ALMOST EVERYWHERE CONVERGENCE OF SERIES IN $L^1$

CIPRIAN DEMETER

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Abstract. We answer positively a question of J. Rosenblatt (1988), proving the existence of a sequence $(c_i)$ with $\sum_{i=1}^{\infty} |c_i| = \infty$, such that for every dynamical system $(X, \Sigma, m, T)$ and $f \in L^1(X)$, $\sum_{i=1}^{\infty} c_i f(T^i x)$ converges almost everywhere. A similar result is obtained in the real variable context.

1. Introduction

Let $T$ be a (not necessarily invertible) measure preserving transformation on the probability space $(X, \Sigma, m)$. Given a sequence $(c_i)$ we will state some mild conditions under which the series $\sum_{i=1}^{\infty} c_i f(T^i x)$ converges almost everywhere for every $f \in L^1(X)$. In [8] Rosenblatt proved that if $r_i(\omega)$ denotes the Rademacher sequence, then for almost every choice of $\omega$ one gets convergence of the above series with $c_i = r_i(\omega)$, for every $f \in L^p(X)$, $p > 1$. As a natural question, in the end of [8] it is asked whether there exists a sequence $(c_i)$ with $\sum_{i=1}^{\infty} |c_i| = \infty$, such that for every $f \in L^1(X)$, $\sum_{i=1}^{\infty} c_i f(T^i x)$ converges almost everywhere. This question is also motivated by the fact that if one considers the same series associated with an invertible $T$ and a two sided sequence $(c_i)_{i=-\infty}^{\infty}$, then the ergodic Hilbert Transform is an example for which the convergence is known to hold.

The purpose of this paper is twofold. On the one hand it gives a positive answer to the question above, as a consequence of Theorem 1 from [7]. On the other hand, the proof of this theorem (as presented in [7]) is quite long and makes use of the result concerning the convergence of the martingale transform from [4], which does not allow it to be extended to a larger setting. We will give a rather short proof here, based on a different type of argument, which will allow us in turn to prove a slightly more general result.

Given a sequence $C = (c_i)$ we will use the following notation:

$$A_{k,C} f(x) = \sum_{i=2^{k+1}}^{2^{k+1}+1} c_i f(T^i x).$$

When the sequence $C$ is clear from the context, $A_k f(x)$ will be used instead.

**Theorem 1.1.** Let $(c_i)$ be a sequence of positive numbers with the following properties:

(a) The sequence $(ic_i)$ is bounded.
(b) The sequence \((c_i)\) is nonincreasing.
(c) The sequence \(s_k = \sum_{i=2^k+1}^{2^{k+1}} c_i\) satisfies \(\sum_{k=0}^{\infty} |s_{k+1} - s_k| < \infty\).

Then for every bounded sequence \((v_k)\), the operators

\[
S_n f(x) = \sum_{k=1}^{n} v_k (A_k f(x) - A_{k-1} f(x))
\]

converge a.e. for \(f \in L^1(X)\), and converge in norm for \(f \in L^p(X), 1 < p < \infty\).

Remark 1.2. This theorem remains valid if \(2^k\) is replaced with an arbitrary lacunary sequence in the definition of \(A_k\), and the proof does not suffer any serious modification. When \(c_i = \frac{1}{2 \log 2 \log i}\), one recovers the result of Theorem 1 from [7].

From the above, one immediately gets the following:

Theorem 1.3. Let \((c_i)\) be a sequence of positive numbers with the following properties:
(a) The sequence \((ic_i)\) converges to 0.
(b) The sequence \((c_i)\) is nonincreasing.
(c) The sequence \(s_k = \sum_{i=2^k+1}^{2^{k+1}} c_i\) satisfies \(\sum_{k=1}^{\infty} |s_{k+1} - s_k| < \infty\).

Define the sequence \(d_i = c_i(-1)^k\), when \(2^k + 1 \leq i \leq 2^{k+1}\). Then the series

\[
S_n f(x) = \sum_{i=1}^{n} d_i f(T^i x)
\]

converges a.e. for every \(f \in L^1(X)\).

Sequences such as \(\frac{(-1)^{i \log (i+1)}}{i \log i}\), \(\frac{(-1)^{i \log (i+1)}}{i \log \log i}\), etc. in which the logarithmic form is expanded satisfy the requirements (a), (b) and (c) of Theorem 1.3. This proves the following corollary:

Corollary 1.4. There exists a nonsummable sequence \((c_i)\) such that for every \(f \in L^1(X)\), \(\sum_{i=1}^{\infty} c_i f(T^i x)\) converges almost everywhere.

Remark 1.5. An interesting question is whether there exists a choice of signs \(r_i \in \{-1, 1\}\) such that the following modulated one-sided Hilbert Transform

\[
S f(x) = \sum_{i=1}^{\infty} r_i f(T^i x)
\]

converges a.e. for \(f \in L^1(X)\). It appears that this question cannot be addressed by the techniques employed in this paper, and here is the reason why: the proof (based on the machinery of Benedek, Calderón and Panzone) of the weak \((1,1)\) maximal inequality for \(\sup_n |S_n|\) in Theorem 1.1 relies heavily on the fact that the summation index for \(A_k\) runs through a block of lacunary growth; on the other hand, a series such as

\[
S f(x) = \sum_{k=1}^{\infty} (-1)^k \sum_{i=n_k+1}^{n_{k+1}} f(T^i x) \frac{1}{i}
\]

diverges for constant functions when \((n_k)\) is lacunary.
The real variable analogues of the above theorems also hold. For a given \( \psi \) defined on \((0, \infty)\) we will use the notation
\[
D_k, \psi f(x) = \int_{2^{k-1}}^{2^k} \psi(y)f(x-y)dy.
\]
Again when \( \psi \) is clear from the context, \( D_k f(x) \) will be used instead.

**Theorem 1.6.** Let \( \psi : (0, \infty) \to [0, \infty) \) be a function satisfying the following:

(a) The function \( x\psi(x) \) is bounded.

(b) The function \( \psi \) is nonincreasing.

(c) The sequence \( s_k = \int_{1/2^{k+1}}^{1/2^k} \psi(x)dx \) satisfies \( \sum_{k=1}^{\infty} |s_{k+1} - s_k| < \infty \).

Then for every bounded sequence \( \{v_k\} \), the operators
\[
S_n f(x) = \sum_{k=1}^{n} v_k (D_k f(x) - D_{k-1} f(x))
\]
converge a.e. for \( f \in L^1(\mathbb{R}) \), and converge in norm for \( f \in L^p(\mathbb{R}) \), \( 1 < p < \infty \).

This immediately gives

**Theorem 1.7.** Let \( \psi : (0, \infty) \to [0, \infty) \) be a function satisfying the following:

(a) The function \( \lim_{x \to 0} x\psi(x) = 0 \).

(b) The function \( \psi \) is nonincreasing.

(c) The sequence \( s_k = \int_{1/2^{k+1}}^{1/2^k} \psi(x)dx \) satisfies \( \sum_{k=1}^{\infty} |s_{k+1} - s_k| < \infty \).

Define the function \( \theta(x) = (-1)^k \psi(x) \), when \( x \in \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right) \). Then the improper integral
\[
I = \int_0^{\infty} \theta(y) f(x-y)dy
\]
converges for a.e. \( x \), for every \( f \in L^1(\mathbb{R}) \).

**Remark 1.8.** Note again that functions such as \( \psi(x) = \frac{1}{x \log x} \) or \( \psi(x) = \frac{1}{x \log \log x} \) satisfy the requirements of Theorem 1.7. Like in the ergodic theoretic setting, it remains open whether there exists a function \( \theta \) with \( |\theta(x)| = \frac{1}{x} \) for each \( x \in (0, \infty) \), which satisfies the conclusion of Theorem 1.7.

**Remark 1.9.** An example of a function \( \theta \notin L_1[0, \infty) \) such that
\[
I = \int_0^{\infty} \theta(y) f(x-y)dy
\]
converges for a.e. \( x \), for every \( f \in L^1(\mathbb{R}) \), appears in [1]. The kernel there, \( \theta(x) = \chi_{(0, \infty)}(x) \cdot \frac{1}{x} \frac{\sin(\log x)}{\log x} \), is a smooth variant of the one we are using here.

2. **Proofs**

We will use the fundamental results from [6] to get a maximal inequality for the operator \( S^* f(x) = \sup_s S_s f(x) \). Since these results are stated in the real variable context, we need to transfer them in the ergodic theoretic setting. For a measure \( \mu \) on \( \mathbb{Z} \) define the Borel measure \( w \) on \( \mathbb{R} \) by the formula \( w = \mu * \chi_{(0,1)} \) where
\[
\mu \ast \chi_{(0,1)}(x) = \int_{\mathbb{Z}} \chi_{(0,1)}(x-y)d\mu(y) = \sum_{k=-\infty}^{\infty} \chi_{(k,k+1)}(x)\mu(k).
\]
In the following, for any Borel measure $w$ on $\mathbb{R}$ we will denote by $|w| = w^+ - w^-$ the total variation of $w$ while $||w||_1$ will stand for the quantity $|w|(\mathbb{R})$. The same notation will be used for measures on $\mathbb{Z}$. Given a sequence $(w_k)$ of Borel measures on $\mathbb{R}$, the associated maximal operator is defined as $w^*(\psi) = \sup_k |w_k * \psi|$, for each $\psi : \mathbb{R} \to \mathbb{R}$. The Dirac mass concentrated on $\{i\}$ will be denoted by $\delta_i$. The following two lemmas are essentially contained in [2], but the proofs are slightly different in this context, so we will sketch them.

**Lemma 2.1.** Assume that $(\mu_k)$ is a sequence of measures on $\mathbb{Z}$ satisfying

\[
|\hat{\mu}_k(\gamma)| \leq C 2^k |\gamma - 1| \tag{2.1}
\]

and

\[
|\hat{\mu}_k(\gamma)| \leq C (2^k |\gamma - 1|)^{-1}, \gamma \neq 1 \tag{2.2}
\]

for some constant $C$ independent of $k$. Then for some constant $C'$ we also have

\[
|\hat{w}_k(\xi)| \leq C' 2^k |\xi| \tag{2.3}
\]

and

\[
|\hat{w}_k(\xi)| \leq C'(2^k |\xi|)^{-1}, \xi \neq 0 \tag{2.4}
\]

where $(w_k)$ are the corresponding measures on $\mathbb{R}$.

**Proof.** The proof immediately follows from the identity

\[
\hat{w}_k(\xi) = \hat{\mu}_k(2\pi i \xi) e^{2\pi i \xi} - 1, \xi \neq 0.
\]

\[\square\]

**Lemma 2.2.** Let $(\mu_k)$ be a sequence of measures on $\mathbb{Z}$ satisfying

\[
\sum_{k=1}^{\infty} ||\mu_k - \mu_k * \delta_1||_1 < \infty, \tag{2.5}
\]

and let $(w_k)$ denote the corresponding measures on $\mathbb{R}$. Define the integral operator

\[
T^*_{\mathbb{Z}} \phi(l) = \sup_k |T_{\mathbb{Z},k} \phi(l)|
\]

with

\[
T_{\mathbb{Z},k} \phi(l) = \sum_{i=1}^{k} \mu_i * \phi(l) \tag{2.6}
\]

and similarly the differential operator

\[
T^*_{\mathbb{R}} \psi(x) = \sup_k |T_{\mathbb{R},k} \psi(x)|
\]

with

\[
T_{\mathbb{R},k} \psi(x) = \sum_{i=1}^{k} w_i * \psi(x). \tag{2.7}
\]

Then

(i) if $T^*_{\mathbb{R}}$ is bounded in $L^p(\mathbb{R})$ for some $p > 1$, then $T^*_{\mathbb{Z}}$ is bounded in $l^p(\mathbb{Z})$;

(ii) if $T^*_{\mathbb{R}}$ satisfies a weak $(1,1)$ inequality, then so does $T^*_{\mathbb{Z}}$.

**Proof.** We will only prove (i), since the second assertion follows similarly. We have that

\[
\left\| \sup_k \left| \sum_{i=1}^{k} w_i * \psi \right| \right\|_p \leq C_p ||\psi||_p,
\]

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for all $\psi \in L^p(\mathbb{R})$. From here we can prove our result on $l^p(\mathbb{Z})$. Given $\phi \in l^p(\mathbb{Z})$, let

$$
\phi * \chi_{[0,1)} = \sum_{k=-\infty}^{\infty} \chi_{[k,k+1)}(k).
$$

Then $\phi * \chi_{[0,1)}$ is in $L^p(\mathbb{R})$ and in fact $\|\phi\|_{l^p(\mathbb{Z})} = \|\phi * \chi_{[0,1)}\|_{L^p(\mathbb{R})}$. By the maximal inequality above,

$$
\left\| \sup_k \left| \sum_{i=1}^{k} w_i \phi * \chi_{[0,1)} \right| \right\|_p \leq C_p \|\phi\|_p.
$$

All that remains to be proven now is that

$$
\left\| \sup_k \left| \sum_{i=1}^{k} \mu_i \phi * \chi_{[0,1)} \right| \right\|_p \leq C_p \left\{ \|\phi\|_p + \left\| \sup_k \left| \sum_{i=1}^{k} w_i \phi * \chi_{[0,1)} \right| \right\|_p \right\}.
$$

Note that

$$
\left\| \sup_k \left| \sum_{i=1}^{k} \mu_i \phi * \chi_{[0,1)} \right| \right\|_p \leq \left\| \sup_k \left| \sum_{i=1}^{k} w_i \phi * \chi_{[0,1)} \right| \right\|_p + \sum_{i=1}^{\infty} \left| \mu_i \phi * \chi_{[0,1)} - w_i \phi * \chi_{[0,1)} \right|_p.
$$

Now if $l \leq x < l + 1$ for some $l \in \mathbb{Z}$, say $x = l + \epsilon$, then

$$
w_i \phi * \chi_{[0,1)}(x) = \sum_{k} (1 - \epsilon) \mu_i(k - 1) \phi(l - k) + \sum_{k} \epsilon \mu_i(k) \phi(l - k),
$$

while

$$
\mu_i \phi * \chi_{[0,1)}(x) = \sum_{k} \mu_i(k) \phi(l - k).
$$

This immediately proves that

$$
|\mu_i \phi * \chi_{[0,1)}(x) - w_i \phi * \chi_{[0,1)}(x)| \leq |\mu_i - \mu_i \delta_1| * |\phi|_0,
$$

and hence

$$
|\mu_i \phi * \chi_{[0,1)} - w_i \phi * \chi_{[0,1)}|_p \leq ||(\mu_i - \mu_i \delta_1)| * |\phi||_p \leq ||\mu_i - \mu_i \delta_1||_1 ||\phi||_p.
$$

The result now follows from (2.5).

The main ingredient of our proofs is the following fundamental lemma:

**Lemma 2.3.** Let $du_k = f_k dx$ be a sequence of measures on $\mathbb{R}$ and let $T_{\mathbb{R}}$ be as above. Assume the following are satisfied:

$$
\int_{|x|>2|y|} \sup_k \left| \sum_{i=1}^{k} (f_i(x - y) - f_1(x)) \right| dx \leq C',
$$

(2.9)

$$
|w_k|_1 < M,
$$

(2.10)

$$
|\tilde{w}_k(\xi)| \leq C'2^k|\xi|,
$$

(2.11)

$$
|\tilde{w}_k(\xi)| \leq C'(2^k|\xi|)^{-1}, \quad \xi \neq 0,
$$

(2.12)

$$
|w^*(\psi)|_2 \leq C' ||\psi||_2,
$$

(2.13)

$$
\text{supp}(w_k) \subset \{ x \in \mathbb{R} : |x| < 2^{k+1} \}.
$$
for some constants $M$ and $C'$ independent of $k$, $y$ and $\psi$. Then $T^*_k$ is bounded in $L^p(\mathbb{R})$ for $1 < p < \infty$ and satisfies a weak $(1, 1)$ inequality.

\begin{proof}

Conditions (2.9), (2.10), (2.11), (2.12) and (2.13) are the ones used by Duoandikoetxea and Rubio de Francia in Theorem E of [6]. Using their result, we have $\|T^*_k\|_2 \leq C$. This fact together with (2.8) are the conditions needed in Theorem 2 from [3], with $B_1 = \mathbb{R}$ and $B_2 = l^\infty$. The result follows immediately. \qed

Here is the proof of Theorem 1.1.

\begin{proof}

Without loss of generality we can assume that $\|(v_k)\|_{l^\infty} \leq 1$. Define the measures $\mu_k$ on $\mathbb{Z}$ by

$$
\mu_k = v_k \left( \sum_{i=2^k+1}^{2^{k+1}} c_i \delta_i - \frac{s_k}{s_{k-1}} \sum_{i=2^{k-1}+1}^{2^k} c_i \delta_i \right),
$$

and let $w_k$ denote the corresponding measures on $\mathbb{R}$. We will first show that the operator $T^*_k$ associated to these measures is bounded in $L^p(\mathbb{R})$, $p > 1$, and satisfies a weak $(1, 1)$ maximal inequality, as a consequence of Lemma 2.3. Conditions (a) and (b) from Theorem 1.1 are easily seen to imply (2.9) and (2.13). Also, since $s_k \leq 2s_{k-1}$, (2.12) follows as a consequence of (a) and the boundedness of the Hardy-Littlewood maximal operator. In order to prove (2.10) and (2.11) it suffices (according to Lemma 2.1) to prove that $|\hat{\mu}_k(\gamma)| \leq C2^k|\gamma - 1|$ and $|\hat{\mu}_k(\gamma)| \leq C(2^k|\gamma - 1|)^{-1}, \gamma \neq 1$. But

$$
|\hat{\mu}_k(\gamma)| \leq \sum_{i=2^k+1}^{2^{k+1}} |c_i(\gamma^i - 1)| + \frac{s_k}{s_{k-1}} \sum_{i=2^{k-1}+1}^{2^k} |c_i(\gamma^i - 1)|
$$

$$
\leq \sum_{i=2^k+1}^{2^{k+1}} ic_i|\gamma - 1| + \frac{s_k}{s_{k-1}} \sum_{i=2^{k-1}+1}^{2^k} ic_i|\gamma - 1|
$$

$$
\leq C2^k|\gamma - 1|,
$$

while by using Abel’s summation, (a) and (b) we get

$$
|\hat{\mu}_k(\gamma)(\gamma - 1)| \leq \sum_{i=2^k+2}^{2^{k+1}} (c_i - c_{i-1}) \gamma^i + |c_{2^k+1} \gamma^{2^{k+1}+1} - c_{2^{k+1}} \gamma^{2^k+1}|
$$

$$
+ \sum_{i=2^{k-1}+2}^{2^k} (c_i - c_{i-1}) \gamma^i + |c_{2^k+1} \gamma^{2^k+1} - c_{2^{k-1}+1} \gamma^{2^{k-1}+1}|
$$

$$
\leq C2^{-k}.
$$

It only remains to prove (2.8). Obviously

$$
f_k(x) = v_k \left( \sum_{i=2^k+1}^{2^{k+1}} c_i \chi_{[i, i+1)}(x) - \frac{s_k}{s_{k-1}} \sum_{i=2^{k-1}+1}^{2^k} c_i \chi_{[i, i+1)}(x) \right).
$$

Fix a $y$. Note that since $f_k(x) = 0$ when $x < 2$, the integral in (2.8) is only over the set $\{x > 1\}$, so we can assume $x$ is positive and hence $0 < x - y < x < 2(x - y)$. Moreover, for each such $x$ there are at most 2 values of $k$ such that $f_k(x) \neq f_k(x-y)$. Define the sets $D = \{x \in [1, \infty) : x > 4|y|\}$, $A_1 = \{x \in D : \exists k \geq 0 \text{ s.t. } 2^k + 1 \leq$
Since \( x, x - y < 2^{k+1} + 1 \) and \( A_2 = D \setminus A_1 \). Since \( 4|y| < x \), it follows that any \( k \) that is used in the definition of \( A_1 \) must be greater than \( \log_2 |y| \). Note that if \( x \in A_1 \), then

\[
\sup_k \left\{ \sum_{i=1}^{k} (f_i(x - y) - f_i(x)) \right\} \leq 4|c[x - y] - c[x]| \leq 4|c[x] - |y| - 1 - c[x]| + 4|c[x] - |y| - c[x]|.
\]

Hence

\[
\int_{A_1} \sup_k \left\{ \sum_{i=1}^{k} (f_i(x - y) - f_i(x)) \right\} dx < 4 \sum_{i \geq |y| + 1} |c_i - |y| - 1 - c_i| + 4 \sum_{i \geq |y| + 1} |c_i - |y| - c_i| < C'_1 \quad \text{by (a) and (b)}.
\]

Consider now an \( x \in A_2 \). There will exist an \( l \in \mathbb{N} \) such that \( x - y < 2^l + 1 \leq x \), and from the same reasons described above, \( l \geq \log_2 |y| \). Note also that

\[
A_2 \subset \bigcup_{k \geq 0} [2^k + 1, 2^k + 1 + |y|)
\]

and

\[
\sup_k \left\{ \sum_{i=1}^{k} f_i(x) \right\} < 4c[x] < 4c_2.
\]

This implies that

\[
\int_{A_2} \sup_k \left\{ \sum_{i=1}^{k} f_i(x) \right\} dx < 4|y| \sum_{l > \log_2 |y|} c_2l \quad < C'_2 \quad \text{by (a)}.
\]

Similarly one finds that

\[
\int_{A_2} \sup_k \left\{ \sum_{i=1}^{k} f_i(x - y) \right\} dx < C'_3.
\]

This proves that (2.8) is satisfied with \( C' = C'_1 + C'_2 + C'_3 \).

Equation (2.24) is easily seen to be satisfied for the measures \( \mu_k \), based on (a) and (b). Hence according to Lemma 2.2, the operator \( T_Z^p \) is also bounded on \( \ell^1(\mathbb{Z}) \) and satisfies a weak \((1, 1)\) type inequality. Define now the operators \( S_{\mathbb{Z}, n}^p \) on \( \mathbb{Z} \) by

\[
S_{\mathbb{Z}, n}^p \phi(l) = \sum_{k=1}^{n} v_k \left( \sum_{i=2^k+1}^{2^{k+1}} c_i \phi(i + l) - \sum_{i=2^{k-1}+1}^{2^k} c_i \phi(i + l) \right).
\]

Note that

\[
S_{\mathbb{Z}}^p \phi(l) \leq T_{\mathbb{Z}}^p \phi(l) + \sum_{k=1}^{\infty} \left( \frac{s_k}{s_{k-1}} - 1 \right) \sum_{i=2^{k-1}+1}^{2^k} c_i \phi(l + i)
\]

\[
= T_{\mathbb{Z}}^p \phi(l) + M \phi(l).
\]

Since

\[
||M \phi||_1 \leq ||\phi||_1 \sum_{k=0}^{\infty} |s_{k+1} - s_k|,
\]
it follows immediately that $S^*_Z$ is bounded in $l^p(Z)$, $p > 1$, and satisfies a weak $(1,1)$ maximal inequality. By using Calderón’s standard transfer principle, see for example [5], we get the same results for the ergodic operator $S^*_Z$ of Theorem 1.1.

Condition (c) proves that $S_nf(x)$ converges a.e. for $T$ invariant functions, while (a) and (b) prove the convergence for every coboundary $f(x) = g(Tx) - g(x)$. Since these functions span a dense subclass of $L^1(X)$, convergence on the whole $L^1(X)$ follows. The norm convergence follows as a consequence of the Dominated Convergence Theorem. □

Proof of Theorem 1.3. Note that (a) implies that

$$
\sup_{2^k+1 \leq j \leq 2^{k+1}} \left| \sum_{i=2^k+1}^j c_i f(T^i x) \right| = o(1) \frac{1}{2^{k+1}} \sum_{i=1}^{2^{k+1}} |f|(T^i x);
$$

hence

$$
\lim_{k \to \infty} \sup_{2^k+1 \leq j \leq 2^{k+1}} \left| \sum_{i=2^k+1}^j c_i f(T^i x) \right| = 0
$$

for a.e. $x$ and all $f \in L^1(X)$. Using this, the conclusion of Theorem 1.3 follows now as an application of Theorem 1.1 with $v_k = (-1)^k$. □

The proofs of Theorems 1.6 and 1.7 are very similar. The argument is simpler in this case since the transfer lemmas 2.1 and 2.2 are no longer needed.

References


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