{X^*})$ always contains copies of ℓ_1, but there is one made up of norm Lipschitz functions if, and only if, X^* is not separable. More generally we have the following:

Theorem 1.1. *For a Banach space X the following are equivalent:*

i) *X is not Asplund.*
ii) *$C(B_{X^*})$ contains a copy of ℓ_1 made up of norm Lipschitz functions.*

The linear subspace $\mathcal{L}$ of the norm Lipschitz functions of $C(B_{X^*})$ becomes a Banach space when endowed with the norm $|||.||| = \|.\|_\infty + \|.\|_{Lip}$, where $\|.\|_\infty$ is the supremum norm and $\|f\|_{Lip} = \sup\{\frac{|f(x)-f(y)|}{\|x-y\|} : x, y \in B_{X^*}, x \neq y\}$; however we do not use this norm in Theorem 1.1. Indeed, it is easy to see that ℓ_1 embeds isomorphically into $(\mathcal{L}, |||.|||)$ for every nontrivial X. The embedding in Theorem 1.1ii) is with respect to the norm $\|.\|_\infty$ in $C(B_{X^*})$.

Received by the editors March 23, 2004.
2000 *Mathematics Subject Classification.* Primary 46B20, 46B22, 54E99.
This research was supported by a grant of Professor J. Lindenstrauss from the Israel Science Foundation, and by research grant BFM2002-01719, MCyT (Spain).

Theorem 1.1 will be a consequence of a more general result, Theorem 2.1, which is formulated in terms of a $C(K)$ space and fragmentability. In the last section of the paper we shall study the subsets of $C(K)$ which are made up of all the Lipschitz functions with respect to some metric on K. We shall give a characterization for these sets without explicit mention of the metric, obtaining as an application, a new characterization of Radon-Nikodým compacta.

2. Fragmentability and copies of ℓ_1

We shall deal with continuous functions on a compact space in order to describe a more general result. Recall that a topological space K is fragmentable by a metric defined on it if every nonempty subset of K has a nonempty relatively open subset of arbitrarily small diameter with respect to d. Our main result is the following:

Theorem 2.1. *Let K be a compact Hausdorff space and let d be a lower semicontinuous metric defined on it. Then the following are equivalent:*

i) *K is not fragmentable by d.*
ii) *$C(K)$ contains an isomorphic copy of ℓ_1 made up of functions which are Lipschitz with respect to d.*
iii) *There is a bounded sequence $(f_n) \subset C(K)$ equivalent to the canonical basis of ℓ_1 which is equicontinuous with respect to d.*

Proof. The Lipschitz property is understood here to refer to the metric d, thus η-Lipschitz for $\eta \in \mathbb{R}$ will mean that $|f(x) - f(y)| \leq \eta\, d(x, y)$ for all $x, y \in K$.

i) $\Rightarrow$ ii) If K is not d-fragmentable, a Cantor-type construction [6] yields an $\varepsilon > 0$, a closed subset $H \subset K$ and a surjective map $h : H \to \{-1, 1\}^{\mathbb{N}}$ such that $d(h^{-1}(r), h^{-1}(s)) \geq \varepsilon$ for every $r, s \in \{-1, 1\}^{\mathbb{N}}$ with $r \neq s$. Let p_n be the projection on the nth coordinate of $\{-1, 1\}^{\mathbb{N}}$ and consider the function $p_n \circ h$ defined on H which is continuous and $2\varepsilon^{-1}$-Lipschitz. By a result of [8], see also [1], there is a continuous extension f_n of $p_n \circ h$ to K with the same Lipschitz bound $2\varepsilon^{-1}$. The sequence (f_n) is equivalent to the canonical basis of ℓ_1. Indeed, given real numbers (a_n) for $i = 1, \ldots, m$ there is $x \in H$ such that $f_n(x) = sign(a_n)$ and thus

$$\|\sum_{n=1}^{m} a_n f_n\| = \sum_{n=1}^{m} |a_n|,$$

which means that $E = \overline{span}^{\|\cdot\|_\infty}\{f_n : n \in \mathbb{N}\}$ is even isometric to ℓ_1. An easy computation shows that if $f \in E$, then f is $2\varepsilon^{-1}\|f\|$-Lipschitz.

ii) $\Rightarrow$ iii) Let $E \subset C(K)$ be an isomorphic copy of ℓ_1 made up of Lipschitz functions with respect to d and take $E_n \subset E$ as the subset of n-Lipschitz functions. Clearly E_n is norm closed, thus by Baire's category theorem some E_n has nonempty interior. By translation we deduce that the functions of the unit ball of E have a common Lipschitz bound, and thus there is a bounded sequence of $C(K)$ which is equivalent to the canonical basis of ℓ_1 and equicontinuous with respect to d.

iii) $\Rightarrow$ i) Let (f_n) be a bounded ℓ_1-sequence of $C(K)$ which is equicontinuous with respect to d. By a standard reduction argument, there is a metrizable compact L, a continuous surjection $q : K \to L$ and functions g_n defined on L such that $f_n = g_n \circ q$ for every $n \in \mathbb{N}$. The sequence (g_n) is equivalent in $C(L)$ to the basis of ℓ_1, and thus (g_n) contains not pointwise convergent subsequence. By a result of Rosenthal [11] (see also [12, p. 20]), there is a subsequence of (g_n), denoted again (g_n) without loss of generality, and there are real numbers $a < b$, such that

$\bigcap_{n=1}^m A_n \neq \emptyset$ for every $m \in \mathbb{N}$ and every choice of $A_n \in \{g_n^{-1}(-\infty, a], g_n^{-1}[b, +\infty)\}$. Using compactness and composition with q^{-1} we get that $\bigcap_{n=1}^\infty A_n \neq \emptyset$ for every every choice of $A_n \in \{f_n^{-1}(-\infty, a], f_n^{-1}[b, +\infty)\}$. In fact, we have used the reduction to the separable case just to apply Rosenthal's theorem as appearing in quoted references. Consider the closed subset of K given by

$$H = \bigcap_{n=1}^{\infty} \left(f_n^{-1}(-\infty, a] \cup f_n^{-1}[b, +\infty)\right)$$

and define a map $h : H \to \{-1, 1\}^{\mathbb{N}}$ such that the nth coordinate of $h(x)$ is -1 if $f_n(x) \leq a$, and 1 if $f_n(x) \geq b$. The construction ensures that h is continuous and surjective. Take a closed subset $C \subset H$ minimal with respect to the property that $h(C) = \{-1, 1\}^{\mathbb{N}}$. It is easy to see that any relatively open subset of C contains two points x, y such that $f_n(x) \leq a$ and $f_n(y) \geq b$ for some n. We claim that C is not d-fragmentable. Suppose that C is d-fragmentable; then there is point $z \in C$ of continuity from the inherited topology of C to the metric d [10]. Take $r > 0$ such that the oscilation of f_n in $B(z, r)$ is less than $b - a$ for all $n \in \mathbb{N}$. Take U a neighbourhood of z such that $diam(C \cap U) < r$. There are points $x, y \in C \cap U$, so $x, y \in B(z, r)$, and $n \in \mathbb{N}$ such that $f_n(y) - f_n(x) \geq b - a$, which is a contradiction. That proves the claim and thus K is not d-fragmentable. □

Recall that a Banach space X is said to be Asplund if every separable subspace has separable dual.

Proof of Theorem 1.1. A Banach space X is Asplund if, and only if, the dual ball B_{X^*} is norm fragmentable; see [3], then apply Theorem 2.1. □

Remark 2.2. The behaviour of Banach spaces is different with respect to Lipschitz copies of c_0. Indeed, if H is an infinite-dimensional Hilbert space it is not difficult to find an isometric copy of c_0 inside $C(B_H)$ made up of norm Lipschitz functions.

Statement iii) of Theorem 2.1 suggests we study the sequential properties of bounded subsets of $C(K)$ which are equicontinuous with respect to a fragmenting metric. However, it is shown in [2] that certain subsets of functions on a fragmentable compact space verify stronger sequential properties than those given by Rosenthal's ℓ_1-theorem (see in particular [2, Corollary 3.5]). We shall include the following result which is relevant to Theorem 2.1.

Proposition 2.3. *Let $(f_n) \subset C(K)$ be a sequence of functions which are equicontinuous with respect to a metric d fragmenting K. Then the pointwise closure of (f_n) in $\overline{\mathbb{R}}^K$ is a metrizable compact subset.*

Proof. Note that we do not require the metric d be lower semicontinuous. Without loss of generality, the sequence (f_n) may be supposed bounded. Indeed, compose with a homeomorphism from $\overline{\mathbb{R}}$ to $[-1, 1]$, and both continuity and equicontinuty with respect to d are preserved. Consider the pseudometric

$$\rho(x, y) = \sup\{|f_n(x) - f_n(y)| : n \in \mathbb{N}\}$$

which is continuous with respect to d by the equicontinuity of (f_n). As a consequece, K is fragmented by ρ. Passing to a quotient, we may assume that K is a metrizable compactum, and thus we get that K is ρ-separable by using the fragmentability property. Let $H \subset [-1,1]^K$ be the pointwise closure of (f_n). Clearly, H is a pointwise compact set made up of ρ-continuous functions. Since a ρ-dense countable subset of K separates points of H, we deduce that H is metrizable. □

We shall finish the section by applying the results to scattered compact spaces. Note that a compact space K is scattered if, and only if, it is fragmentable by the discrete metric. Since the discrete metric makes Lipschitz any bounded function and it is lower semicontinuous with respect to any topology on K, we get the following well-known result.

Corollary 2.4. *A compact Hausdorff space K is scattered if, and only if, $C(K)$ does not contain a copy of ℓ_1.*

Proposition 2.3 implies the following result of Meyer [9].

Corollary 2.5. *Let K be a scattered compact Hausdorff space. Then any Baire function on K is of the first Baire class.*

3. The lattice of Lipschitz functions

It is easy to see that the set of η-Lipschitz functions is lattice closed, that is, the functions $\max\{f,g\}$ and $\min\{f,g\}$ are η-Lipschitz whenever the functions f and g are η-Lipschitz. Note that the subset of $C(K)$ made up of functions which are Lipschitz with respect to some metric d on K can be given a Banach lattice structure with the norm $\|\!|.|\!\|$ defined in the Introduction.

Lemma 3.1. *Let K be a compact Hausdorff space and let $B \subset C(K)$ be a bounded closed symmetric convex which separates points of K, is lattice closed and contains a nontrivial constant function. Define a lower semicontinuous metric d on K by the formula*

$$d(x,y) = \sup\{|f(x) - f(y)| : f \in B\}.$$

Then $L = \bigcup_{n=1}^{\infty} nB$ is the set of continuous functions on K which are Lipschitz with respect to d.

Proof. We may assume without loss of generality that $B \subset B_{C(K)}$. Let $\delta > 0$ such that $t\mathbf{1} \in B$ for all $t \in [-\delta, \delta]$. It is clear that $L = \bigcup_{n=1}^{\infty} nB$ is the linear space spaned by B and L is a lattice. Given $f \in C(K)$, which is Lipschitz with respect to d, we shall prove that $f \in L$. We may assume without loss of generality that $f \in B_{C(K)}$ and f is 1-Lipschitz, thus $|f(x) - f(y)| \leq d(x,y)$ for every $x, y \in K$. Fix a pair of points $x, y \in K$ with $x \neq y$. There is $g \in B$ such that $d(x,y) \leq 2(g(x) - g(y))$. We may take $\lambda \in [-2,2]$ such that $f(x) - f(y) = \lambda(g(x) - g(y))$. Take $\eta = f(x) - \lambda g(x) = f(y) - \lambda g(y)$. We have $|\eta| \leq 3$, and so the function $p = \lambda g + \eta\mathbf{1}$ belongs to $D = (2 + 3\delta^{-1})B$, and verifies that $p(x) = f(x)$ and $p(y) = f(y)$. Applying the method of [4, p. 146] to prove the Stone-Weierstrass theorem, for every $\varepsilon > 0$ there is $h \in D$ such that $\|f - h\| < \varepsilon$ because D is lattice closed, and thus $f \in D$ because it is closed. □

Theorem 3.2. *Let $L \subset C(K)$ be a dense linear lattice which contains the constant functions. Then L is the subset of functions which are Lipschitz with respect to some (lower semicontinuous) metric d on K if, and only if, L is an $\mathcal{F}_\sigma$ and L is complete for a finer lattice norm. In that case, any two complete finer lattice norms on L are equivalent.*

Proof. If L is the lattice of the functions which are Lipschitz with respect to some metric d on K, then it is complete with the lattice norm $\|\!|.|\!\|$ defined in the Introduction, and $L = \bigcup_{n=1}^{\infty} F_n$ where $F_n = \{f \in L : f \text{ is } n\text{-Lipschitz}\}$.

For the converse, let $\|\!|.|\!\|$ be a complete lattice norm on L finer than $\|.\|_\infty$. Put $L \cap B_{C(K)} = \bigcup_{n=1}^{\infty} F_n$ where each F_n is $\|.\|_\infty$-closed. Since $\|\!|.|\!\|$ is finer that the restriction of $\|.\|_\infty$ to L, we get that $L \cap B_{C(K)}$ is $\|\!|.|\!\|$-closed and every F_n is $\|\!|.|\!\|$-closed. Baire's Theorem gives that some F_n has nonempty $\|\!|.|\!\|$-interior, say $C = \{f \in L : \|\!|f - g|\!\| < r\} \subset L$ for some $g \in L$ and $r > 0$. As F_n is $\|.\|_\infty$-closed, we have $\overline{C}^{\|.\|_\infty} \subset F_n \subset L$. By translation we deduce that

$$B = \overline{\{f \in L : \|\!|f|\!\| \leq 1\}}^{\|.\|_\infty}$$

is a subset of L which also is $\|.\|_\infty$-bounded, $\|.\|_\infty$-closed, lattice closed, symmetric, convex and contains a nontrivial constant function. As $L = \bigcup_{n=1}^{\infty} nB$, then B also separates points of K. By the previous lemma, L is a lattice of Lipschitz functions. Observe that the norm given by the lemma and $\|\!|.|\!\|$ are comparable, thus the Open Mapping theorem concludes that they are equivalent. □

Recall that a compact Hausdorff space is said to be Radon-Nikodým if there is a lower semicontinuous metric d which fragments K; see [10]. We have the following characterization.

Corollary 3.3. *A compact Hausdorff space K is Radon-Nikodým if, and only if, there exists a dense $\mathcal{F}_\sigma$ lattice $L \subset C(K)$ containing the constant functions which is complete for a finer lattice norm and such that every bounded sequence in L has a pointwise Cauchy subsequence.*

Proof. A sequence in L is bounded if, and only if, it is $\|.\|_\infty$-bounded and equi-Lipschitz with respect to some lower semicontinuous metric d given by the former theorem. The result follows from Theorem 2.1 and Rosenthal's ℓ_1-theorem. □

Acknowledgements

I would like to express my gratitude to Professor Joram Lindenstrauss for his hospitality during my stay at the Einstein Institute of Mathematics in the academic year 2002/03.

References

[1] Y. Benyamini, J. Lindenstrauss, *Geometric Nonlinear Functional Analysis. Vol. 1*, American Mathematical Society Colloquium Publications 48, 2000. MR1727673 (2001b:46001)

[2] B. Cascales, I. Namioka, J. Orihuela, The Lindelöf property in Banach spaces, *Studia Math.* **154** (2003), 165-192. MR1949928 (2003m:54028)

[3] R. Deville, G. Godefroy, V. Zizler, *Smoothness and Renorming in Banach Spaces*, Pitman Monog. and Surveys 64, 1993. MR1211634 (94d:46012)

[4] G. Choquet, *Topology*, Academic Press, New York-London, 1966. MR0193605 (33:1823)

[5] R.C. James A separable somewhat reflexive Banach space with nonseparable dual, *Bull. Amer. Math. Soc.* **80** (1974), 738-743. MR0417763 (54:5811)

[6] J.E. Jayne, I. Namioka, C.A. Rogers, Norm fragmented weak* compact sets, *Collect. Math.* **41** (1990), 161-188. MR1149650 (93d:46035)

[7] J. Lindenstrauss, C. Stegall, Examples of separable spaces which do not contain ℓ_1 and whose duals are non-separable, *Studia Math.* **54** (1975), 81-105. MR0390720 (52:11543)

[8] E. Matouskova, Extensions of continuous and Lipschitz functions, *Canad. Math. Bull.* **43** (2000), 208-217. MR1754025 (2001c:54009)

[9] P.R. Meyer, The Baire order problem for compact spaces, *Duke Math. J.* **33** (1966), 33-39. MR0190897 (32:8307)

[10] I. Namioka, Radon-Nikodým compact spaces and fragmentability, *Mathematika* **34** (1989), 258-281. MR0933504 (89i:46021)

[11] H. Rosenthal, A characterization of Banach spaces containing ℓ_1, *Proc. Nat. Acad. Sci. U.S.A.* **71** (1974), 2411-2413. MR0358307 (50:10773)

[12] S. Todorcevic, *Topics in Topology*, Lecture Notes in Mathematics 1652, Springer-Verlag, Berlin, 1997. MR1442262 (98g:54002)

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Givat Ram, 91904, Jerusalem, Israel

Current address: Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Espinardo, Murcia, Spain

E-mail address: matias@um.es