SEMIGLOBAL RESULTS FOR $\overline{\partial}$ ON A COMPLEX SPACE WITH ARBITRARY SINGULARITIES

JOHN ERIK FORNÆSS, NILS ØVRELID, AND SOPHIA VASSILIADOU

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ABSTRACT. We obtain some $L^2$-results for the $\overline{\partial}$ operator on forms that vanish to high order on the singular set of a complex space.

1. Introduction

Let $X$ be a pure $n$-dimensional reduced Stein space, and let $A$ be a lower-dimensional complex analytic subset with empty interior containing $X_{\text{sing}}$. Let $\Omega$ be an open relatively compact Stein domain in $X$ and let $K = \overline{\Omega}^X$ be the holomorphic convex hull of $\overline{\Omega}$ in $X$. Since $X$ is Stein and $K = \overline{K_X}$, $K$ has a neighborhood basis of Oka-Weil domains in $X$ ([8]). Let $\Omega_0$ be an Oka-Weil neighborhood of $K$ in $X$, $X_0 \subset \subset X$. Then $X_0$ can be realized as a holomorphic subvariety of an open polydisk in some $\mathbb{C}^*$. Set $\Omega^* = \Omega \setminus A$. Let $d_A$ be the distance to $A$, relative to an embedding of $X_0$ in $\mathbb{C}^*$, and let $| |$ and $dV$ denote the induced norm on $\Lambda^* T^* \Omega^*$, resp. the volume element (different embeddings of neighborhoods of $\overline{\Omega}$ in $\mathbb{C}^*$ give rise to equivalent distance functions and norms). For a measurable $(p, q)$-form $u$ on $\Omega^*$ let $||u||^2_{\Omega^*} := \int_{\Omega^*} |u|^2 d_A^{-n} dV$.

In this paper we address the question of whether we can solve the equation $\overline{\partial} u = f$ in $\Omega^*$ for a $\overline{\partial}$-closed $(p, q)$-form $f$ on $\Omega^*$ that vanishes “to high order” on $A$. Our main result is the following theorem:

Theorem 1.1. Let $X$, $\Omega$ be as above. For every $N_0 \geq 0$, there exists $N \geq 0$ such that if $f$ is a $\overline{\partial}$-closed $(p, q)$-form on $\Omega^*$, $q > 0$, with $\|f\|_{N, \Omega} < \infty$, there is $v \in L^2_{p, q-1}(\Omega^*)$ solving $\overline{\partial} v = f$, with $\|v\|_{N_0, \Omega'} < \infty$ for every $\Omega' \subset \subset \Omega$. For each $\Omega' \subset \subset \Omega$, there is a solution of this kind satisfying $\|v\|_{N_0, \Omega'} \leq C \|f\|_{N, \Omega}$, where $C$ is a positive constant that depends only on $\Omega'$, $N$, $N_0$.

When $A \cap \overline{\Omega}$ is a finite subset of $\overline{\Omega}$ with $b\Omega \cap A = \emptyset$, $\Omega$ is Stein and $\overline{\Omega}$ has a Stein neighborhood, we obtain the following corollary of Theorem 1.1:

Corollary 1.2. With $N_0$, $N$ as in Theorem 1.1 and for $f$ a $\overline{\partial}$-closed $(p, q)$-form on $\Omega^*$, $q > 0$, with $\|f\|_{N, \Omega} < \infty$, there is a solution $u$ of $\overline{\partial} u = f$ on $\Omega^*$ with $\|u\|_{N_0, \Omega} \leq c \|f\|_{N, \Omega}$, $c$ independent of $f$. In other words, we obtain a weighted $L^2$ estimate for $u$ on all of $\Omega$.
Theorem 1.1 extends to the case when \( \Omega \) is just holomorphically convex and contains a maximal compact subvariety \( B \) that is contained in \( A \). It also extends to the case of \((p, q)\) forms on \( X^* \) with values in a holomorphic vector bundle \( E \) over \( X \). Theorem 1.1 and Corollary 1.2 can be used to construct analytic objects with prescribed behaviour on the maximal, positive-dimensional compact subvariety \( B \) of a holomorphically convex manifold. We also expect them to be useful in studying the obstructions to solving a \((p, q)\)-forms on \( \Omega \) with only "normal crossing singularities", i.e. near each \( x_0 \in \tilde{A} \) there are local holomorphic coordinates \((z_1, \ldots, z_n)\) in terms of which \( \tilde{A} \) is given by \( h(z) = z_1 \cdots z_m = 0 \), where \( 1 \leq m \leq n \), \( ii) \pi : \tilde{X} \setminus \tilde{A} \to X \setminus A \) is a biholomorphism, \( iv) \pi \) is proper.

The existence of such a map follows from the fact that a) every reduced, complex space can be desingularized and, b) every reduced, closed complex subspace of a complex manifold admits an embedded desingularization (the exact statements and proofs can be found in [1], [2]).

Let \( \tilde{\Omega} := \pi^{-1}(\Omega) \). We give \( \tilde{X} \) a real analytic metric \( \sigma \) and we consider the corresponding distance function \( d_\sigma(x) = \text{dist}(x, \tilde{A}) \), volume element \( dV_\sigma \) and norms on \( \Lambda^* \tilde{X} \) and \( \Lambda^* \tilde{X}^* \). Let \( J \) denote the ideal sheaf of \( \tilde{A} \) in \( \tilde{X} \) and \( \Omega^p \) the sheaf of holomorphic \( p \) forms on \( \tilde{X} \). We introduce some auxiliary sheaves (denoted by \( L_{p,q} \)) on \( \tilde{X} \). For every open subset \( U \) of \( \tilde{X} \), let \( L_{p,q}(U) \) be

\[
L_{p,q}(U) := \{ u \in L^2_{p,q}(U); \, \overline{\partial} u \in L^2_{p,q+1}(U) \}
\]

and for each open subset \( V \subset U \), let \( r_U^V : L_{p,q}(U) \to L_{p,q}(V) \) be the obvious restriction maps. Then the map \( u \to \overline{\partial} u \) defines an \( \mathcal{O}_{\tilde{X}} \)-homomorphism \( \overline{\partial} : L_{p,q} \to L_{p,q+1} \) and the sequence

\[
0 \to \Omega^p \to L_{p,0} \to L_{p,1} \to \cdots \to L_{p,n} \to 0
\]

is exact by the local Poincaré lemma for \( \overline{\partial} \). Since each \( L_{p,q} \) is closed under multiplication by smooth cut-off functions we have a fine resolution of \( \Omega^p \). In the same way, since \( J \) is locally generated by one function, then the sequence

\[
0 \to J^k \Omega^p \to J^k L_{p,0} \to \cdots \to J^k L_{p,n} \to 0
\]

is a fine resolution of \( J^k \Omega^p \). Here, \( u \in (L_j \Omega^p)_x \) if it can be locally written as \( h^k u_0 \) where \( h \) generates \( J_x \) and \( u_0 \in (L_{p,q})_x \). So we can interpret the sheaf cohomology groups \( H^q(\tilde{\Omega}, (J^k \Omega^p)|_a) \) as

\[
H^q(\tilde{\Omega}, (J^k \Omega^p)|_a) \cong \frac{\ker(\overline{\partial} : J^k L_{p,q}(\tilde{\Omega}) \to J^k L_{p,q+1}(\tilde{\Omega}))}{\text{Im}(\overline{\partial} : J^k L_{p,q-1}(\tilde{\Omega}) \to J^k L_{p,q}(\tilde{\Omega}))}.
\]
Inspired by Grauert’s Satz 1, Section 4 in [3] (Grauert’s result corresponds to the case where \( A \) is a finite set), we were led to the vanishing of a canonical morphism between certain sheaf cohomology groups of the above type. More precisely we were able to show the following:

**Proposition 1.3.** For \( q > 0 \) and \( k \geq 0 \) given, there exists a natural number \( \ell, \ell \geq k \), such that the map

\[
i_* : H^q(\tilde{\Omega}, J^\ell \Omega^p) \to H^q(\tilde{\Omega}, J^k \Omega^p),
\]

induced by the inclusion \( i : J^\ell \Omega^p \to J^k \Omega^p \), is the zero map.

Proposition 1.3 will play a key role in the proof of Theorem 1.1. Here is an outline for the proof of this theorem. Given \( N_0 \) we shall choose appropriately \( k, N \). Starting with an \( f \) as in Theorem 1.1 we shall show that the pullback \( \pi^* f \) satisfies

\[
\int_\tilde{\Omega} |\pi^* f|^2 \sigma^{-N_1} d\tilde{V}_\sigma \leq C \int_{\tilde{\Omega}^*} |f|^2 \sigma^{-N} dV,
\]

for a suitable \( 0 < N_1 < N \) and \( \tilde{\Omega} \pi^* f = 0 \) on \( \tilde{\Omega} \).

Using (2) and the way we have chosen \( N, k \) we can show that \( \pi^* f \in J^\ell L_{p,q}(\tilde{\Omega}) \). By Proposition 1.3, this will imply that the equation \( \tilde{\Omega} v = \pi^* f \) has a solution in \( J^k L_{p,q-1}(\tilde{\Omega}) \). Since \( |h(x)| \leq C d_{\tilde{x}}(x) \) on compacts in the set where \( h \) generates \( J \), it will follow that

\[
\int_{\tilde{\Omega}^*} |v|^2 \sigma^{-2k} d\tilde{V}_\sigma < \infty,
\]

where \( \tilde{\Omega} = \pi^{-1}(\Omega)' \) and \( \Omega' \subset \Omega \). Then \( (\pi^{-1})^* v = f \) on \( \Omega^* \) and the final step will be to show that

\[
\int_{\Omega^*} |(\pi^{-1})^* v|^2 \sigma^{-N_0} dV \leq C \int_{\tilde{\Omega}^*} |v|^2 \sigma^{-2k} d\tilde{V}_\sigma.
\]

The paper is organized as follows: In section 2 we prove Proposition 1.3. Section 3 contains the estimates for the pullback of forms under \( \pi \) and \( \pi^* \). In Section 4 we prove Theorem 1.1. The proof of Corollary 1.2 is contained in section 5. Finally in section 6 we discuss some generalizations of Theorem 1.1 and Corollary 1.2.

2. **Proof of Proposition 1.3**

Following Grauert [5], we consider more generally the coherent analytic sheaves \( \mathcal{S} \) on \( \tilde{X} \) that are torsion free i.e. sheaves with the property

\[
T(\mathcal{S})_x = 0 \quad \text{for all} \quad x \in \tilde{X},
\]

where \( T(\mathcal{S})_x = \{ g_x \in \mathcal{S}_x : \ f_x \cdot g_x = 0 \quad \text{for some} \ f_x \neq 0, \ f_x \in \mathcal{O}_x \} \).

We shall show (Lemma 2.1) that when \( \mathcal{S} \) is coherent and torsion free and \( i : J^l \mathcal{S} \to \mathcal{S} \) is the inclusion homomorphism, then the induced map \( i_{\tilde{\Omega},*} : H^q(\tilde{\Omega}, J^l \mathcal{S}) \to H^q(\tilde{\Omega}, \mathcal{S}) \) is zero when \( q > 0 \) and \( l \) is big enough. In order to exploit the idea that analytic sheaf cohomology on \( \Omega \) is concentrated over \( \tilde{A} \), the exceptional set of the resolution, we need to introduce the higher direct image sheaves, denoted by \( R^p pi_* \mathcal{S} \), of an analytic sheaf \( \mathcal{S} \) on \( \tilde{X} \), \( q \geq 0 \) and recall some basic facts about them. For \( q \geq 0 \) and \( \mathcal{S} \) an \( \mathcal{O}_{\tilde{X}} \)-module, the higher direct image sheaves of \( \mathcal{S} \) are the sheaves...
on $X$, associated to the presheaf

$$P : U \to H^q(\pi^{-1}(U), S),$$

where $U$ is open in $X$.

When $\phi : S \to S'$ is an $O_X$-homomorphism the induced maps $\phi_* : H^q(\pi^{-1}(U), S) \to H^q(\pi^{-1}(U), S')$, $U$ open in $X$, determine a sheaf homomorphism $\phi : R^q\pi_* S \to R^q\pi_* S'$ on $X$. For future reference, we recall the $O_X$-module structure on $R^q\pi_* S$. Given $U$ an open subset of $X$, $f \in O_X(U)$, we define a map $f_U \cdot : S_{\pi(U)} \to S_{\pi(U)}$ described by $(f_U \cdot)s_x = (f \circ \pi)x \cdot s_x$, $x \in \pi^{-1}(U)$, $s_x \in S_x$ and let $(f_U \cdot)_* : H^q(\pi^{-1}(U), S) \to H^q(\pi^{-1}(U), S)$ be the induced map on cohomology.

We can then define a map $O_X(U) \times H^q(\pi^{-1}(U), S) \to H^q(\pi^{-1}(U), S)$ that sends $(f, c) \in O_X(U) \times H^q(\pi^{-1}(U), S)$ to $(f_U \cdot)_*c$. It is easy to check that it is a morphism of presheaves $O_X(-) \times H^q(\pi^{-1}(-), S) \to H^q(\pi^{-1}(-), S)$ which extends naturally to a morphism on the associated sheaves $O_X \times R^q\pi_* S \to R^q\pi_* S$.

The main theorem in Grauert [6], says that the direct image sheaves $R^q\pi_* S$ are coherent $O_X$-modules, when $S$ is a coherent $O_X$-module and $q \geq 0$. Since $\Omega$ is a Stein domain, Satz 5, Section 2 in [6], gives that the natural map $\pi_* : H^q(\overline{\Omega}, S_{\overline{\Omega}}) \to \Gamma(\overline{\Omega}, R^q\pi_* S_{\overline{\Omega}})$ is an isomorphism. This fact and the following lemma will enable us to finish the proof of Proposition 1.3.

**Lemma 2.1.** For each $q > 0$ and for each coherent, torsion-free $O_X$-module $S$ there exists a $T \in \mathbb{N}$ such that $i_{\overline{\Omega},*} : H^q(\overline{\Omega}, J^T S) \to H^q(\overline{\Omega}, S)$ is the zero map, where $i : J^T S \hookleftarrow S$ is the inclusion map.

**Proof.** We shall prove the lemma using downward induction on $q > 0$. Observe that $\overline{\Omega}$ is an $n$-dimensional complex manifold with no compact $n$-dimensional connected components since it is obtained by blow-ups from a pure $n$-dimensional Stein space $\Omega$. It follows from the Main Theorem in Siu [12] that $H^u(\overline{\Omega}, S) = 0$ for every coherent $O_X$-module $S$. Hence, the statement is true for $q = n$ and any $T \in \mathbb{N}$.

When $q > 0$, $\text{Supp}R^q\pi_* S$ is contained in $A$. The annihilator ideal $A'$ of $R^q\pi_* S$ is coherent and by Cartan’s Theorem A there exist functions $f_1, \ldots, f_L \in A'(X)$ that generate each stalk $A'_x$ in a neighborhood of $\overline{\Omega}$. Let $A$ be the $O_X$-ideal generated by $\tilde{f}_j = f_j \circ \pi$, $1 \leq j \leq L$. A crucial observation, which will be useful later, is that $(f_j)_{\overline{\Omega},*} : H^q(\overline{\Omega}, S_{\overline{\Omega}}) \to H^q(\overline{\Omega}, S_{\overline{\Omega}})$ are zero for all $j$, $1 \leq j \leq L$, $q > 0$. To see this, consider the following commutative diagram:

$$
\begin{array}{ccc}
H^q(\overline{\Omega}, S_{\overline{\Omega}}) & \xrightarrow{(f_j)_{\overline{\Omega},*}} & H^q(\overline{\Omega}, S_{\overline{\Omega}}) \\
\cong & & \cong \\
R^q\pi_* S(\overline{\Omega}) & \xrightarrow{(f_j)_{\overline{\Omega},*}} & R^q\pi_* S(\overline{\Omega})
\end{array}
$$

The vertical maps are isomorphisms, due to Satz 5, Section 2, in [6]. Recalling the way $O_X$ acts on $R^q\pi_* S$ and using the fact that the $f_j$’s are in the annihilator ideal of $R^q\pi_* S$, we conclude that $(f_j)_{\overline{\Omega},*} = 0$. Hence, due to the commutativity of the above diagram $(f_j)_{\overline{\Omega},*}$ is zero.

Let $Z(A)$ (resp. $Z(A')$) denote the zero variety of $A$ (resp. $A'$). Since $Z(A') = \text{Supp}R^q\pi_* S$ is contained in $A$, we have that $Z(A)$ is contained in $\overline{A}$ near $\overline{\Omega}$. Thus by Rückert’s Nullstellensatz for ideal sheaves (see Theorem, page 82 in [6]), we have $J^\mu \subset A$ on $\Omega$ for some $\mu \in \mathbb{N}$. Consider the surjection $\phi : S^{[L]} \to AS$ given by
Let $\sigma$ be a real analytic metric on $\tilde{X}$, let $|\cdot|_{x,\sigma}$ denote the pointwise norm of an element of $\bigwedge^r T_x \tilde{X}$ or $\bigwedge^r T_x^* \tilde{X}$ for some $r > 0$ with respect to the metric $\sigma$ and let $d_{\tilde{A}}$ be the distance to $\tilde{A}$ in $\tilde{X}$. Let $d_A$ denote the distance to $A$ relative to an embedding of a neighborhood $X_0$ of $\overline{\Omega}$ in $\mathbb{C}^s$ and let $|\cdot|_y$ denote the pointwise norm of an element in $\bigwedge^r T_y (X_0 \setminus X_{\text{sing}})$ for some $r > 0$, with respect to the restriction of the pull back of the euclidean metric in $\mathbb{C}^s$ to $X_0 \setminus X_{\text{sing}}$. Let $dV, dV_y$ denote the volume forms on $X_0 \setminus X_{\text{sing}}$ and $\tilde{X}$. The map $\pi : \tilde{X} \setminus \tilde{A} \rightarrow X \setminus A$ is a biholomorphism of complex manifolds. It induces a linear isomorphism $\pi_* : \bigwedge^r T_x (\tilde{X} \setminus \tilde{A}) \rightarrow \bigwedge^r T_{\pi(x)} (X \setminus A)$ for $x \notin \tilde{A}$. 

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Lemma 3.1. We have for \( x \in \tilde{\Omega} \setminus \tilde{A} \), \( v \in \bigwedge^r T_x(\Omega) \)

\[
6 \quad c' d^d_{A}(x) \leq d_A(\pi(x)) \leq C' d_{\tilde{A}}(x),
\]

\[
7 \quad c d^d_{\tilde{A}}(x) |v|_{x,\sigma} \leq |\pi_*(v)|_{\pi(x)} \leq C |v|_{x,\sigma}
\]

for some positive constants \( c', c, C', t, M \), where \( c, C, M \) may depend on \( r \).

For an \( r \)-form \( a \) in \( \Omega^* \) set \( |\pi^*a|_{x,\sigma} := \max\{|\langle a_{\pi(x)}, \pi_* v \rangle| : |v|_{x,\sigma} \leq 1, v \in \bigwedge^r T_x(\Omega \setminus \tilde{A})\} \), where by \( \langle \cdot, \cdot \rangle \) we denote the pairing of an \( r \)-form with a corresponding tangent vector. Using (7) we obtain

\[
8 \quad c d^d_{\tilde{A}}(x) |a|_{\pi(x)} \leq |\pi^*a|_{x,\sigma} \leq C |a|_{\pi(x)}
\]
on \( \tilde{\Omega} \), for some positive constant \( M \).

Proof. The right-hand side inequalities in the above estimates are obvious consequences of the differentiability of \( \pi \), while the left-hand side inequalities are consequences of the Lojasiewicz inequalities (see for example [10], or [11], Chapter 4, Theorem 4.1) in the following form:

Lemma 3.2. Let \( f \) be a real analytic, real-valued function defined in an open set \( V \) in \( \mathbb{R}^d \). Let \( Z_f = \{ x \in V : f(x) = 0 \} \). Then, for every compact \( K \subset V \), there exist positive constants \( c, m \) such that

\[
|f(x)| \geq c d(x, Z_f)^m
\]

when \( x \in K \).

Lemma 3.2 generalizes easily to the case when \( V \) is an open subset in a real analytic manifold and the distance is defined by a real analytic Riemannian metric.

To prove the left-hand side inequality in (6) let \( f : \tilde{X} \times A \to \mathbb{R} \) be given by

\[
f(x, z) = |\pi(x) - z|^2
\]

and \( K := \tilde{\Omega} \times \text{(compact neighborhood of } \Omega \cap A) \).

Clearly \( Z_f \subset \tilde{A} \times A \). When \( x \in \tilde{\Omega} \) and \( z \) is the nearest point to \( \pi(x) \) in \( A \), we have

\[
f(x, z) = |\pi(x) - z|^2 = d(\pi(x), A)^2 \geq c d((x, z), Z_f)^m \geq c d_{\tilde{A}}(x)^m.
\]

If we write \( m = 2t \) for some \( t > 0 \) constant, then we obtain from this last estimate the left-hand side inequality in (6).

To prove the left-hand side inequality in (7), we consider the unit sphere bundle \( S^r(\tilde{X}) \) in \( \bigwedge^r T\tilde{X} \). Recall \( \tilde{X} \) has a real analytic metric \( \sigma \) so \( S^r(\tilde{X}) \) becomes a real analytic manifold. We choose a metric on \( S^r(\tilde{X}) \) such that the projection \( p : S^r(\tilde{X}) \to \tilde{X} \) is distance decreasing. For \( \nu = (x, \xi_x) \) on the unit sphere bundle \( S^r(\tilde{X}) \), we set \( f(\nu) := |\pi_* \xi_x|^2_{\pi(p(\nu))} \) and let \( K := p^{-1}(\overline{\Omega}) \). Clearly, \( Z_f \subset p^{-1}(\tilde{A}) \).

It follows that \( |\pi_* \xi_x|^2_{\pi(p(\nu))} = f(\nu) \geq c d(\nu, Z_f)^R \geq c d(p(\nu), \tilde{A})^R \) when \( \nu \in K \). Write \( 2M = R \) for some \( M > 0 \) constant. The general case follows by applying this last inequality to \( \frac{\nu}{|\nu|} \) for \( \nu \neq 0, v \in \bigwedge^r T_x(\Omega) \).

Estimate (8) will be derived from (7) and the following remark:

Remark. Let \( T : V \to W \) be a linear isomorphism of normed spaces such that \( \|Tv\| \geq c\|v\| \) for \( v \in V \) and \( c > 0 \) constant. Then \( B_W(0, c) \subset T(B_V(0, 1)) \), where by \( B_V(0, 1) \) we denote the unit ball in \( V \) and \( B_W(0, c) \) is the ball in \( W \), centered at 0 and having radius \( c \).
Using (7) and applying the above remark to \( \pi_\ast : \Lambda^r T_x(\tilde{\Omega} \setminus \tilde{A}) \to \Lambda^r T_{\pi(x)}(\Omega \setminus A) \)
we obtain for \( x \in \Omega \setminus A \), \( a_{\pi(x)} \in \Lambda^r T_{\pi(x)}(\Omega \setminus A) \):
\[
|\pi^* a|_{x, \sigma} = \max \{ |(a_{\pi(x)}, \pi_* v)| : |v|_{x, \sigma} \leq 1, v \in \Lambda^r T_x(\tilde{\Omega} \setminus \tilde{A}) \} \\
\geq \max \{ |(a_{\pi(x)}, w)| : |w|_{\pi(x)} \leq c d_{\tilde{A}}(x) \}, w \in \Lambda^r T_{\pi(x)}(\Omega \setminus A) \} \\
= c d_{\tilde{A}}(x) |a|_{\pi(x)}.
\]

This result applies in particular to the volume form in \( \Omega \setminus A \) and gives
\[
(9) \quad c_1 d_{\tilde{A}}(x)^M d\tilde{V}_{x, \sigma} \leq (\pi^* dV)_x \leq C_1 d\tilde{V}_{x, \sigma}.
\]

4. Proof of Theorem 1.1

Given \( N_0 \in \mathbb{N} \), choose \( k \geq M + t \frac{M}{N_0} \geq 0 \), with \( t, M \) as in Lemma 3.1. Then by Proposition 1.3, there exists \( \ell \geq k \) such that \( H^q(\tilde{\Omega}, J^k \Omega^p) \to H^q(\tilde{\Omega}, J^k \Omega^p) \) is the zero homomorphism. Choose \( N \in \mathbb{N} \) such that \( N \geq 2n\ell + M_1 \), where \( M_1 \) is as in (3).

The proof of Theorem 1.1 will be based on the following change of variables result:

**Lemma 4.1.** Let \( W, W' \) be orientable, Riemannian manifolds and let \( F : W \to W' \) be an orientation-preserving diffeomorphism. Let \( dV, dV' \) denote the corresponding volume elements of \( W, W' \), respectively. For \( f \in L^1(W', dV') \) we have
\[
(10) \quad \int_W f dV' = \int_W (f \circ F) F^*(dV').
\]

Since \( \pi : \tilde{\Omega} \setminus \tilde{A} \to \Omega \setminus A \) is a biholomorphism and orientation-preserving map—as long as we choose appropriate orientations on \( \Omega \setminus A \), \( \tilde{\Omega} \setminus \tilde{A} \)—for any \( f \) satisfying \( \|f\|_{N, \Omega^*} < \infty \) we have (by applying Lemma 4.1)
\[
\int_{\Omega \setminus A} |f|^2 d_{\tilde{A}}^N dV = \int_{\tilde{\Omega} \setminus \tilde{A}} |f|^2_{\pi(x)} d_{\tilde{A}}(\pi(x))^{-N} (\pi^* dV)_x.
\]

Using the fact that
\[
|f|_{\pi(x)} \geq C^{-1} |\pi^* f|_{x, \sigma} \quad \text{(right-hand side of (8))},
\]
\[
d_{\tilde{A}}(\pi(x))^{-1} \geq C'\ell^{-1} d_{\tilde{A}}(x) \quad \text{(right-hand side of (6))},
\]
\[
(\pi^* dV)_{x, \sigma} \geq c_1 d_{\tilde{A}}^{M_1}(x) d\tilde{V}_{x, \sigma} \quad \text{(left-hand side of (9))},
\]

we obtain
\[
\|f\|^2_{N, \Omega^*} \geq c'' \int_{\tilde{\Omega} \setminus \tilde{A}} |\pi^* f|^2_{\tilde{A}} d_{\tilde{A}}^{M_1-N} d\tilde{V}_{x, \sigma}
\]

for some \( c'' > 0 \) constant. Since \( N \) was chosen such that \( N \geq M_1 \), we see that \( \overline{\partial} \pi^* f = 0 \) on \( \tilde{\Omega} \). It is not hard to show that \( \pi^* f \in J^k \mathcal{L}_{p,q}(\tilde{\Omega}) \). By Proposition 1.3 we know that there exists \( v \in J^k \mathcal{L}_{p,q-1}(\tilde{\Omega}) \) such that \( \overline{\partial} v = \pi^* f \) in \( \tilde{\Omega} \). Set \( u := (\pi^{-1})^* v \).
Then \( \overline{\partial} u = f \) in \( \Omega^* \) and for any \( \Omega' \subset \subset \Omega \) we have
\[
\int_{\Omega'} |u|^2 d^{N_0}_A dV = \int_{\Omega' \setminus \overline{A}} |u|_{\pi(x)}^2 d^{N_0}_A (\pi(x)) \pi^* (dV) \leq \int_{\Omega' \setminus \overline{A}} d^{-tN_0-2M} |v|^2_{x,\sigma} d\tilde{V}_{x,\sigma} \leq \int_{\Omega' \setminus \overline{A}} d^{-2k} |v|^2_{x,\sigma} d\tilde{V}_{x,\sigma} < \infty.
\]

To pass from the 1st line to the 2nd one we use the fact that
\[
|u|_{\pi(x)} \leq c^{-1} d^{-M}_A (x) |v|_{x,\sigma}, \quad d^{N_0}_A (\pi(x)) \leq c' - N_0 d^{-tN_0}_A (x)
\]
and that \( (\pi^* dV)_{x,\sigma} \leq C_1 d\tilde{V}_{x,\sigma} \).

To conclude the proof of Theorem 1.1 we shall need the following lemma:

**Lemma 4.2.** Let \( M \) be a complex manifold and let \( E \) and \( F \) be Frechet spaces of differential forms (or currents) of type \( (p,q-1), (p,q) \), whose topologies are finer (possibly strictly finer) than the weak topology of currents. Assume that for every \( f \in F \), the equation \( \overline{\partial} u = f \) has a solution \( u \in E \). Then, for every continuous seminorm \( p \) on \( E \), there is a continuous seminorm \( q \) on \( F \) such that the equation \( \overline{\partial} u = f \) has a solution with \( p(u) \leq q(f) \) for every \( f \in F \), \( q(f) > 0 \).

**Proof.** Set \( G = \{ (u, f) \in E \times F : \overline{\partial} u = f \} \). Then \( G \) is closed in \( E \times F \). To see this, let \( (u_\nu, f_\nu) \in G \) with \( u_\nu \to u \in E \), \( f_\nu \to f \) in \( F \). For test forms \( \phi \in C^0_0(n-p, n-q)(M) \) we get
\[
\int_M f \wedge \phi = \lim_{\nu \to \infty} \int_M f_\nu \wedge \phi = \lim_{\nu \to \infty} (-1)^{p+q} \int_M u_\nu \wedge \overline{\partial} \phi = (-1)^{p+q} \int_M u \wedge \overline{\partial} \phi,
\]
so \( \overline{\partial} u = f \) weakly.

Thus, \( G \) is a Frechet space and the bounded surjection \( \pi_2 : G \to F \) is open must be open. The set \( \pi_2(\{(u, v) \in G : p(u) < 1\}) \) is an open neighborhood of \( 0 \) in \( F \), and contains \( \{ f : q(f) \leq 1 \} \) for some continuous seminorm \( q \). Let \( f \in F \), \( 0 < q(f) = c \). Then \( q(c^{-1} f) = 1 \), so by the previous argument there exists a solution \( c^{-1} u \) satisfying \( \overline{\partial} (c^{-1} u) = c^{-1} f \) with \( p(c^{-1} u) < 1 \), i.e. \( p(u) < c = q(f) \). \( \square \)

When \( F \) is a Banach space with norm \( \| \cdot \| \), we conclude that, given a seminorm \( p \), there is a constant \( C > 0 \) such that \( \{ f : \| f \| \leq C^{-1} \} \subset \overline{G}(\{ u : p(u) \leq 1 \}) \), so \( \overline{\partial} u = f \) has a solution \( u \) with \( p(u) \leq C \| f \| \). Applying this result to our situation, we see that if \( \overline{\partial} f = 0 \), \( \| f \|_{\Omega, N} < \infty \) and \( \Omega_0 \subset \subset \Omega \), we obtain a solution \( u \) to \( \overline{\partial} u = f \) in \( L^2_{p,q-1}(\Omega^*) \) with \( \| u \|_{\Omega_0, N_0} \leq c \| f \|_{\Omega, N} \).

5. Applications of Theorem 1.1

We apply Theorem 1.1 to the case where \( A \cap \overline{\Omega} \) is a finite subset of \( \overline{\Omega} \) with \( \partial \Omega \cap \partial A = \emptyset \), \( \Omega \subset \subset X \) is Stein and \( \overline{\Omega} \) has a Stein neighborhood \( \Omega^* \).

**Proposition 5.1.** With \( N_0, N \) as in Theorem 1.1 and \( \overline{\partial} f = 0 \) on \( \Omega^* \) and \( \| f \|_{\Omega, N} < \infty \), there is a solution \( u \) of \( \overline{\partial} u = f \) on \( \Omega^* \) with \( \| u \|_{\Omega, N_0} \leq c \| f \|_{\Omega, N} \), \( c \) independent of \( f \). In other words, we obtain a weighted \( L^2 \) estimate for \( u \) on all of \( \Omega \).
PROOF. Choosing $\Omega_0 \subset \subset \Omega$ containing $A \cap \Omega$, we have a solution $u_0$ in $L^2_{\text{loc}}(\Omega^*)$ with $\|u_0\|_{\Omega_0,N_0} \leq c\|f\|_{\Omega,N}$. We introduce a cut-off function $\chi \in C^\infty(X)$ such that $\chi = 1$ on $X \setminus \Omega_0$ but $\chi = 0$ near $A \cap \Omega$. Set $f_1 = \overline{\partial}(\chi u_0)$. Clearly, $\|f_1\|_{L^2(\Omega)} \leq c\|f\|_{\Omega,N}$ and $f_1 = 0$ near $\Omega_0 \cap A$.

Let $\pi: \tilde{X} \to X$ be a desingularization of $X$ and consider the equation $\overline{\partial}v = \pi^* f_1$ on $\tilde{\Omega}$. Let $\tilde{\Omega}_0 := \pi^{-1}(\Omega_0)$. The equation $\overline{\partial}v = \pi^* f_1$ is solvable in $L^2_{p,q-1}(\tilde{\Omega}_0)$. We can assume that $\tilde{\Omega}$ can be exhausted by smoothly bounded strongly pseudoconvex domains $\tilde{\Omega}_j := \{z \in \tilde{\Omega}; \phi < c_j\}$, where $c_j$ are real numbers, $\phi$ is an exhaustion function for $\tilde{\Omega}$, of class $C^3(\tilde{\Omega})$, strictly plurisubharmonic outside a compact subset, and also that $b\tilde{\Omega}_0$ is smooth and strongly pseudoconvex and contained in each $\tilde{\Omega}_j$. To each $\tilde{\Omega}_j$ we apply Theorem 3.4.6 in \cite{9} and we obtain a solution $\tilde{v}_j$ to the equation $\overline{\partial}\tilde{v}_j = \pi^* f_1$ in $\tilde{\Omega}_j$ with

$$\int_{\tilde{\Omega}_j} |v_j|^2 e^{-\phi} d\tilde{V}_\sigma \leq C \int_{\tilde{\Omega}} |\pi^* f_1|^2 d\tilde{V}_\sigma,$$

where $C$ is a positive constant independent of $j, f$ (this follows from a careful inspection of the proof of Theorem 3.4.6 in \cite{9}).

Consider the trivial extensions $v_j^*$ of $v_j$ outside $\tilde{\Omega}_j$. Let $v$ be a weak limit of $v_j^*$. Then

$$\int_{\tilde{\Omega}} |v|^2 e^{-\phi} d\tilde{V}_\sigma \leq C \int_{\tilde{\Omega}} |\pi^* f_1|^2 d\tilde{V}_\sigma$$

and $\overline{\partial}v = \pi^* f_1$ in $\tilde{\Omega}$. So there is a solution $v$ satisfying $\|v\|_{L^2(\tilde{\Omega})} \leq c\|f_1\|$. Then $w := (\pi^{-1})^* v$ satisfies $\overline{\partial}w = f_1$ in $\Omega^*$ but we no longer have control of its $L^2$-norm near $A \cap \Omega$. Choose another cut-off function $\chi_0$ such that $\chi_0 = 1$ on $\text{supp} \chi$ but $\chi_0 = 0$ near $\Omega_0 \cap A$. Then

$$\overline{\partial}((1 - \chi)u_0) + \chi_0 (\pi^{-1})^* v = (1 - \chi) f - \overline{\partial} \chi \wedge u_0 + \overline{\partial} \chi_0 \wedge (\pi^{-1})^* v + \chi f + \overline{\partial} \chi \wedge u_0 = f + \overline{\partial} \chi_0 \wedge (\pi^{-1})^* v.$$

Finally we may solve $\overline{\partial}v_1 = \overline{\partial} \chi_0 \wedge (\pi^{-1})^* v$ in $\Omega^*$ (apply Theorem 1.1 to the trivial extension of $\overline{\partial} \chi_0 \wedge (\pi^{-1})^* v$ in $\Omega^*$):

$$\|v_1\|_{\Omega,N_0} \leq C \|\overline{\partial} \chi_0 \wedge (\pi^{-1})^* v\|_{\Omega,N} \leq c' \|\overline{\partial} \chi_0 \wedge (\pi^{-1})^* v\|_{L^2(\Omega)} \leq C\|f\|_{\Omega,N}$$

since $\overline{\partial} \chi = 0$ near $A$. Thus, $u := (1 - \chi)u_0 + \chi_0 (\pi^{-1})^* v - v_1$ is a solution with the required estimate.

6. GENERALIZATIONS

Theorem 1.1 and Corollary 1.2 also extend to the case when $\Omega$ is a relatively compact domain in a complex space $X$ of pure dimension $n$ with strictly pseudoconvex boundary. We know that $\Omega$ contains a maximal positive-dimensional compact variety $B$ and let $A$ be a nowhere open analytic subvariety of $X$ containing $X_{\text{sing}}$ and $B$. Then Theorem 1.1 carries over verbatim to the case described above. The proof needs the following modifications: Let $\Omega 

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desingularization of $X_0$ such that $\pi^{-1}(A)$ is a hypersurface with normal crossings. To obtain a proof of Proposition 1.3 (vanishing cohomology), we need to consider direct images $R^q(\phi \circ \pi)_* S$ on the Stein space $X_1$ and their annihilator ideal $A$ for $S$ coherent on $\tilde{X}$. Then, the proof carries over.

Corollary 1.2, for the case when $X_{\text{sing}} \cap b\Omega$ is empty and with $A = B \cup (X_{\text{sing}} \cap \Omega)$, follows exactly as above.

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Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109-1109
E-mail address: fornaess@umich.edu

Department of Mathematics, University of Oslo, P.B 1053 Blindern, Oslo, N-0316 Norway
E-mail address: nilsov@math.uio.no

Department of Mathematics, Georgetown University, Washington, DC 20057
E-mail address: sv46@georgetown.edu

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