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QUASI-HYPERBOLIC PLANES IN HYPERBOLIC GROUPS

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ABSTRACT. The hyperbolic plane $\mathbb{H}^2$ admits a quasi-isometric embedding into every hyperbolic group which is not virtually free.

The purpose of this note is to prove the following theorem which answers a question posed by P. Papasoglu:

Theorem 1. *The hyperbolic plane $\mathbb{H}^2$ admits a quasi-isometric embedding into a hyperbolic group if and only if the group is not virtually free.*

A map $f\colon X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is called a *quasi-isometric embedding* if there exist constants $\lambda \geq 1$ and $K \geq 0$ such that

$$\frac{1}{\lambda} d_X(x,y) - K \leq d_Y(f(x), f(y)) \leq \lambda d_X(x,y) + K$$

for all $x, y \in X$. A group is *virtually free* if it contains a free subgroup of finite index. We refer to [9] for the definition of hyperbolic groups and related concepts from the theory of Gromov hyperbolic spaces. Every Gromov hyperbolic space X has a boundary $\partial_\infty X$ which carries a class of canonical *visual metrics*. These metrics are bi-Lipschitz equivalent to distance functions of the form

$$d_{w,\epsilon}(a,b) = \exp(-\epsilon(a,b)_w), \quad a, b \in \partial_\infty X,$$

where $w \in X$ is a base point, $\epsilon > 0$ is sufficiently small, and $(a,b)_w$ denotes the Gromov product of the points a and b with respect to w (cf. [9, Ch. 7]).

Corollary 2. *The boundary of a hyperbolic group (equipped with any visual metric) contains a quasi-circle if and only if the group is not virtually free.*

By definition a *quasi-circle* is a metric circle which admits a quasisymmetric parametrization by the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ (see [10] for the definition and basic facts about quasisymmetric maps). Since the boundary of a virtually free group is totally disconnected, the "only if" part of the corollary is obvious.

One of the main ingredients in the proof of the theorem is a result by Tukia [14] which insures the existence of quasi-arcs with given endpoints inside certain subsets of $\mathbb{R}^n$ (a *quasi-arc* is a quasisymmetric image of the interval $[0,1]$). The authors thank Juha Heinonen for drawing their attention to Tukia's paper, which allowed them to substantially shorten the proof of the next proposition.

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To state this proposition, we need two more definitions. A metric space Z is *doubling* if there exists a constant $N \in \mathbb{N}$ with the following property: If B is an arbitrary ball in Z and R is its radius, then B can be covered by N balls of radius $R/2$. The metric space Z is *linearly connected* if there exists a constant L such that for all $x, y \in Z$ there is a connected subset $S \subset Z$ of diameter at most $Ld(x, y)$ containing $\{x, y\}$.

Proposition 3. *If X is a complete, doubling, and linearly connected metric space, then any two distinct points in X are the endpoints of a quasi-arc.*

Proof. Let d denote the metric on X, and pick $\alpha \in (0, 1)$. Since X is doubling, there exists $n \in \mathbb{N}$ such that the "α-snowflaked" metric space (X, d^α) can be embedded into $\mathbb{R}^n$ (equipped with the usual metric) by a bi-Lipschitz mapping (this follows from Assouad's Embedding Theorem [1, 2.6. Prop.]; see [10, Thm. 12.2] for the version of this theorem used here). Let Z denote the image of such an embedding. Then Z is complete and linearly connected, since X has these properties. Hence any two distinct points in Z are the endpoints of a quasi-arc in Z (this is [14, Thm 1A] expressed in our terminology; see the introduction of [14] for a discussion). Since quasi-arcs in Z pull back to quasi-arcs in X, the result follows. □

Proposition 4. *If G is a 1-ended hyperbolic group, then $\partial_\infty G$ equipped with any visual metric d is compact, doubling, connected, and linearly connected.*

Proof. It is easy to show that $\partial_\infty G$ is compact [9, p. 123, 9. Prop.] and doubling [4, Sect. 9]. Since the group G is 1-ended, its boundary $\partial_\infty G$ is connected.

It remains to prove linear connectedness (note that this is a stronger quantitative version of local connectedness which was established in this context in [2, Prop. 3.3]). Given two points x and y in a metric space (Z, d), and $\lambda > 0$, a *λ-chain from x to y* is a sequence of points $x = z_1, \ldots, z_k = y$ such that $d(z_i, z_{i+1}) \leq \lambda$ for all $1 \leq i < k$. The *length* of a λ-chain is the number of points in the chain.

Lemma 5. *There is a number $N \in \mathbb{N}$ such that for all $x, y \in \partial_\infty G$ there is a $\frac{1}{2}d(x, y)$-chain of length at most N from x to y.*

Proof. If not, there are sequences $\{x_k\}$ and $\{y_k\}$ in $\partial_\infty G$ such that the shortest $\frac{1}{2}d(x_j, y_j)$-chain from x_j to y_j has length j. The boundary $\partial_\infty G$ is compact and connected, so clearly $r_j := d(x_j, y_j) \to 0$ as $j \to \infty$. In view of the doubling property, the sequence $(\partial_\infty G, \frac{1}{r_j}d, x_j)$ of pointed metric spaces subconverges to a limit (W, d_W, x_∞) with respect to pointed Gromov-Hausdorff convergence [7, Thm. 8.1.10]. We can then find a point $y_\infty \in W$ such that $d_W(x_\infty, y_\infty) = 1$ and there is no λ-chain from x_∞ to y_∞ for any $\lambda < \frac{1}{2}$. This implies that W is not connected. By [3, Lemma 5.2], the limit space W is homeomorphic to $\partial_\infty G \setminus \{z\}$ for some $z \in \partial_\infty G$, and so z is a "global cut point" of $\partial_\infty G$.

On the other hand, it is a well-known (and deep) fact if $\partial_\infty G$ is connected, then $\partial_\infty G$ has no global cut points (see [13], [6, Thm. 9.3], [5, Cor. 0.3]). This is a contradiction. □

Now suppose x and y are arbitrary points in $\partial_\infty G$. By the lemma we can find a $\frac{1}{2}d(x, y)$-chain $S_1 = \{z_1, \ldots, z_k\}$ which joins x to y and has length $k \leq N$. Now define S_2 by adding, for each $1 \leq i < k$, the points in a $\frac{1}{2}d(z_i, z_{i+1})$-chain joining z_i to z_{i+1}. Repeating this process inductively, we obtain a nested sequence of sets $S_1 \subset \ldots \subset S_j \subset \ldots$. The closure S of the union $\bigcup_j S_j$ will be a connected set containing x and y whose diameter does not exceed $Ld(x, y)$, where L is a constant independent of x and y. This shows that $\partial_\infty G$ is linearly connected. □

The proofs of Theorem 1 *and Corollary* 2. We first assume that G a hyperbolic group which is not virtually free, and prove that there is a quasi-isometric embeddding $\mathbb{H}^2 \to G$ and a quasi-circle in $\partial_\infty G$. Every hyperbolic group is finitely presentable [9, p. 76, 17. Prop.]. Hence there is a finite graph of groups decomposition of G where all edge groups are finite, and all vertex groups have at most one end [8, Theorem 6.2.14]. Since G is not virtually free, one of the vertex groups G_0 is 1-ended [8, Theorem 6.2.12]. The group G_0 is quasi-isometrically embedded in G, since this is true for every vertex group in a graph of groups decomposition with finite edge groups [11, Rem. 3.6]. This implies that G_0 is also a hyperbolic group. So without loss of generality we may assume that G itself is 1-ended.

Let $\partial_\infty G$ denote the boundary of G equipped with a visual metric. By Proposition 4, the hypotheses of Proposition 3 are satisfied for $\partial_\infty G$. Hence there is a quasisymmetric map $[0, 1] \to \partial_\infty G$. Since $[0, 1]$ is quasisymmetrically homeomorphic to the boundary of a hyperbolic half-plane $\mathbb{H}^2_+ \subset \mathbb{H}^2$, we conclude that there is a quasi-isometric embedding $\mathbb{H}^2_+ \to G$ (see the proof of Prop. 4.2 in [12], for example). In particular, one can quasi-isometrically embed arbitrarily large balls $B \subset \mathbb{H}^2$ into G with uniform constants for the quasi-isometric embeddings. By precomposing with isometries in $\mathbb{H}^2$, post-composing with left translations in the group G, and applying a compactness argument based on the Arzelà-Ascoli Theorem, we can obtain a quasi-isometric embedding $\mathbb{H}^2 \to G$ as a limit. A quasi-isometric embedding of a Gromov hyperbolic space X into a Gromov hyperbolic space Y induces a quasisymmetric embedding of $\partial_\infty X$ into $\partial_\infty Y$ (see [4, Thm. 6.5], where this is essentially proved); since $\partial_\infty \mathbb{H}^2$ is quasisymmetrically equivalent to $\mathbb{S}^1$, we deduce that the boundary $\partial_\infty G$ contains a quasi-circle.

Now suppose G is virtually free. It follows that $\partial_\infty G$ is totally disconnected, and therefore cannot contain a quasi-circle. This then implies that there is no quasi-isometric embedding $\mathbb{H}^2 \to G$.

This completes the proofs of the theorem and corollary. □

Remarks. There are various open questions that are related to our theorem. For example, Papasoglu has asked if every one-ended finitely presented group G contains a quasi-plane—the image of a uniform embedding $P \to G$, where P is a complete Riemannian plane of bounded geometry. A problem due to Gromov is whether every 1-ended hyperbolic group G is the target of a homomorphism $\phi : S \to G$, where S is a surface group and ϕ does not factor through a free group.

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