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SEMIGLOBAL RESULTS FOR $\overline{\partial}$ ON A COMPLEX SPACE WITH ARBITRARY SINGULARITIES

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Abstract. We obtain some L^2-results for the $\overline{\partial}$ operator on forms that vanish to high order on the singular set of a complex space.

1. Introduction

Let X be a pure n-dimensional reduced Stein space, and let A be a lower-dimensional complex analytic subset with empty interior containing X_{sing}. Let Ω be an open relatively compact Stein domain in X and let $K = \widehat{\overline{\Omega}_X}$ be the holomorphic convex hull of $\overline{\Omega}$ in X. Since X is Stein and $K = \widehat{K_X}$, K has a neighborhood basis of Oka-Weil domains in X ([8]). Let X_0 be an Oka-Weil neighborhood of K in X, $X_0 \subset\subset X$. Then X_0 can be realized as a holomorphic subvariety of an open polydisk in some $\mathbb{C}^s$. Set $\Omega^* = \Omega \setminus A$. Let d_A be the distance to A, relative to an embedding of X_0 in $\mathbb{C}^s$, and let $|\ |$ and dV denote the induced norm on $\Lambda^{\cdot} T^*\Omega^*$, resp. the volume element (different embeddings of neighborhoods of $\overline{\Omega}$ in $\mathbb{C}^s$ give rise to equivalent distance functions and norms). For a measurable (p,q) form u on Ω^* let $\|u\|^2_{N,\Omega} := \int_{\Omega^*} |u|^2 d_A^{-N} dV$.

In this paper we address the question of whether we can solve the equation $\overline{\partial} u = f$ in Ω^* for a $\overline{\partial}$-closed (p,q) form f on Ω^* that vanishes to "high order" on A. Our main result is the following theorem:

Theorem 1.1. *Let X, Ω be as above. For every $N_0 \geq 0$, there exists $N \geq 0$ such that if f is a $\overline{\partial}$-closed (p,q)-form on $\Omega^*, q > 0$, with $\|f\|_{N,\Omega} < \infty$, there is $v \in L^{2,\,\text{loc}}_{p,q-1}(\Omega^*)$ solving $\overline{\partial} v = f$, with $\|v\|_{N_0,\Omega'} < \infty$ for every $\Omega' \subset\subset \Omega$. For each $\Omega' \subset\subset \Omega$, there is a solution of this kind satisfying $\|v\|_{N_0,\Omega'} \leq C\|f\|_{N,\Omega}$, where C is a positive constant that depends only on Ω', N, N_0.*

When $A \cap \overline{\Omega}$ is a finite subset of $\overline{\Omega}$ with $b\Omega \cap A = \emptyset$, Ω is Stein and $\overline{\Omega}$ has a Stein neighborhood, we obtain the following corollary of Theorem 1.1:

Corollary 1.2. *With N_0, N as in Theorem 1.1 and for f a $\overline{\partial}$-closed (p,q)-form on $\Omega^*, q > 0$, with $\|f\|_{N,\Omega} < \infty$, there is a solution u of $\overline{\partial} u = f$ on Ω^* with $\|u\|_{\Omega,N_0} \leq c\|f\|_{\Omega,N}$, c independent of f. In other words, we obtain a weighted L^2 estimate for u on all of Ω.*

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Theorem 1.1 extends to the case when Ω is just holomorphically convex and contains a maximal compact subvariety B that is contained in A. It also extends to the case of (p,q) forms on X^* with values in a holomorphic vector bundle E over X. Theorem 1.1 and Corollary 1.2 can be used to construct analytic objects with prescribed behaviour on the maximal, positive-dimensional compact subvariety B of a holomorphically convex manifold. We also expect them to be useful in studying the obstructions to solving $\overline{\partial}$ on a deleted neighborhood of an isolated singular point of a complex analytic set of dimension bigger than 2. The power series arguments that were used in the surface case in [3], [4] might be replaced by the solution of a Cousin problem with L^2 bounds, which exists by our results when a finite number of obstructions vanish.

We have not managed to find a proof of Theorem 1.1 based on transcendental L^2 methods. Instead, our arguments are based on resolution of singularities, combined with cohomological arguments in the spirit of Grauert [5]. More precisely, let $\pi : \tilde{X} \to X$ be a holomorphic surjection with the following properties:

i) $\widetilde{X}$ is an n-dimensional complex manifold,

ii) $\tilde{A} = \pi^{-1}(A)$ is a hypersurface in $\tilde{X}$ with only "normal crossing singularities", i.e. near each $x_0 \in \tilde{A}$ there are local holomorphic coordinates $(z_1, \dots, z_n)$ in terms of which $\tilde{A}$ is given by $h(z) = z_1 \cdots z_m = 0$, where $1 \le m \le n$,

iii) $\pi : \widetilde{X} \setminus \tilde{A} \to X \setminus A$ is a biholomorphism,

iv) π is proper.

The existence of such a map follows from the fact that a) every reduced, complex space can be desingularized and, b) every reduced, closed complex subspace of a complex manifold admits an embedded desingularization (the exact statements and proofs can be found in [1], [2]).

Let $\tilde{\Omega} := \pi^{-1}(\Omega)$. We give $\tilde{X}$ a real analytic metric σ and we consider the corresponding distance function $d_{\tilde{A}}(x) = \operatorname{dist}(x, \tilde{A})$, volume element $d\tilde{V}_\sigma$ and norms on $\Lambda^{\cdot} T\tilde{X}$ and $\Lambda^{\cdot} T^*\tilde{X}$. Let J denote the ideal sheaf of $\tilde{A}$ in $\tilde{X}$ and Ω^p the sheaf of holomorphic p forms on $\tilde{X}$. We introduce some auxiliary sheaves (denoted by $\mathcal{L}_{p,q}$) on $\tilde{X}$. For every open subset U of $\tilde{X}$, let $\mathcal{L}_{p,q}(U)$ be

$$\mathcal{L}_{p,q}(U) := \{u \in L^{2,\,\mathrm{loc}}_{p,q}(U);\ \ \overline{\partial} u \in L^{2,\,\mathrm{loc}}_{p,q+1}(U)\}$$

and for each open subset $V \subset U$, let $r^U_V : \mathcal{L}_{p,q}(U) \to \mathcal{L}_{p,q}(V)$ be the obvious restriction maps. Then the map $u \to \overline{\partial} u$ defines an $\mathcal{O}_{\tilde{X}}$-homomorphism $\overline{\partial} : \mathcal{L}_{p,q} \to \mathcal{L}_{p,q+1}$ and the sequence

$$0 \to \Omega^p \to \mathcal{L}_{p,0} \to \mathcal{L}_{p,1} \to \cdots \to \mathcal{L}_{p,n} \to 0$$

is exact by the local Poincaré lemma for $\overline{\partial}$. Since each $\mathcal{L}_{p,q}$ is closed under multiplication by smooth cut-off functions we have a fine resolution of Ω^p. In the same way, since J is locally generated by one function, then the sequence

$$0 \to J^k\Omega^p \to J^k\mathcal{L}_{p,0} \to \cdots \to J^k\mathcal{L}_{p,n} \to 0$$

is a fine resolution of $J^k\Omega^p$. Here, $u \in (J^k\mathcal{L}_{p,q})_x$ if it can be locally written as $h^k\, u_0$ where h generates J_x and $u_0 \in (\mathcal{L}_{p,q})_x$. So we can interpret the sheaf cohomology groups $H^q(\tilde{\Omega}, (J^k\Omega^p)_{|\Omega})$ as

$$H^q(\tilde{\Omega}, (J^k\Omega^p)_{|\Omega}) \cong \frac{\ker(\overline{\partial} : J^k\mathcal{L}_{p,q}(\tilde{\Omega}) \to J^k\mathcal{L}_{p,q+1}(\tilde{\Omega}))}{\operatorname{Im}(\overline{\partial} : J^k\mathcal{L}_{p,q-1}(\tilde{\Omega}) \to J^k\mathcal{L}_{p,q}(\tilde{\Omega}))}. \tag{1}$$

Inspired by Grauert's Satz 1, Section 4 in [5] (Grauert's result corresponds to the case where A is a finite set), we were led to the vanishing of a canonical morphism between certain sheaf cohomology groups of the above type. More precisely we were able to show the following:

Proposition 1.3. *For $q > 0$ and $k \geq 0$ given, there exists a natural number ℓ, $\ell \geq k$, such that the map*

$$i_* : H^q(\tilde{\Omega}, J^\ell \Omega^p) \to H^q(\tilde{\Omega}, J^k \Omega^p),$$

induced by the inclusion $i : J^\ell \Omega^p \to J^k \Omega^p$, is the zero map.

Proposition 1.3 will play a key role in the proof of Theorem 1.1. Here is an outline for the proof of this theorem. Given N_0 we shall choose appropriately k, N. Starting with an f as in Theorem 1.1 we shall show that the pullback $\pi^* f$ satisfies

$$\int_{\tilde{\Omega}} |\pi^* f|^2_\sigma d_{\tilde{A}}^{-N_1} d\tilde{V}_\sigma \leq C \int_{\Omega^*} |f|^2 d_A^{-N} dV, \tag{2}$$

for a suitable $0 < N_1 < N$ and $\overline{\partial}\pi^* f = 0$ on $\tilde{\Omega}$.

Using (2) and the way we have chosen N, k we can show that $\pi^* f \in J^\ell \mathcal{L}_{p,q}(\tilde{\Omega})$. By Proposition 1.3, this will imply that the equation $\overline{\partial} v = \pi^* f$ has a solution in $J^k \mathcal{L}_{p,q-1}(\tilde{\Omega})$. Since $|h(x)| \leq C d_{\tilde{A}}(x)$ on compacts in the set where h generates J, it will follow that

$$\int_{\widetilde{\Omega}'} |v|^2_\sigma d_{\tilde{A}}^{-2k}(x) d\tilde{V}_\sigma < \infty,$$

where $\widetilde{\Omega}' = \pi^{-1}(\Omega')$ and $\Omega' \subset\subset \Omega$. Then $\overline{\partial}((\pi^{-1})^* v) = f$ on Ω^* and the final step will be to show that

$$\int_{\Omega'^*} |(\pi^{-1})^* v|^2 d_A^{-N_0} dV \leq c \int_{\widetilde{\Omega}'} |v|^2_\sigma d_{\tilde{A}}^{-2k} d\tilde{V}_\sigma. \tag{3}$$

The paper is organized as follows: In section 2 we prove Proposition 1.3. Section 3 contains the estimates for the pullback of forms under π and π^{-1}. In Section 4 we prove Theorem 1.1. The proof of Corollary 1.2 is contained in section 5. Finally in section 6 we discuss some generalizations of Theorem 1.1 and Corollary 1.2.

2. Proof of Proposition 1.3

Following Grauert [5], we consider more generally the coherent analytic sheaves $\mathcal{S}$ on $\tilde{X}$ that are torsion free i.e. sheaves with the property

$$T(\mathcal{S})_x = 0 \quad \text{for all } \ x \in \tilde{X}, \tag{4}$$

where $T(\mathcal{S})_x = \{g_x \in \mathcal{S}_x : \ \ f_x \cdot g_x = 0 \ \text{ for some } f_x \neq 0, \ f_x \in \mathcal{O}_x\}$.

We shall show (Lemma 2.1) that when $\mathcal{S}$ is coherent and torsion free and $i : J^t\mathcal{S} \to \mathcal{S}$ is the inclusion homomorphism, then the induced map $i_{\tilde{\Omega},*} : H^q(\tilde{\Omega}, J^t\mathcal{S}) \to H^q(\tilde{\Omega}, \mathcal{S})$ is zero when $q > 0$ and t is big enough. In order to exploit the idea that analytic sheaf cohomology on $\tilde{\Omega}$ is concentrated over $\tilde{A}$, the exceptional set of the resolution, we need to introduce the higher direct image sheaves, denoted by $R^q\pi_*\mathcal{S}$, of an analytic sheaf $\mathcal{S}$ on $\tilde{X}$, $q \geq 0$ and recall some basic facts about them. For $q \geq 0$ and $\mathcal{S}$ an $\mathcal{O}_{\tilde{X}}$-module, the higher direct image sheaves of $\mathcal{S}$ are the sheaves

on X, associated to the presheaf

$$P : U \to H^q(\pi^{-1}(U), \mathcal{S}),$$

where U is open in X.

When $\phi : \mathcal{S} \to \mathcal{S}'$ is an $\mathcal{O}_{\tilde{X}}$- homomorphism the induced maps $\phi_* : H^q(\pi^{-1}(U), \mathcal{S}) \to H^q(\pi^{-1}(U), \mathcal{S}')$, U open in X, determine a sheaf homomorphism $\phi_\# : R^q\pi_*\mathcal{S} \to R^q\pi_*\mathcal{S}'$ on X. For future reference, we recall the $\mathcal{O}_X$-module structure on $R^q\pi_*\mathcal{S}$. Given U an open subset of X, $f \in \mathcal{O}_X(U)$, we define a map $f_U\bullet : \mathcal{S}_{|_{\pi^{-1}(U)}} \to \mathcal{S}_{|_{\pi^{-1}(U)}}$ described by $(f_U\bullet)s_x = (f \circ \pi)_x \cdot s_x$, $x \in \pi^{-1}(U)$, $s_x \in \mathcal{S}_x$ and let $(f_U\bullet)_* : H^q(\pi^{-1}(U), \mathcal{S}) \to H^q(\pi^{-1}(U), \mathcal{S})$ be the induced map on cohomology. We can then define a map $\mathcal{O}_X(U) \times H^q(\pi^{-1}(U), \mathcal{S}) \to H^q(\pi^{-1}(U), \mathcal{S})$ that sends $(f, c) \in \mathcal{O}_X(U) \times H^q(\pi^{-1}(U), \mathcal{S})$ to $(f_U\bullet)_*c$. It is easy to check that it is a morphism of presheaves $\mathcal{O}_X(-) \times H^q(\pi^{-1}(-), \mathcal{S}) \to H^q(\pi^{-1}(-), \mathcal{S})$ which extends naturally to a morphism on the associated sheaves $\mathcal{O}_X \times R^q\pi_*\mathcal{S} \to R^q\pi_*\mathcal{S}$.

The main theorem in Grauert [6], says that the direct image sheaves $R^q\pi_*\mathcal{S}$ are coherent $\mathcal{O}_X$-modules, when $\mathcal{S}$ is a coherent $\mathcal{O}_{\tilde{X}}$-module and $q \geq 0$. Since Ω is a Stein domain, Satz 5, Section 2 in [6], gives that the natural map $\pi_q : H^q(\tilde{\Omega}, \mathcal{S}_{|_{\tilde{\Omega}}}) \to \Gamma(\Omega, R^q\pi_*\mathcal{S}_{|_\Omega})$ is an isomorphism. This fact and the following lemma will enable us to finish the proof of Proposition 1.3.

Lemma 2.1. *For each $q > 0$ and for each coherent, torsion-free $\mathcal{O}_{\tilde{X}}$-module $\mathcal{S}$ there exists a $T \in \mathbb{N}$ such that $i_{\tilde{\Omega},*} : H^q(\tilde{\Omega}, J^T\mathcal{S}) \to H^q(\tilde{\Omega}, \mathcal{S})$ is the zero map, where $i : J^T\mathcal{S} \hookrightarrow \mathcal{S}$ is the inclusion map.*

Proof. We shall prove the lemma using downward induction on $q > 0$. Observe that $\tilde{\Omega}$ is an n-dimensional complex manifold with no compact n-dimensional connected components since it is obtained by blow-ups from a pure n-dimensional Stein space Ω. It follows from the Main Theorem in Siu [12] that $H^n(\tilde{\Omega}, \mathcal{S}) = 0$ for every coherent $\mathcal{O}_{\tilde{X}}$-module $\mathcal{S}$. Hence, the statement is true for $q = n$ and any $T \in \mathbb{N}$.

When $q > 0$, $\mathrm{Supp} R^q\pi_*\mathcal{S}$ is contained in A. The annihilator ideal $\mathcal{A}'$ of $R^q\pi_*\mathcal{S}$ is coherent and by Cartan's Theorem A there exist functions $f_1, \cdots, f_L \in \mathcal{A}'(X)$ that generate each stalk $\mathcal{A}'_z$ in a neighborhood of $\overline{\Omega}$. Let $\mathcal{A}$ be the $\mathcal{O}_{\tilde{X}}$-ideal generated by $\tilde{f}_j = f_j \circ \pi$, $1 \leq j \leq L$. A crucial observation, which will be useful later, is that $(\tilde{f}_j)_{\tilde{\Omega},*} : H^q(\tilde{\Omega}, \mathcal{S}_{|_{\tilde{\Omega}}}) \to H^q(\tilde{\Omega}, \mathcal{S}_{|_{\tilde{\Omega}}})$ are zero for all j, $1 \leq j \leq L$, $q > 0$. To see this, consider the following commutative diagram:

$$\begin{array}{ccc} H^q(\tilde{\Omega}, \mathcal{S}_{|_{\tilde{\Omega}}}) & \xrightarrow{(\tilde{f}_j)_{\tilde{\Omega},*}} & H^q(\tilde{\Omega}, \mathcal{S}_{|_{\tilde{\Omega}}}) \\ \cong\downarrow & & \cong\downarrow \\ R^q\pi_*\mathcal{S}(\Omega) & \xrightarrow{(f_j)_{\Omega,\#}} & R^q\pi_*\mathcal{S}(\Omega) \end{array}$$

The vertical maps are isomorphisms, due to Satz 5, Section 2, in [6]. Recalling the way $\mathcal{O}_X$ acts on $R^q\pi_*\mathcal{S}$ and using the fact that the f_j's are in the annihilator ideal of $R^q\pi_*\mathcal{S}$, we conclude that $(f_j)_{\Omega,\#} = 0$. Hence, due to the commutativity of the above diagram $(\tilde{f}_j)_{\tilde{\Omega},*}$ is zero.

Let $Z(\mathcal{A})$ (resp. $Z(\mathcal{A}')$) denote the zero variety of $\mathcal{A}$ (resp. $\mathcal{A}'$). Since $Z(\mathcal{A}') = \mathrm{Supp} R^q\pi_*\mathcal{S}$ is contained in A, we have that $Z(\mathcal{A})$ is contained in $\tilde{A}$ near $\overline{\tilde{\Omega}}$. Thus by Rückert's Nullstellensatz for ideal sheaves (see Theorem, page 82 in [7]), we have $J^\mu \subset \mathcal{A}$ on $\tilde{\Omega}$ for some $\mu \in \mathbb{N}$. Consider the surjection $\phi : \mathcal{S}^{\oplus L} \to \mathcal{A}\mathcal{S}$ given by

$(s_1, \cdots, s_L) \to \sum_1^L \tilde{f}_j s_j$ and set $K = \ker\phi$. Clearly, K is torsion free, whenever $\mathcal{S}$ is. By definition the sequence

$$0 \to K \xrightarrow{i} \mathcal{S}^{\oplus L} \xrightarrow{\phi} \mathcal{A}\mathcal{S} \to 0 \tag{5}$$

is exact, and it follows from (4) and the fact that J is locally generated by one element that

$$0 \to J^a K \xrightarrow{i} J^a \mathcal{S}^{\oplus L} \xrightarrow{\phi} J^a \mathcal{A}\mathcal{S} \to 0$$

is also exact for any $a \in \mathbb{N}$.

Taking all the above into consideration we obtain the following commutative diagram:

$$\begin{tikzcd}
 & H^q(\tilde{\Omega}, J^{a+\mu}\mathcal{S}) \arrow[d] & \\
 & H^q(\tilde{\Omega}, J^a\mathcal{A}\mathcal{S}) \arrow[r, "\delta"] \arrow[d] & H^{q+1}(\tilde{\Omega}, J^a K) \arrow[d, "i_1"'] \\
H^q(\tilde{\Omega}, \mathcal{S})^{\oplus L} \arrow[r, "\phi_{\tilde{\Omega},*}"] \arrow[dr, "\chi"] & H^q(\tilde{\Omega}, \mathcal{A}\mathcal{S}) \arrow[r, "\delta"] \arrow[d, "i_2"] & H^{q+1}(\tilde{\Omega}, K) \\
 & H^q(\tilde{\Omega}, \mathcal{S}) &
\end{tikzcd}$$

where the third row is exact (as part of the long exact cohomology sequence that arises from (5)) and the vertical maps are induced by sheaf inclusions. The map χ is defined to be $\chi := i_2 \circ \phi_{\tilde{\Omega},*}$ and we can show that $\chi(c_1, \cdots, c_L) = \sum_{j=1}^L (\tilde{f}_j)_{\tilde{\Omega},*} c_j$, where $c_j \in H^q(\tilde{\Omega}, \mathcal{S})$, $1 \le j \le L$. It follows from the induction hypothesis for $(q+1)$ applied to the coherent, torsion-free sheaf K that there exists an integer a large enough such that $i_1 = 0$. Then, for an element $\sigma \in H^q(\tilde{\Omega}, \mathcal{A}\mathcal{S})$ that comes from $H^q(\tilde{\Omega}, J^{a+\mu}\mathcal{S})$, we have $\delta\sigma = 0$, so $\sigma = \phi_{\tilde{\Omega},*}(\sigma_1, \cdots, \sigma_L)$, $\sigma_j \in H^q(\tilde{\Omega}, \mathcal{S})$, $1 \le j \le L$. By the crucial observation above and the way χ is defined, we conclude that χ is the zero map. Hence $i_2(\sigma) = i_2 \circ \phi_{\tilde{\Omega},*}(\sigma_1, \cdots, \sigma_L) = \sum_{j=1}^L (\tilde{f}_j)_{\tilde{\Omega},*} \sigma_j = 0$. Thus, for $i : J^{a+\mu}\mathcal{S} \hookrightarrow \mathcal{S}$ the inclusion map, we have that $i_{\tilde{\Omega},*} : H^q(\tilde{\Omega}, J^{a+\mu}\mathcal{S}) \to H^q(\tilde{\Omega}, \mathcal{S})$ is the zero map. $\square$

Choosing as $\mathcal{S} := J^k\Omega^p$ we obtain Proposition 1.3.

3. Pointwise estimates for the pull back of forms under π, π^{-1}

Let σ be a real analytic metric on $\tilde{X}$, let $|\ |_{x,\sigma}$ denote the pointwise norm of an element of $\bigwedge^r T_x\tilde{X}$ or $\bigwedge^r T_x^*\tilde{X}$ for some $r > 0$ with respect to the metric σ and let $d_{\tilde{A}}$ be the distance to $\tilde{A}$ in $\tilde{X}$. Let d_A denote the distance to A relative to an embedding of a neighborhood X_0 of $\overline{\Omega}$ in $\mathbb{C}^s$ and let $|\ |_y$ denote the pointwise norm of an element in $\bigwedge^r T_y(X_0 \setminus X_{\text{sing}})$ for some $r > 0$, with respect to the restriction of the pull back of the euclidean metric in $\mathbb{C}^s$ to $X_0 \setminus X_{\text{sing}}$. Let dV, $d\tilde{V}_\sigma$ denote the volume forms on $X_0 \setminus X_{\text{sing}}$, and $\tilde{X}$. The map $\pi : \tilde{X} \setminus \tilde{A} \to X \setminus A$ is a biholomorphism of complex manifolds. It induces a linear isomorphism $\pi_* : \bigwedge^r T_x(\tilde{X} \setminus \tilde{A}) \to \bigwedge^r T_{\pi(x)}(X \setminus A)$ for $x \notin \tilde{A}$.

Lemma 3.1. *We have for* $x \in \widetilde{\Omega} \setminus \tilde{A},\ v \in \bigwedge^r T_x(\widetilde{\Omega})$

$$c'\ d^t_{\tilde{A}}(x) \quad \leq \quad d_A(\pi(x)) \leq C'\ d_{\tilde{A}}(x), \tag{6}$$

$$c\ d^M_{\tilde{A}}(x)\ |v|_{x,\sigma} \quad \leq \quad |\pi_*(v)|_{\pi(x)} \leq C\ |v|_{x,\sigma} \tag{7}$$

for some positive constants c', c, C', C, t, M, *where* c, C, M *may depend on* r.

For an r*-form* a *in* Ω^* *set* $|\pi^* a|_{x,\sigma} := max\{|\langle a_{\pi(x)}, \pi_* v\rangle|\ ;\ |v|_{x,\,\sigma} \leq 1,\ v \in \bigwedge^r T_x(\tilde{\Omega}\setminus\tilde{A})\}$, *where by* $\langle , \rangle$ *we denote the pairing of an* r*-form with a corresponding tangent vector. Using* (7) *we obtain*

$$c\ d^M_{\tilde{A}}(x)\ |a|_{\pi(x)} \leq |\pi^* a|_{x,\sigma} \leq C\ |a|_{\pi(x)} \tag{8}$$

on $\tilde{\Omega}$, *for some positive constant* M.

Proof. The right-hand side inequalities in the above estimates are obvious consequences of the differentiability of π, while the left-hand side inequalities are consequences of the Lojasiewicz inequalities (see for example [10], or [11], Chapter 4, Theorem 4.1) in the following form:

Lemma 3.2. *Let let* f *be a real analytic, real-valued function defined in an open set* V *in* $\mathbb{R}^d$. *Let* $Z_f = \{x \in V;\ f(x) = 0\}$. *Then, for every compact* $K \subset V$, *there exist positive constants* c, m *such that*

$$|f(x)| \geq c\ d(x, Z_f)^m$$

when $x \in K$.

Lemma 3.2 generalizes easily to the case when V is an open subset in a real analytic manifold and the distance is defined by a real analytic Riemannian metric.

To prove the left-hand side inequality in (6) let $f : \tilde{X} \times A \to \mathbb{R}$ be given by $f(x, z) = |\pi(x) - z|^2$ and $K := \overline{\widetilde{\Omega}} \times$ (compact neighborhood of $\overline{\Omega} \cap A$).

Clearly $Z_f \subset \tilde{A} \times A$. When $x \in \overline{\widetilde{\Omega}}$ and z is the nearest point to $\pi(x)$ in A, we have

$$f(x, z) = |\pi(x) - z|^2 = d(\pi(x), A)^2 \geq c\ d((x, z), Z_f)^m \geq c\ d_{\tilde{A}}(x)^m.$$

If we write $m = 2t$ for some $t > 0$ constant, then we obtain from this last estimate the left-hand side inequality in (6).

To prove the left-hand side inequality in (7), we consider the unit sphere bundle $S^r(\tilde{X})$ in $\bigwedge^r T\tilde{X}$. Recall $\tilde{X}$ has a real analytic metric σ so $S^r(\tilde{X})$ becomes a real analytic manifold. We choose a metric on $S^r(\tilde{X})$ such that the projection $p : S^r(\tilde{X}) \to \tilde{X}$ is distance decreasing. For $\nu = (x, \xi_x)$ on the unit sphere bundle $S^r(\tilde{X})$, we set $f(\nu) := |\pi_*\xi_x|^2_{\pi(p(\nu))}$ and let $K := p^{-1}(\overline{\widetilde{\Omega}})$. Clearly, $Z_f \subset p^{-1}(\tilde{A})$. It follows that $|\pi_*\xi_x|^2_{\pi(p(\nu))} = f(\nu) \geq c\ d(\nu, Z_f)^R \geq c\ d(p(\nu), \tilde{A})^R$ when $\nu \in K$. Write $2M = R$ for some $M > 0$ constant. The general case follows by applying this last inequality to $\frac{v}{|v|_x}$ for $v \neq 0$, $v \in \bigwedge^r T_x(\widetilde{\Omega})$.

Estimate (8) will be derived from (7) and the following remark:

Remark. Let $T : V \to W$ be a linear isomorphism of normed spaces such that $\|Tv\| \geq c\|v\|$ for $v \in V$ and $c > 0$ constant. Then $B_W(0, c) \subset T(B_V(0, 1))$, where by $B_V(0, 1)$ we denote the unit ball in V and $B_W(0, c)$ is the ball in W, centered at 0 and having radius c.

Using (7) and applying the above remark to $\pi_* : \bigwedge^r T_x(\tilde{\Omega}\setminus\tilde{A}) \to \bigwedge^r T_{\pi(x)}(\Omega\setminus A)$ we obtain for $x \in \tilde{\Omega}\setminus\tilde{A},\ a_{\pi(x)} \in \bigwedge^r T^*_{\pi(x)}(\Omega\setminus A)$:

$$\begin{aligned} |\pi^* a|_{x,\sigma} &= \max\{\ |\langle a_{\pi(x)}, \pi_* v\rangle|\ ;\ |v|_{x,\sigma} \le 1, v \in \textstyle\bigwedge^r T_x(\tilde{\Omega}\setminus\tilde{A})\} \\ &\ge \max\{\ |\langle a_{\pi(x)}, w\rangle|\ ;\ |w|_{\pi(x)} \le c\, d_{\tilde{A}}(x)^M,\ w \in \textstyle\bigwedge^r T_{\pi(x)}(\Omega\setminus A)\} \\ &= c\, d_{\tilde{A}}(x)^M |a|_{\pi(x)}. \end{aligned}$$

This result applies in particular to the volume form in $\Omega\setminus A$ and gives

$$c_1\, d_{\tilde{A}}(x)^{M_1} d\tilde{V}_{x,\sigma} \le (\pi^* dV)_x \le C_1\, d\tilde{V}_{x,\sigma}. \tag{9}$$

4. Proof of Theorem 1.1

Given $N_0 \in \mathbb{N}$, choose $k \ge M + t\,\frac{N_0}{2} \ge 0$, with t, M as in Lemma 3.1. Then by Proposition 1.3, there exists $\ell \ge k$ such that $H^q(\tilde{\Omega}, J^\ell\Omega^p) \to H^q(\tilde{\Omega}, J^k\Omega^p)$ is the zero homomorphism. Choose $N \in \mathbb{N}$ such that $N \ge 2n\ell + M_1$, where M_1 is as in (9).

The proof of Theorem 1.1 will be based on the following change of variables result:

Lemma 4.1. *Let W, W' be orientable, Riemannian manifolds and let $F : W \to W'$ be an orientation-preserving diffeomorphism. Let dV, dV' denote the corresponding volume elements of W, W', respectively. For $f \in L^1(W', dV')$ we have*

$$\int_{W'} f dV' = \int_W (f\circ F)\ F^*(dV'). \tag{10}$$

Since $\pi : \tilde{\Omega}\setminus\tilde{A} \to \Omega\setminus A$ is a biholomorphism and orientation-preserving map—as long as we choose appropriate orientations on $\Omega\setminus A$, $\tilde{\Omega}\setminus\tilde{A}$—for any f satisfying $\|f\|_{N,\Omega^*} < \infty$ we have (by applying Lemma 4.1)

$$\int_{\Omega\setminus A} |f|^2\, d_A^{-N}\, dV = \int_{\tilde{\Omega}\setminus\tilde{A}} |f|^2_{\pi(x)}\, d_A(\pi(x))^{-N} (\pi^* dV)_x.$$

Using the fact that

$$\begin{aligned} |f|_{\pi(x)} &\ge C^{-1}|\pi^* f|_{x,\sigma} && \text{(right-hand side of (8))}, \\ d_A(\pi(x))^{-1} &\ge C'^{-1} d_{\tilde{A}}^{-1}(x) && \text{(right-hand side of (6))}, \\ (\pi^* dV)_{x,\sigma} &\ge c_1 d_{\tilde{A}}^{M_1}(x)\, d\tilde{V}_{x,\sigma} && \text{(left-hand side of (9))}, \end{aligned}$$

we obtain

$$\|f\|^2_{N,\Omega^*} \ge c'' \int_{\tilde{\Omega}\setminus\tilde{A}} |\pi^* f|^2_{x,\sigma}\, d_{\tilde{A}}^{M_1 - N} d\tilde{V}_{x,\sigma}$$

for some $c'' > 0$ constant. Since N was chosen such that $N \ge M_1$, we see that $\overline{\partial}\pi^* f = 0$ on $\tilde{\Omega}$. It is not hard to show that $\pi^* f \in J^\ell\mathcal{L}_{p,q}(\tilde{\Omega})$. By Proposition 1.3 we know that there exists $v \in J^k\mathcal{L}_{p,q-1}(\tilde{\Omega})$ such that $\overline{\partial} v = \pi^* f$ in $\tilde{\Omega}$. Set $u := (\pi^{-1})^* v$.

Then $\overline{\partial} u = f$ in Ω^* and for any $\Omega' \subset\subset \Omega$ we have

$$\begin{aligned}\int_{\Omega'} |u|^2 d_A^{-N_0} dV &= \int_{\widetilde{\Omega'}\setminus\tilde{A}} |u|^2_{\pi(x)}\, d_A^{-N_0}(\pi(x))\, \pi^*(dV) \\ &\lesssim \int_{\widetilde{\Omega'}\setminus\tilde{A}} d_{\tilde{A}}^{-tN_0-2M} |v|^2_{x,\sigma}\, d\tilde{V}_{x,\sigma} \\ &\lesssim \int_{\widetilde{\Omega'}\setminus\tilde{A}} d_{\tilde{A}}^{-2k} |v|^2_{x,\sigma}\, d\tilde{V}_{x,\sigma} < \infty.\end{aligned}$$

To pass from the 1st line to the 2nd one we use the fact that

$$|u|_{\pi(x)} \le c^{-1} d_{\tilde{A}}^{-M}(x)\, |v|_{x,\,\sigma}, \quad d_A^{-N_0}(\pi(x)) \le c'^{-N_0} d_{\tilde{A}}^{-tN_0}(x)$$

and that $(\pi^* dV)_{x,\sigma} \le C_1\, d\tilde{V}_{x,\,\sigma}$.

To conclude the proof of Theorem 1.1 we shall need the following lemma:

Lemma 4.2. *Let M be a complex manifold and let E and F be Frechet spaces of differential forms (or currents) of type $(p, q-1)$, (p, q), whose topologies are finer (possibly strictly finer) than the weak topology of currents. Assume that for every $f \in F$, the equation $\overline{\partial} u = f$ has a solution $u \in E$. Then, for every continuous seminorm p on E, there is a continuous seminorm q on F such that the equation $\overline{\partial} u = f$ has a solution with $p(u) \le q(f)$ for every $f \in F$, $q(f) > 0$.*

Proof. Set $G = \{(u, f) \in E \times F : \overline{\partial} u = f\}$. Then G is closed in $E \times F$. To see this, let $(u_\nu, f_\nu) \in G$ with $u_\nu \to u$ in E, $f_\nu \to f$ in F. For test forms $\phi \in C^\infty_{0,(n-p,n-q)}(M)$ we get

$$\begin{aligned}\int_M f \wedge \phi &= \lim_{\nu\to\infty} \int_M f_\nu \wedge \phi \\ &= \lim_{\nu\to\infty} (-1)^{p+q} \int_M u_\nu \wedge \overline{\partial}\phi = (-1)^{p+q} \int_M u \wedge \overline{\partial}\phi\end{aligned}$$

so $\overline{\partial} u = f$ weakly.

Thus, G is a Frechet space and the bounded surjection $\pi_2 : G \to F; (u, f) \to f$ must be open. The set $\pi_2(\{(u, v) \in G : p(u) < 1\})$ is an open neighborhood of 0 in F, and contains $\{f : q(f) \le 1\}$ for some continuous seminorm q. Let $f \in F$, $0 < q(f) = c$. Then $q(c^{-1} f) = 1$, so by the previous argument there exists a solution $c^{-1} u$ satisfying $\overline{\partial}(c^{-1} u) = c^{-1} f$ with $p(c^{-1} u) < 1$, i.e. $p(u) < c = q(f)$. $\square$

When F is a Banach space with norm $\|.\|$, we conclude that, given a seminorm p, there is a constant $C > 0$ such that $\{f : \|f\| \le C^{-1}\} \subset \overline{\partial}(\{u : p(u) \le 1\})$, so $\overline{\partial} u = f$ has a solution u with $p(u) \le C\|f\|$. Applying this result to our situation, we see that if $\overline{\partial} f = 0$, $\|f\|_{\Omega,N} < \infty$ and $\Omega_0 \subset\subset \Omega$, we obtain a solution u to $\overline{\partial} u = f$ in $L^{2,\mathrm{loc}}_{p,q-1}(\Omega^*)$ with $\|u\|_{\Omega_0,N_0} \le c\|f\|_{\Omega,N}$.

5. Applications of Theorem 1.1

We apply Theorem 1.1 to the case where $A \cap \overline{\Omega}$ is a finite subset of $\overline{\Omega}$ with $b\Omega \cap A = \emptyset$, $\Omega \subset\subset X$ is Stein and $\overline{\Omega}$ has a Stein neighborhood Ω'.

Proposition 5.1. *With N_0, N as in Theorem 1.1 and $\overline{\partial} f = 0$ on Ω^* and $\|f\|_{\Omega,N} < \infty$, there is a solution u of $\overline{\partial} u = f$ on Ω^* with $\|u\|_{\Omega,N_0} \le c\|f\|_{\Omega,N}$, c independent of f. In other words, we obtain a weighted L^2 estimate for u on all of Ω.*

Proof. Choosing $\Omega_0 \subset\subset \Omega$ containing $A \cap \Omega$, we have a solution u_0 in $L^{2,\mathrm{loc}}_{p,q-1}(\Omega^*)$ with $\|u_0\|_{\Omega_0,N_0} \leq c\|f\|_{\Omega,N}$. We introduce a cut-off function $\chi \in C^\infty(X)$ such that $\chi = 1$ on $X \setminus \Omega_0$ but $\chi = 0$ near $A \cap \Omega$. Set $f_1 = \overline{\partial}(\chi u_0)$. Clearly, $\|f_1\|_{L^2(\Omega)} \leq c\|f\|_{\Omega,N}$ and $f_1 = 0$ near $\Omega \cap A$.

Let $\pi : \tilde{X} \to X$ be a desingularization of X and consider the equation $\overline{\partial} v = \pi^* f_1$ on $\tilde{\Omega}$. Let $\tilde{\Omega}_0 := \pi^{-1}(\Omega_0)$. The equation $\overline{\partial} v = \pi^* f_1$ is solvable in $L^2_{p,q-1}(\tilde{\Omega}_0)$. We can assume that $\tilde{\Omega}$ can be exhausted by smoothly bounded strongly pseudoconvex domains $\tilde{\Omega}_j := \{z \in \tilde{\Omega};\ \phi < c_j\}$, where c_j are real numbers, ϕ is an exhaustion function for $\tilde{\Omega}$, of class $C^3(\tilde{\Omega})$, strictly plurisubharmonic outside a compact subset, and also that $b\tilde{\Omega}_0$ is smooth and strongly pseudoconvex and contained in each $\tilde{\Omega}_j$. To each $\tilde{\Omega}_j$ we apply Theorem 3.4.6 in [9] and we obtain a solution v_j to the equation $\overline{\partial} v_j = \pi^* f_1$ in $\tilde{\Omega}_j$ with

$$\int_{\tilde{\Omega}_j} |v_j|^2\, e^{-\phi}\, d\tilde{V}_\sigma \leq C \int_{\tilde{\Omega}} |\pi^* f_1|^2\, d\tilde{V}_\sigma,$$

where C is a positive constant independent of j, f (this follows from a careful inspection of the proof of Theorem 3.4.6 in [9]).

Consider the trivial extensions v_j^o of v_j outside $\tilde{\Omega}_j$. Let v be a weak limit of v_j^o. Then

$$\int_{\tilde{\Omega}} |v|^2\, e^{-\phi}\, d\tilde{V}_\sigma \leq C \int_{\tilde{\Omega}} |\pi^* f_1|^2\, d\tilde{V}_\sigma$$

and $\overline{\partial} v = \pi^* f_1$ in $\tilde{\Omega}$. So there is a solution v satisfying $\|v\|_{L^2(\tilde{\Omega})} \leq c\|f_1\|$. Then $w := (\pi^{-1})^* v$ satisfies $\overline{\partial} w = f_1$ in Ω^* but we no longer have control of its L^2-norm near $A \cap \Omega$. Choose another cut-off function χ_0 such that $\chi_0 = 1$ on suppχ but $\chi_0 = 0$ near $\Omega \cap A$. Then

$$\begin{aligned}
\overline{\partial}((1-\chi)u_0 \quad &+ \quad \chi_0(\pi^{-1})^* v) \\
&= \quad (1-\chi)f - \overline{\partial}\chi \wedge u_0 + \overline{\partial}\chi_0 \wedge (\pi^{-1})^* v \\
&+ \quad \chi\, f + \overline{\partial}\chi \wedge u_0 \\
&= \quad f + \overline{\partial}\chi_0 \wedge (\pi^{-1})^* v.
\end{aligned}$$

Finally we may solve $\overline{\partial} v_1 = \overline{\partial}\chi_0 \wedge (\pi^{-1})^* v$ in Ω'^* (apply Theorem 1.1 to the trivial extension of $\overline{\partial}\chi_0 \wedge (\pi^{-1})^* v$ in Ω'):

$$\|v_1\|_{\Omega,N_0} \leq c\, \|\overline{\partial}\chi_0 \wedge (\pi^{-1})^* v\|_{\Omega',N} \leq c'\, \|\overline{\partial}\chi_0 \wedge (\pi^{-1})^* v\|_{L^2(\Omega)} \leq C\|f\|_{\Omega,N}$$

since $\overline{\partial}\chi = 0$ near A. Thus, $u := (1-\chi)u_0 + \chi_0\, (\pi^{-1})^* v - v_1$ is a solution with the required estimate. □

6. Generalizations

Theorem 1.1 and Corollary 1.2 also extend to the case when Ω is a relatively compact domain in a complex space X of pure dimension n with strictly pseudoconvex boundary. We know that Ω contains a maximal positive-dimensional compact variety B and let A be a nowhere open analytic subvariety of X containing X_{sing} and B. Then Theorem 1.1 carries over verbatim to the case described above. The proof needs the following modifications: Let $\overline{\Omega} \subset X_0$ be a neighborhood with strictly pseudoconvex boundary and maximal positive dimensional compact subvariety B. Take the Remmert reduction $\phi : X_0 \to X_1$ so that X_1 is Stein, $\phi(B) = B_1$ is finite and $\phi : X_0 \setminus B_0 \to X_1 \setminus B_1$ is a biholomorphism. Let $\pi : \tilde{X}_0 \to X_0$ be a

desingularization of X_0 such that $\pi^{-1}(A)$ is a hypersurface with normal crossings. To obtain a proof of Proposition 1.3 (vanishing cohomology), we need to consider direct images $R^q(\phi \circ \pi)_* \mathcal{S}$ on the Stein space X_1 and their annihilator ideal $\mathcal{A}$ for $\mathcal{S}$ coherent on $\tilde{X}$. Then, the proof carries over.

Corollary 1.2, for the case when $X_{\text{sing}} \cap b\Omega$ is empty and with $A = B \cup (X_{\text{sing}} \cap \Omega)$, follows exactly as above.

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