POINTWISE UNIFORMLY ROTUND NORMS

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(Communicated by Jonathan M. Borwein)

Abstract. It is shown that some properties of compact spaces \( K \), such as carrying a strictly positive measure or being descriptive, are closely related to renormings of \( C(K) \) or \( C(K)^* \), respectively, by pointwise uniformly rotund norms.

Let \( X \) be a Banach space. If \( F \) is a closed, weak* dense subspace of \( X^* \), then a norm \( \| \cdot \| \) on \( X \) is said to be \( F \)-uniformly rotund (UR\(^F \)) if \( \lim_{n \to \infty} f(x_n - y_n) = 0 \) for every \( f \in F \) and every \( x_n, y_n \in X \) such that
\[
\|x_n\| = \|y_n\| = 1 \quad \text{and} \quad \lim_{n \to \infty} \|x_n + y_n\| = 2.
\]
The norm is called pointwise uniformly rotund (p-UR) if it is UR\(^F \) for some weak* dense \( F \subseteq X^* \) (see [20], [19]). In particular, the norm on \( X = Y^* \) is called weak* uniformly rotund if it is UR\(^Y \) with the canonical embedding \( Y \subseteq X^* = Y^{**} \). The norm \( \| \cdot \| \) is called uniformly rotund in every direction (URED) if \( \lim_{n \to \infty} \|x_n - y_n\| = 0 \) for every \( x_n, y_n \in X \) such that \( \|x_n\| = \|y_n\| = 1 \), \( \lim_{n \to \infty} \|x_n + y_n\| = 2 \), and \( x_n - y_n \in \text{span}\{z_0\} \) for some \( z_0 \in X \).

A measure \( \mu \) on a compact space \( K \) is said to be strictly positive if \( \mu(U) > 0 \) for every nonempty open set \( U \subseteq K \). A compact space \( K \) is called a uniform Eberlein compact if \( K \) is homeomorphic to a weakly compact set in a Hilbert space [3]. A family \( \mathfrak{N} \) of subsets of a compact space \( K \) is said to be a network if every open set in \( K \) is a union of members of \( \mathfrak{N} \). A compact space \( K \) is descriptive if there are closed sets \( A_n \subseteq K \) and a network \( \mathfrak{N} = \bigcup_n \mathfrak{N}_n \) such that every \( \mathfrak{N}_n \) consists of relatively open and pairwise disjoint sets in \( A_n \) [13, Lemma 3.1]. A compact space \( (K, \tau) \) is fragmentable, if there is a metric \( \rho \) on \( K \) such that for every \( \varepsilon > 0 \) and every nonempty subset \( M \subseteq K \) there exists a \( \tau \)-open set \( \Omega \subseteq K \) such that \( M \cap \Omega \) is nonempty and has \( \rho \)-diameter less than \( \varepsilon \) (17, 16). A Banach space \( X \) is weakly compactly generated if there is a weakly compact set \( K \subseteq X \) such that \( X = \text{span}K \). For unexplained terms used in this paper we refer to [7] and [9].

Clearly, every p-UR norm is URED. URED norms are used in fixed point theory; see e.g. [3]. It turned out that p-UR norms can be used in characterizing some properties of compact spaces as follows.

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Received by the editors March 25, 2003.

2000 Mathematics Subject Classification. Primary 46B03, 46B26, 46E05.

Key words and phrases. Pointwise uniformly rotund norm, strictly positive measure, uniform Eberlein compacts, descriptive compacts, fragmentability.

This research was supported by NSERC 7926, FS Chia Ph.D. Scholarship for 2002/2003 and GAUK 277/2001, written as part of the author’s Ph.D. thesis under the supervision of Professor N. Tomczak-Jaegermann and Professor V. Zizler.

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**Theorem 1.** The space $C(K)$ of continuous functions on a compact space $K$ admits an equivalent pointwise uniformly rotund norm if and only if $K$ carries a strictly positive Radon probability.

**Theorem 2.** (a) For a compact space $K$, the space $C(K)^*$ admits a pointwise uniformly rotund (in general nondual) norm if and only if the space $L_1(\mu)$ is separable for every Radon probability on $K$.

(b) If $K$ is a descriptive compact space, then $C(K)^*$ admits an equivalent dual pointwise uniformly rotund norm.

(c) There is a nondescriptive (fragmentable) compact space $K$ such that $C(K)^*$ admits an equivalent dual pointwise uniformly rotund norm.

(d) If $K$ is a fragmentable compact space, then $C(K)^*$ admits an equivalent pointwise uniformly rotund norm. Consequently, the space $L_1(\mu)$ is separable for every Radon probability $\mu$ on a fragmentable compact $K$.

**Theorem 3.** Let $\mu$ be a finite measure. Then $L_1(\mu)$ admits an equivalent pointwise uniformly rotund norm if and only if $L_1(\mu)$ is separable.

**Theorem 4.** If a Banach space $X$ admits an equivalent pointwise uniformly rotund norm, then every weakly compact subset of $X$ is a uniform Eberlein compact.

For any finite measure $\mu$, the space $L_1(\mu)$ admits an equivalent URED norm by [13]; see also [5] Theorem 7.5.2). Consequently, by Theorem 3, nonseparable $L_1(\mu)$ admits an equivalent URED norm and no p-UR norm. This is connected to [20] Problem 1. Moreover, every weakly compact subset of $L_1(\mu)$ is a uniform Eberlein compact [1] Section 4. Thus the converse of Theorem 3 does not hold even in WCG spaces. This is connected to [1] Problem 2.9.

There are fragmentable compact spaces such that $C(K)^*$ admits no dual strictly convex norm (e.g. $[0, \omega_1]$; see [5] Theorem 7.5.2)) and thus no dual p-UR norm (cf. Theorem 2(b) and (d)). It was proved in [21] Theorem 2 that $L_1(\mu)$ is separable for every Radon probability $\mu$ on a compact subset of the first Baire class. Thus split interval $S(I)$ is a nonfragmentable compact satisfying the conclusion of Theorem 2(d). By [13] and Kakutani’s Theorem, $C(K)^*$ admits an equivalent URED norm for every compact $K$. The space $C([0,1][0,1])^*$ does not admit an equivalent p-UR norm, as $L_1(\lambda)$ is nonseparable, where $\lambda$ is a product of Lebesgue measures on $[0,1]$.

By [13], if $C(K)^*$ admits a dual weak* locally uniformly rotund norm, then $K$ is descriptive. Thus by Theorem 2(b), $C(K)^*$ admits an equivalent dual p-UR norm. By [7] Theorem 5.3.1, if $C(K)^*$ admits a dual strictly convex norm, then $K$ is fragmentable and thus, by Theorem 2(d), $C(K)^*$ admits an equivalent p-UR norm. We do not know if in this case $C(K)^*$ admits an equivalent dual p-UR norm.

As shown in [14], there is a reflexive Banach space that does not admit any equivalent norm that is uniformly rotund in every direction. Thus this space does not admit any equivalent p-UR norm, although it admits an equivalent dual locally uniformly rotund norm.

**Proof of Theorem 3.** By Šmulian’s type theorem [5] Theorem 2.6.7], if the norm $\|\cdot\|$ on a Banach space $X$ is UR$^F$, then the limit

$$
\lim_{t \to 0} \frac{\|f + tg\|^- - \|f\|^*}{t}
$$

exists for every $g \in X^*$, $\|g\|^* = 1$ and is uniform in $f \in F$, $\|f\|^* = 1$, where $\|\cdot\|^*$ is the dual norm to $\|\cdot\|$. In particular, the norm $\|\cdot\|^*$ is uniformly Gâteaux smooth on
F}. By [8], the dual unit ball \( B_{F^*} \) is a uniform Eberlein compact in weak* topology of \( F^* \). Hence, by [2], \( F \) is a subspace of weakly compactly generated space \( C(B_{F^*}) \).

For a given weak* dense subspace \( F \subset X^* \), let an operator \( T : X \to F^* \) be given by \( T = r \circ i \), where \( i : X \to X^{**} \) is the canonical inclusion and \( r : X^{**} \to F^* \) is the canonical restriction. The operator \( T \) is one-to-one and \( \sigma(X, X^*) = \sigma(F^*, F) \) continuous. Since \( B_{F^*} \) is a uniform Eberlein compact in \( \sigma(F^*, F) \) topology, \( T(K) \) is a uniform Eberlein compact for every weakly compact set \( K \subset X \). Hence \( K \) is a uniform Eberlein compact, and the proof of Theorem 3 is finished.

Note that if \( F \) admits a uniformly Gâteaux smooth norm, then \( F^* \) admits a weak* uniformly rotund norm (see [5, Theorem 2.6.7]), and thus the norm \( ||.|| \) on \( X \) defined by

\[
||x||^2 = ||x||^2 + ||Tx||^2
\]

is an equivalent UR \( F \) norm.

**Proof of Theorem 1.** Let \( \mu \) be a strictly positive Radon probability measure on \( K \). Then the identity map \( I : C(K) \to L_2(\mu) \) is one-to-one and with a dense range. Thus the norm \( ||.|| \) defined on \( C(K) \) by

\[
||f||^2 = ||f||^2 + ||f||_{L^2(\mu)}^2
\]

is an equivalent UR \( F \) norm, where \( F = \text{span} \{f : L_2(\mu) \subset C(K)\}^\ast \).

Conversely, if \( C(K) \) admits an equivalent UR \( F \) norm, then \( F \) is a subspace of a weakly compactly generated space (see the proof of Theorem 1). Thus \( \ell_1(G) \) is not a subspace of \( F \) for any uncountable set \( G \); see [9, Chapter 11]. By [17, Lemma 1.3], there is a Radon probability \( \mu \) on \( K \) such that \( F \subset L_1(\mu) \subset C(K)^\ast \). Note that the measure \( \mu \) is strictly positive as \( F \) is weak* dense in \( C(K)^\ast \). This concludes the proof of Theorem 1.

**Proof of Theorem 3.** If \( L_1(\mu) \) is separable, then it admits an equivalent p-UR norm with the same proof as that of [5, Corollary 2.6.9]. Assume that \( L_1(\mu) \) is nonseparable and admits an equivalent UR \( F \) norm. We claim that \( F \) is norm separable. This means that \( L_1(\mu)^\ast \) is weak* separable, which is a contradiction with [5, Theorem 11.3].

To prove our claim, let us identify \( L_1(\mu)^\ast \equiv L_\infty(\mu) \) with \( C(\Omega) \), where \( \Omega \) is a Stonian space for measure \( \mu \) (see [11, Appendix B] for details). Since the measure \( \mu \) is finite, the space \( L_1(\mu)^\ast \) admits a strictly positive uniformly rotund norm. By Theorem 1, \( \Omega \) carries a strictly positive probability measure. In particular, \( \Omega \) has a property ccc, that is, every collection of pairwise disjoint open sets of \( \Omega \) is countable. Thus we only need to prove the following fact, which is a version of [17, Theorem 4.5(a) and Proposition 4.7].

**Fact 5.** Let \( \Omega \) be a compact space with property ccc and let \( X \subset C(\Omega) \) be isomorphic to a subspace of a weakly compactly generated space. Then \( X \) is separable.

**Proof.** By [7, Theorem 7.2.2], there exists a Markushevich basis of \( X \), i.e., a biorthogonal system \( \{x_\gamma, f_\gamma\}_{\gamma \in \Gamma} \subset X \times X^* \) such that \( \text{span} \{x_\gamma ; \gamma \in \Gamma\} = X \) and \( \{f_\gamma ; \gamma \in \Gamma\} \) separates points of \( X \). We may and do assume that \( \|x_\gamma\| = 3 \). By [10], there exists a decomposition of \( \Gamma = \bigcup_{n=1}^\infty \Gamma_n \) such that, for every \( n \in \mathbb{N} \),

\[
\emptyset \neq \{x_\gamma ; \gamma \in \Gamma_n\} \subset B_{X^{**}} \setminus \{x_\gamma ; \gamma \in \Gamma_n\} \subset B_{X^{**}}.
\]
Take $n$ such that $\Gamma_n$ is uncountable and define open sets $U_\gamma \subset \Omega$ by $U_\gamma = \{ \omega \in \Omega : |x_\gamma(\omega)| > 2 \}$ for $\gamma \in \Gamma_n$. Since $\Omega$ has ccc, there is a sequence $\{ \gamma_i \}_{i=1}^\infty \subset \Gamma_n$ such that $\bigcap_{i=1}^\infty U_{\gamma_i} \neq \emptyset$ [14, Lemma 4.2]. Thus there is $\omega \in \bigcap_{i=1}^\infty U_{\gamma_i}$ such that $|x_{\gamma_i}(\omega)| > 2$ for every $i \in \mathbb{N}$, a contradiction with (2). Thus $X$ is separable. This concludes the proof of Fact $[3]$ and the proof of Theorem $[3]$ is complete.

**Remark.** After submission, we learned that Theorem $[3]$ was proved by a different method in [6, Theorem 2.11].

**Proof of Theorem [2](a).** Theorem follows easily from Theorem $[3]$ and the Kakutani’s Theorem; see e.g. [15].

**Proof of Theorem [2](b).** Let $||.||_1$ be the canonical dual norm on $C(K)^*$. Fix the family $\mathcal{R} = \bigcup_{n=1}^\infty \mathcal{R}_n$, given by the definition of descriptivity of $K$. Consider $\mathcal{R} \subset C(K)^{**}$ by the action $N(\mu) = \mu(N)$ for $N \in \mathcal{R}$ and $\mu \in C(K)^*$. Let $F = \text{span}\mathcal{R}$. We will show that there is an equivalent dual UR norm on $C(K)^*$.

We claim that $F$ is weak* dense in $C(K)^{**}$. Indeed, $\mu(G) = 0$ for all open $G \subset K$ whenever $\mu(N) = 0$ for all $N \in \mathcal{R}$, as $\mathcal{R}$ is a $\sigma$-isolated network consisting of relatively open pairwise disjoint sets.

Define a norm $||.||$ on $C(K)^*$ in four steps, similarly as in [13] Proof of Theorem 3.3. First, for every $n \in \mathbb{N}$, define a convex function $F_n$ on $C(K)^*$ by

$$F_n(\mu)^2 = \sum_{N \in \mathcal{R}_n} |\mu|(N)^2.$$  

The function $F_n$ is weak* lower semi-continuous on $C(A_n)^*$. Second, for every $n, m \in \mathbb{N}$, define a weak* lower semi-continuous seminorm $||.||_{m,n}$ on $C(K)^*$ by

$$||\mu||_{m,n}^2 = \inf \left\{ ||\mu - u||_1^2 + m^{-1}F_n(u)^2 ; u \in C(A_n)^* \right\}.$$  

Third, define an equivalent dual norm on $C(K)^*$ by

$$||\mu||_+^2 = ||\mu||_1^2 + \sum_{m,n \in \mathbb{N}} 2^{-m-n}||\mu||_{m,n}^2.$$  

**Claim 6.**

\begin{equation}
\lim_{\omega \to \infty} (\mu_\omega - \nu_\omega)(N) = 0,
\end{equation}

for all $n \in \mathbb{N}, N \in \mathcal{R}_n$ and all positive measures $\mu_\omega, \nu_\omega \in C(K)^*, \omega \in \mathbb{N}$, such that $||\mu_\omega||_1 \leq 1, ||\nu_\omega||_1 \leq 1$, and

\begin{equation}
\lim_{\omega \to \infty} 2||\mu_\omega||_+^2 + 2||\nu_\omega||_+^2 - ||\mu_\omega + \nu_\omega||_+^2 = 0.
\end{equation}

Once the claim is proved, finally define a norm $||.||$ by

\begin{equation}
||\mu||^2 = \inf \left\{ ||\mu_1||_+^2 + ||\mu_2||_+^2 ; \mu_1 \in M(K), \mu_1 \geq 0, \mu = \mu_1 - \mu_2 \right\}.
\end{equation}

Using the compactness argument, it follows from the weak* lower semicontinuity of $||.||_+$ that the infimum in (5) is attained for every $\mu \in C(K)^*$ and that the norm $||.||$ is an equivalent dual norm on $C(K)^*$. Thus [23] holds whenever $||\mu_\omega||_+ = 1 = ||\nu_\omega||$ and $\lim_{\omega \to \infty} ||\mu_\omega + \nu_\omega|| = 2$. Hence the norm $||.||$ is UR^E.
Proof of Claim. Fix \( n \in \mathbb{N} \) and \( N \in \mathfrak{N}_n \). From (13) and a convexity argument,
\[
\lim_{\omega \to \infty} 2\|\mu_\omega\|_{m,n}^2 + 2\|\nu_\omega\|_{m,n}^2 - \|\mu_\omega + \nu_\omega\|_{m,n}^2 = 0,
\]
for every \( m \in \mathbb{N} \). From a compactness argument, for every \( \omega, m \in \mathbb{N} \), there are positive measures \( u_\omega^{m,n}, v_\omega^{m,n} \in C(A_n)^* \) such that
\[
\|\mu_\omega\|_{m,n}^2 = \|\mu_\omega - u_\omega^{m,n}\|^2 + m^{-1} F_n(u_\omega^{m,n})^2 \quad \text{and} \quad \|\nu_\omega\|_{m,n}^2 = \|\nu_\omega - v_\omega^{m,n}\|^2 + m^{-1} F_n(v_\omega^{m,n})^2.
\]
Consequently,
\[
F_n(u_\omega^{m,n}) \leq m\|\mu_\omega\|_{m,n} \leq m\|\mu_\omega\|_1 \leq m
\]
and similarly \( F_n(v_\omega^{m,n}) \leq m \). By passing to a subsequence, we may assume that
\[
\lim_{\omega \to \infty} \|\mu_\omega\|_{m,n} = d_{m,n} = \lim_{\omega \to \infty} \|\nu_\omega\|_{m,n}.
\]
The sequence \( \{\|\mu\|_{m,n}\}_{m=1}^{\infty} \) is nonincreasing for every measure \( \mu \in C(K)^* \). Thus there is \( d_n = \lim_{m \to \infty} d_{m,n} \). Choose \( \varepsilon > 0 \) and let \( m_0 \in \mathbb{N} \) be such that \( d_{m_0,n} < d_n + \varepsilon \). We will estimate \( |(\mu_\omega - \nu_\omega)(N)| \) by
\[
|\mu_\omega - u_\omega^{m_0,n})(N)| + |(u_\omega^{m_0,n} - v_\omega^{m_0,n})(N)| + |(v_\omega^{m_0,n} - \nu_\omega)(N)|.
\]
By a convexity argument and (6), (7), (8),
\[
\lim_{\omega \to \infty} 2F_n(u_\omega^{m_0,n})^2 + 2F_n(v_\omega^{m_0,n})^2 - F_n(u_\omega^{m_0,n} + v_\omega^{m_0,n})^2 = 0.
\]
Since \( u_\omega^{m_0,n} \) and \( v_\omega^{m_0,n} \) are positive measures, by a convexity argument again
\[
\lim_{\omega \to \infty} |(u_\omega^{m_0,n} - v_\omega^{m_0,n})(N)| = 0.
\]
In order to estimate \( |(\mu_\omega - u_\omega^{m_0,n})(N)| \), consider a measure
\[
u = \mu_\omega|\mathcal{N} + u_\omega^{m_0,n}|_{K \setminus \mathcal{N}}
\]
in the definition of \( \|\mu_\omega\|_{m,n} \), where \( \mu|A \) means the restriction of \( \mu \) on \( A \subset K \). We get
\[
\|\mu_\omega\|_{m,n}^2 \leq \|(\mu_\omega - u_\omega^{m_0,n})|_{K \setminus \mathcal{N}}\|^2 + m^{-1} F_n(\mu_\omega|\mathcal{N} + u_\omega^{m_0,n}|_{K \setminus \mathcal{N}})^2
\]
\[
\leq \|(\mu_\omega - u_\omega^{m_0,n})|_{K \setminus \mathcal{N}}\|^2 + m^{-1} (F_n(\mu_\omega|\mathcal{N}) + F_n(u_\omega^{m_0,n}|_{K \setminus \mathcal{N}}))^2
\]
\[
\leq \|(\mu_\omega - u_\omega^{m_0,n})|_{K \setminus \mathcal{N}}\|^2 + m^{-1} (\mu_\omega(N) + F_n(u_\omega^{m_0,n}))^2
\]
\[
\leq \|(\mu_\omega - u_\omega^{m_0,n})|_{K \setminus \mathcal{N}}\|^2 + m^{-1} (1 + m_0)^2.
\]
Thus, for all \( m \in \mathbb{N} \),
\[
\limsup_{\omega \to \infty} \|(\mu_\omega - u_\omega^{m_0,n})|_{K \setminus \mathcal{N}}\|^2 \geq \lim_{\omega \to \infty} \|\mu_\omega\|_{m,n}^2 - m^{-1} (1 + m_0)^2,
\]
\[
\limsup_{\omega \to \infty} \|(\mu_\omega - u_\omega^{m_0,n})|_{K \setminus \mathcal{N}}\|^2 \geq d_{m,n}^2 - m^{-1} (1 + m_0)^2, \quad \text{and}
\]
\[
\limsup_{\omega \to \infty} \|(\mu_\omega - u_\omega^{m_0,n})|_{K \setminus \mathcal{N}}\|^2 \geq d_n^2.
\]
For all \( \omega \in \mathbb{N} \) we have
\[
|(\mu_\omega - u_\omega^{m_0,n})(N)| \leq \|(\mu_\omega - u_\omega^{m_0,n})|_{\mathcal{N}}\|_1
\]
\[
= \|\mu_\omega - u_\omega^{m_0,n}\|_1 - \|(\mu_\omega - u_\omega^{m_0,n})|_{K \setminus \mathcal{N}}\|_1
\]
\[
\leq \|\mu_\omega\|_{m_0,n} - \|(\mu_\omega - u_\omega^{m_0,n})|_{K \setminus \mathcal{N}}\|_1.
\]
The union above is a union of disjoint open sets and $|F|$.

**Claim 7.** Suppose that on a tree $T$ there is an increasing function $g : T \to \mathbb{R}$ which is constant on no strictly increasing sequence in $T$. Then there is an equivalent dual $p$-UR norm on $C_0(T)^*$.

**Proof of Claim 7.** The space $C_0(T)^*$ can be identified with $\ell_1(T)$ with the canonical dual norm $\|\mu\|_1 = \sum_{t \in T^*} |\mu(t)|$. Let us define $T^+$ as the set of successors and $T_0$ as the set of all $t \in T^+$ such that $g(t) > g(t^-)$. We may modify the function $g$ so that it takes rational values at all points of $T_0$.

We will show that there is an equivalent dual UR$^F$ norm where

$$F = \text{span}\left\{\{(s); s \in T^+\} \cup \{|s, \infty); s \in T_0\} \cup \{T\}\right\} \subset \ell_\infty(T).$$

We claim that $F$ is weak*-dense in $C_0(T)^{**}$. To prove it, let $\mu \in C(T)^*$ be such that $\mu(f) = 0$ for all $f \in F$. We want to show that $\mu(t) = 0$ for all $t \in T$. Choose $t \in T$ and put $A(t) = \{u; u \in (t, \infty), g(u) = g(t)\}$ and $B(t) = \min\{u \in (t, \infty); g(u) > g(t)\}$. We have

$$(t, \infty) = \bigcup_{u \in A(t)} \{u\} \cup \bigcup_{u \in B(t)} [u, \infty).$$

The union above is a union of disjoint open sets and $|\mu|$ is nonzero at most on countably many of them. Hence $\mu((t, \infty)) = 0$. Thus $\mu((t, \infty)) = 0$ for all $t \in T^+$. Since

$$(0, t) = T \setminus \bigcup_{s \leq t} \bigcup_{r \in s^+ \setminus (0, t)} [r, \infty),$$

we have that $\mu(((0, t)) = 0$ for all $t \in T$. Every limit element $t \in T$ is a limit of a sequence (of elements of $T_0$), thus $\mu(((0, t)) = 0$ for all $t \in T$. Hence $\mu(\{t\}) = 0$ for all $t \in T$.

For every $q \in \mathbb{Q}$, the wedges $[s, \infty)$ with $s \in T_0$ and $g(s) = q$ are disjoint, so we can define an equivalent dual norm on $C(T)^*$ by

$$\|\mu\|_F^2 = \|\mu\|_1^2 + \sum_{s \in T^+} \|\mu\{s\}\|^2_1 + \sum_{q \in \mathbb{Q}} c_q \left( \sum_{s \in T_0 \cap q^{-1}(q)} \|\mu\{s, \infty\}\|^2_1 \right),$$

where $c_q$ are some positive constants.

Let $\mu_n, \nu_n \in C_0(T)^*$ be positive elements such that $\|\mu_n\| \leq 1$, $\|\nu_n\| \leq 1$ and

$$\lim_{n \to \infty} 2\|\mu_n\|^2_1 + 2\|\nu_n\|^2_1 - \|\mu_n + \nu_n\|^2_1 = 0.$$

A standard convexity argument shows that

$$\lim_{n \to \infty} (\mu_n - \nu_n)(T) = 0, \lim_{n \to \infty} (\mu_n - \nu_n)(s) = 0,$$

for all $s \in T^+$, and

$$\lim_{n \to \infty} (\mu_n - \nu_n)((s, \infty)) = 0,$$

for any $s \in T_0$. Thus the norm $\|\cdot\|$ defined by (5) is UR$^F$. This concludes the proof of Claim 7.
Now, let $\Lambda$ be a tree defined in [11 Section 10] and let $K$ be its Alexandroff compactification. Then $C(K)^*$ admits an equivalent dual p-UR norm by Claim 7. The space $C(K)^*$ does not admit any equivalent dual locally uniformly rotund norm, since $C(K)$ does not admit an equivalent Fréchet smooth norm [11 Corollary 10.9]. Thus $K$ is not a descriptive compact space by [13 Corollary 4.9]. The proof of Theorem 2(c) is complete.

Proof of Theorem 2(d). Let $K$ be a fragmentable compact. By [3] Theorem 5.1.9 and Proof of Theorem 5.1.12(iii)], there is a family $U = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ of subsets of $K$ such that

1. $\mathcal{U}$ is a separating family, i.e. if $x \neq y \in K$, then there is $U \in \mathcal{U}$ such that $\#U \cap \{x, y\} = 1$;
2. $\mathcal{U}$ is a network;
3. for every $n \in \mathbb{N}$, $\mathcal{U}_n$ is an open partitioning, i.e. $\mathcal{U}_n = \{U_\xi; \xi < \xi_n\}$ is well ordered such that $U_\xi$ is contained and is relatively open in $K \setminus (\bigcup_{\eta < \xi} U_\eta)$ for every $\xi < \xi_n$ and $K = \bigcup_{\xi < \xi_n} U_\xi$;
4. for every $U \in \mathcal{U}_{m+1}$ there is $V \subset \mathcal{U}_n$ such that $U \subset V$.

As $\mathcal{U}_n$ is an open partitioning, it follows that

$$\sum_{U \in \mathcal{U}_n} \mu(U) = \mu(K).$$

Define equivalent norms on $C(K)^*$

$$\|\mu\|^2 = |\mu|^2(K) + \sum_{n=1}^{\infty} 2^{-n} \sum_{U \in \mathcal{U}_n} |\mu|^2(U)$$

and

$$\|\mu\|^2 = \inf\{\|\mu_1\|^2 + \|\mu_2\|^2; \mu_1, \mu_2 \in C(K)^*, \mu_i \geq 0, \mu = \mu_1 - \mu_2\}.$$  \hspace{1cm} (9)

From a definition of a norm $\|\cdot\|$ it follows that $\|\mu\|^2 = \|\mu^+\|^2 + \|\mu^-\|^2$. Let $F = \text{span}\{U; U \in \mathcal{U}\} \subset C(K)^{**}$. We will show that the norm $\|\cdot\|$ is UR$^F$. Note that $F \subset C(K)^{**}$ is weak* dense. Indeed, assume $\mu(U) = 0$ for all $U \in \mathcal{U}$ and let $G \subset K$ be an open set. Since $\mathcal{U}$ is a network, we have $G = \bigcup_{n} (\bigcup_{\mathcal{U}_n})$, where for every $n \in \mathbb{N}$, $\mathcal{U}'_n$ is a subfamily of $\mathcal{U}_n$. Moreover, by condition (4), we may assume that $\mathcal{U}'_n \cap \mathcal{U}'_m = \emptyset$ for $m \neq n$. Thus

$$\mu(G) = \mu\left(\bigcup_{n} (\bigcup_{\mathcal{U}'_n})\right) = \sum_{n} \mu\left(\bigcup_{\mathcal{U}'_n}\right) = \sum_{n} \sum_{U \in \mathcal{U}'_n} \mu(U) = 0,$$

where the third equality hold as $\mathcal{U}'_n$’s are relatively open partitioning. Thus, by a convexity argument, the norm $\|\cdot\|$ is UR$^F$.

The proof of Theorem 2(d) is complete.

References


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