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ALGEBRAIC ISOMORPHISMS AND $\mathcal{J}$-SUBSPACE LATTICES

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Abstract. The class of $\mathcal{J}$-lattices was originally defined in the second author's thesis and subsequently by Longstaff, Nation, and Panaia. A subspace lattice $\mathcal{L}$ on a Banach space X which is also a $\mathcal{J}$-lattice is called a *$\mathcal{J}$-subspace lattice*, abbreviated JSL. It is demonstrated that every single element of $Alg\mathcal{L}$ has rank at most one. It is also shown that $Alg\mathcal{L}$ has the strong finite rank decomposability property. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be subspace lattices that are also JSL's on the Banach spaces X_1 and X_2, respectively. The two properties just referred to, when combined, show that every algebraic isomorphism between $Alg\mathcal{L}_1$ and $Alg\mathcal{L}_2$ preserves rank. Finally we prove that every algebraic isomorphism between $Alg\mathcal{L}_1$ and $Alg\mathcal{L}_2$ is quasi-spatial.

1. Introduction

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be Banach algebras. An *algebraic isomorphism*, φ say, from $\mathcal{A}_1$ onto $\mathcal{A}_2$ is a linear bijection that is multiplicative in the sense that for all $a, b \in \mathcal{A}_1$, $\varphi(ab) = \varphi(a)\varphi(b)$. If $\mathcal{A} = \mathcal{A}_1 = \mathcal{A}_2$, then φ is said to be an *algebraic automorphism* of $\mathcal{A}$.

This paper is concerned with Banach algebras arising as the set of operators that leave invariant every element of a JSL on a real or complex Banach space. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are algebras of operators on the Banach spaces X_1 and X_2, respectively, an algebraic isomorphism $\varphi\colon \mathcal{A}_1 \to \mathcal{A}_2$ is said to be *spatial* (or *spatially induced*) if there exists a bicontinuous linear bijection, S say, from X_1 onto X_2 such that $\varphi(A) = SAS^{-1}$, for every $A \in \mathcal{A}_1$. Algebraic isomorphisms need not be spatial and need not even preserve rank (see for example [2, Example 5.1]).

In 1977 Lambrou [6] introduced the (strictly) weaker notion of quasi-spatiality. With $\mathcal{A}_1$ and $\mathcal{A}_2$ as in the preceding paragraph, an algebraic isomorphism, φ say, is said to be *quasi-spatial* (or *quasi-spatially induced*), if there exists a closed, densely defined, injective linear transformation, S say, from X_1 onto a dense subset of X_2, with the properties that

(i) if x belongs to the domain of S, then Ax belongs to the domain of S, for every$A \in \mathcal{A}_1$,
(ii) if x belongs to the domain of S, then $\varphi(A)Sx = SAx$, for every $A \in \mathcal{A}_1$.

Quasi-spatiality of algebraic isomorphisms has been studied in [2, 3, 4, 6] and in [17, 18].

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An element s of an (abstract) algebra A is called *single* if, whenever $asb = 0$ with $a, b \in A$, then $as = 0$ or $sb = 0$. (The interested reader may wish to consult [13, 14, 15] for further references on single elements.) It is clear that the property of being 'single' is an algebraic one and thus is preserved by an algebraic isomorphism. It is easy to show that an operator of rank one is a single element of any operator algebra containing it. Single elements need not have rank one. Indeed, for any positive integer n, an operator algebra and a single element of rank n in it can be exhibited [8, 13]. Theorem 3.1 below shows that element S of $Alg\mathcal{L}$, where $\mathcal{L}$ is a JSL on a Banach space X, is single if and only if S has rank one.

Let $\mathcal{A}$ denote an arbitrary operator algebra on a Banach space X. Then $\mathcal{A}$ is said to have the *strong finite rank decomposability property* if for every positive integer n every operator of rank n in $\mathcal{A}$ can be written as a sum of n rank one operators in $\mathcal{A}$. This 'strong finite rank decomposability property' and the weaker 'finite rank decomposability property', which simply requires that every non-zero finite-rank operator of $\mathcal{A}$ can be decomposed as a finite sum of rank one operators in $\mathcal{A}$, has been studied by several authors. The reader may wish to consult [19] for further references (see also [1, 16]). Of particular relevance here is the fact that $Alg\mathcal{L}$, where $\mathcal{L}$ is a JSL, has the strong finite rank decomposability property (Theorem 3.2). This fact, when combined with the result that non-zero singles are rank one, shows that every algebraic isomorphism from $Alg\mathcal{L}_1$ onto $Alg\mathcal{L}_2$, for $\mathcal{L}_1$, $\mathcal{L}_2$ JSL's on the Banach spaces X_1, X_2, respectively, is rank preserving (Theorem 3.3). This in turn is used to demonstrate that every algebraic isomorphism from $Alg\mathcal{L}_1$ onto $Alg\mathcal{L}_2$ is quasi-spatially induced (Theorem 3.4).

2. Preliminaries

Throughout what follows X will denote a real or complex Banach space, with topological dual X^*. The terms *operator* on X and *subspace* of X shall mean, respectively, bounded linear mapping of X into itself, and closed linear manifold of X. For non-zero vectors $e^* \in X^*$ and $f \in X$, the rank one operator defined by $x \mapsto e^*(x)f$ is denoted by $e^* \otimes f$. Clearly $(e^* \otimes f)^* = \hat{f} \otimes e^*$, where $\hat{f}$ is the image of f under the canonical map of X into X^{**}.

If $\mathcal{F}$ is a collection of subspaces of X, $Alg\mathcal{F}$ denotes the set of operators on X leaving every member of $\mathcal{F}$ invariant. It is obvious that both 0 and I belong to $Alg\mathcal{F}$. It is not difficult to show that $Alg\mathcal{F}$ is closed in the operator norm and is thus a unital Banach algebra. Let $\{L_\gamma\}_{\gamma\in\Gamma}$ be a collection of subspaces, where Γ is some index set. The smallest subspace that contains every L_γ will be denoted by $\bigvee_{\gamma\in\Gamma} L_\gamma$. The largest subspace that is contained in every L_γ will be denoted by $\bigcap_{\gamma\in\Gamma} L_\gamma$. The latter is simply the intersection of the L_γ's; the former is just the closed linear span of $\bigcup_{\gamma\in\Gamma} L_\gamma$. A collection $\mathcal{L}$ of subspaces of X is called a *subspace lattice* on X if it contains (0) and X, and is complete in the sense that it is closed under closed linear spans and intersections of families of subspaces of arbitrary cardinality.

If L is a subspace of X, its annihilator is denoted by $L^\perp$. Thus $L^\perp = \{e^* \in X^* | e^*(f) = 0, \text{ for every } f \in L\}$. Dually, if M is a subspace of X^*, its pre-annihilator is denoted by ${}^\perp M$. Thus ${}^\perp M = \{f \in X | e^*(f) = 0, \quad \text{for every } e^* \in M\}$. It is clear from the definitions that $X^\perp = (0)$ and $(0)^\perp = X^*$. A corollary of the Hahn-Banach

Theorem shows that ${}^{\perp}\left(L^{\perp}\right) = L$. Also, the subspace $({}^{\perp}M)^{\perp}$ is the weak* closure of M, so $M \subseteq ({}^{\perp}M)^{\perp}$. If $\{L_\gamma\}_{\gamma\in\Gamma}$ is a collection of subspaces of X, then it is not too difficult to deduce that $\bigcap_{\gamma\in\Gamma} L_\gamma^{\perp} = \left(\bigvee_{\gamma\in\Gamma} L_\gamma\right)^{\perp}$ and that $\bigvee_{\gamma\in\Gamma} L_\gamma^{\perp} \subseteq \left(\bigcap_{\gamma\in\Gamma} L_\gamma\right)^{\perp}$. In fact $\left(\bigcap_{\gamma\in\Gamma} L_\gamma\right)^{\perp}$ is the weak* closure of $\bigvee_{\gamma\in\Gamma} L_\gamma^{\perp}$. Thus, if X is reflexive, we have $M = ({}^{\perp}M)^{\perp}$ and $\bigvee_{\gamma\in\Gamma} L_\gamma^{\perp} = \left(\bigcap_{\gamma\in\Gamma} L_\gamma\right)^{\perp}$.

In any complete lattice L the operation '$_-$' is defined by $a_- = \bigvee\{b \in L : a \not\leq b\}$ $(a \in L)$. The set of elements $a \in L$ satisfying $a \neq 0$ and $a_- \neq 1$ is called the set of *$\mathcal{J}$-elements* of L and is denoted by $\mathcal{J}(L)$.

Definition 2.1 ([17])**.** An abstract complete lattice L is called a $\mathcal{J}$-lattice if

(1) $\bigvee\{a : a \in \mathcal{J}(L)\} = 1$,
(2) $\bigwedge\{a_- : a \in \mathcal{J}(L)\} = 0$,
(3) $a \vee a_- = 1$, for every $a \in \mathcal{J}(L)$,
(4) $a \wedge a_- = 0$, for every $a \in \mathcal{J}(L)$.

The class of $\mathcal{J}$-lattices was defined in [17] and subsequently discussed in [11, 12]. The following result will be useful for what follows:

Lemma 2.1 (Longstaff [10] (see also [5]))**.** *If $\mathcal{L}$ is a subspace lattice on a real or complex normed space, the rank one operator $e^* \otimes f$ belongs to $Alg\mathcal{L}$ if and only if there is an element $J \in \mathcal{J}$ such that $f \in J$ and $e^* \in (J_-)^{\perp}$.*

The series of lemmas which immediately follow are slight extensions of results given in [8]. Lambrou's hypotheses have been weakened, but the conclusions retained, with the result that they hold for every JSL. The proof of each lemma is precisely the same as that given in [8]. The proofs have been repeated here for the reader's convenience.

Lemma 2.2 (Lambrou)**.** *Let $\mathcal{L}$ be a subspace lattice on a Banach space X that satisfies conditions (1) and (2) of Definition 2.1 above. Let T be an element of $Alg\mathcal{L}$.*

(i) *If $RT = 0$ for all rank one operators $R \in Alg\mathcal{L}$, then $T = 0$.*
(ii) *If $TR = 0$ for all rank one operators $R \in Alg\mathcal{L}$, then $T = 0$.*

Proof. Suppose that $RT = 0$ for every rank one operator R in $Alg\mathcal{L}$. Let $L \in \mathcal{J}(\mathcal{L})$, and let $R = e^* \otimes f$, with $0 \neq f \in L$ and $e^* \in (L_-)^{\perp}$. Given $RT = 0$, then $T^*e^* = 0$. This shows that $\bigvee\left\{(L_-)^{\perp} | L \in \mathcal{J}(\mathcal{L})\right\}$ is contained in ker T^*. Since (by (2)) the weak* closure of $\bigvee\left\{(L_-)^{\perp} | L \in \mathcal{J}(\mathcal{L})\right\}$ is X^* and T^* is weak* continuous, it follows that $T^* = 0$ and so $T = 0$.

Suppose now that $TR = 0$, for every rank one operator R of $Alg\mathcal{L}$. Then $Tf = 0$, for every $f \in L$ and for every $L \in \mathcal{J}(\mathcal{L})$. Condition (1) now shows that $T = 0$. $\square$

Lemma 2.3 (Lambrou)**.** *Let $\mathcal{L}$ be a subspace lattice on a Banach space X that satisfies conditions (1) and (2) of Definition 2.1 above. Let S be an element of*

$Alg\mathcal{L}$. Then S is single if and only if for rank one operators R_1, R_2 of $Alg\mathcal{L}$ the condition $R_1SR_2 = 0$ implies $R_1S = 0$ or $SR_2 = 0$.

Proof. Necessity follows from the definition of a single element. For sufficiency, suppose that the stated condition concerning rank one operators is satisfied but $ASB = 0$, $AS \neq 0$ and $SB \neq 0$ for A and B in $Alg\mathcal{L}$. By Lemma 2.2 there exist rank one operators R_1 and R_2 such that $R_1AS \neq 0$ and $SBR_2 \neq 0$. The proof is completed by observing that $R_1ASBR_2 = 0$ with R_1A and BR_2 rank one operators, a contradiction. □

Lemma 2.4 (Erdös)**.** *Let $\mathcal{L}$ be a subspace lattice on a Banach space X that satisfies conditions (1) and (2) of Definition 2.1 above. Let S be a non-zero single element of $Alg\mathcal{L}$. Then there exists $M \in \mathcal{J}(\mathcal{L})$ such that $S_{|M}$ is non-zero. Moreover, for any $L \in \mathcal{J}(\mathcal{L})$, the rank of the operator $S_{|L}$ is at most one.*

Proof. The first part of the conclusion of the statement in the lemma is true for any non-zero operator in $Alg\mathcal{L}$. This follows directly from condition (1). Now let $L \in \mathcal{J}(\mathcal{L})$. Choose non-zero vectors $f, g \in L$. From Lemma 2.2 there exists a rank one operator, R say, in $Alg\mathcal{L}$ such that $RS \neq 0$. Since RS is rank one, it follows that there exist scalars α and β not both zero such that $RS(\alpha f + \beta g) = 0$. Choose $e^* \in (L_-)^\perp$ non-zero. Then,

$$RS(e^* \otimes (\alpha f + \beta g)) = 0.$$

Since S is single, it follows from Lemma 2.3 that $S(\alpha f + \beta g) = 0$ as required. □

Lemma 2.5 (Lambrou)**.** *Let $\mathcal{L}$ be a subspace lattice on a Banach space X that satisfies conditions (1) and (2) of Definition 2.1 above. Let S be a non-zero single element of $Alg\mathcal{L}$. Then there exists $M \in \mathcal{J}(\mathcal{L})$ such that $S^*_{|(M_-)^\perp}$ is non-zero. Moreover, for any $L \in \mathcal{J}(\mathcal{L})$ the rank of the operator $S^*_{|(L_-)^\perp}$ is at most one.*

Proof. The first part of the conclusion of the statement in the lemma holds for any non-zero operator in $Alg\mathcal{L}$. This follows directly from condition (2) above, the weak* density of $\bigvee_{L \in \mathcal{J}(\mathcal{L})} (L_-)^\perp$ in X^* and the continuity of S^* with respect to the weak* topology on X^*. Now let $L \in \mathcal{J}(\mathcal{L})$. Choose non-zero vectors $e^*, g^* \in (L_-)^\perp$. By Lemma 2.2 there exists a rank one operator in $Alg\mathcal{L}$, R say, such that $SR \neq 0$. Since SR has rank one, so has R^*S^*. Thus there exist scalars α and β not both zero such that $R^*S^*(\alpha e^* + \beta g^*) = 0$. Let f be any non-zero vector in L. Then

$$R^*S^*\left((\alpha e^* + \beta g^*) \otimes f\right)^* = R^*S^*\left(\hat{f} \otimes (\alpha e^* + \beta g^*)\right) = 0,$$

which shows that

$$\left((\alpha e^* + \beta g^*) \otimes f\right) SR = 0.$$

Since S is single and $SR \neq 0$, it follows that $\left((\alpha e^* + \beta g^*) \otimes f\right) S = 0$ and $S^*\left(\alpha e^* + \beta g^*\right) = 0$ as required. □

3. Main results

Let L be an abstract complete lattice. An element a of L is said to be an *atom* of L if $b \in L$ and $0 \leq b \leq a$ implies $b = 0$ or $b = a$.

Lemma 3.1. *Let L be an abstract complete that satisfies condition (3) or (4) of Definition 2.1 above. Then every element of $\mathcal{J}(L)$ is an atom of L.*

Proof. Suppose that $\mathcal{L}$ satisfies condition (4). Let $a \in \mathcal{J}(L)$, and suppose that a is not an atom of $\mathcal{J}(L)$. Then there exists a non-zero element of L, b say, such that $b < a$. Then $a \not\leq b$ with the consequence that $b \leq a \wedge a_- = 0$. So $b = 0$, and this is a contradiction.

Next suppose that L satisfies condition (3). Also assume that $a \in \mathcal{J}(L)$ is not an atom. Thus there exists a non-zero element of L, c say, such that $c < a$. It is clear that $c \leq a_-$ (otherwise $a \leq c$). Now $c_- \leq a_-$, and therefore $c_- \neq 1$ and $c \in \mathcal{J}(L)$. This shows that $1 = c \vee c_- \leq a_-$, which is a contradiction. $\square$

Lemma 3.2. *Let L be an abstract complete that satisfies condition (3) of Definition 2.1 above. Let $a \in \mathcal{J}(L)$. If $a_- < b$ with $b \in L$, then $b = 1$.*

Proof. Let $a \in \mathcal{J}(L)$, and let $b \in L$ satisfy $a_- < b$. Clearly $a \leq b$ (otherwise $b \leq a_-$). Thus $1 = a \vee a_- \leq b$; so $b = 1$. $\square$

Theorem 3.1. *Let $\mathcal{L}$ be a JSL on the Banach space X. Then every single element of $Alg\mathcal{L}$ has rank at most one.*

Proof. Assume that there exists a single element S in $Alg\mathcal{L}$ with rank greater than one. It follows immediately from Lemma 3.1 that every element of $\mathcal{J}(\mathcal{L})$ is an atom of $\mathcal{L}$. From Lemma 2.5 there exists an atom $M \in \mathcal{L}$ such that $S^*_{|(M_-)^\perp}$ is non-zero. Since S is assumed to have rank at least two, from Lemma 2.4 and condition (1) there exists an element $N \in \mathcal{J}(\mathcal{L})$ with $N \neq M$ such that $S_{|N} \neq 0$. Clearly $M \not\subseteq N$, and thus $(M_-)^\perp \subseteq N^\perp$. Choose $g^* \in (M_-)^\perp$ such that $S^*g^* \neq 0$. Choose $h \in M$ non-zero. Thus $g^* \otimes h \in Alg\mathcal{L}$ and $(g^* \otimes h)S \neq 0$. Choose $f \in N$ such that $Sf \neq 0$, and choose $e \in (N_-)^\perp$ non-zero as well. Now this shows that $S(e^* \otimes f) \neq 0$. Observe however that

$$(g^* \otimes h)S(e^* \otimes f) = g^*(Sf)(e^* \otimes h) = 0.$$

But this contradicts the fact that S was taken to be a single element of $Alg\mathcal{L}$. $\square$

Theorem 3.2 (Panaia [17] (see also [9])). *Let $\mathcal{L}$ be a subspace lattice on a Banach space X that satisfies conditions (1) and (3) of Definition 2.1 above. Then $Alg\mathcal{L}$ has the strong finite rank decomposability property.*

Proof. Let $F \in Alg\mathcal{L}$ have rank n. The first observation which can be made is that the cardinality of the set $\{L \in \mathcal{J}(\mathcal{L}) | F(L) \neq (0)\}$ is at most n. For, suppose that the cardinality was greater than n. Then there exist non-zero vectors $y_1, y_2, \ldots, y_n, y_{n+1}$ and subspaces $L_1, L_2, \ldots, L_n, L_{n+1}$ such that $y_j \in F(L_j)$ and $L_j \in \mathcal{J}(\mathcal{L})$, for $1 \leq j \leq n+1$. Since $\mathcal{L}$ satisfies condition (3), Lemma 3.1 shows that each L_j $(j = 1, 2, \ldots, n+1)$ is an atom of $\mathcal{L}$. Thus the set $\{y_1, y_2, \ldots, y_n, y_{n+1}\}$ is linearly independent. (To see this, assume without loss of generality that $y_1 = \sum_{j=2}^{n+1} \alpha_j y_j$, with not all the $\alpha_j = 0$. Then $y_1 \in \left(\bigvee_{j=2}^{n+1} L_j\right) \subseteq L_{1-}$, so $L_1 \cap L_{1-} \neq (0)$, a contradiction.) The fact that the set $\{y_1, y_2, \ldots, y_n, y_{n+1}\}$ is linearly independent now contradicts the assumption made that the rank of the operator F was n. Let the cardinality of $\{L \in \mathcal{J}(\mathcal{L}) | F(L) \neq (0)\}$ be m, and let $L_1, L_2, \ldots, L_m$ be distinct elements of $\mathcal{J}(\mathcal{L})$ such that $F(L_j) \neq (0)$, for $1 \leq j \leq m$. Then

$$\begin{aligned} F(X) &= F\left(\bigvee\{L|L \in \mathcal{J}(\mathcal{L})\}\right) &&= F\left(\overline{\sum_{L\in\mathcal{J}(\mathcal{L})} L}\right) \\ &\subseteq \overline{F\left(\sum_{L\in\mathcal{J}(\mathcal{L})} L\right)} &&= F\left(\sum_{L\in\mathcal{J}(\mathcal{L})} L\right) \\ &= \sum_{L\in\mathcal{J}(\mathcal{L})} F(L) &&= \sum_{j=1}^{m} F(L_j). \end{aligned}$$

This shows that $F(X) = \sum_{j=1}^{m} F(L_j)$. For each $1 \leq j \leq m$, let $\{x_{1,j}, x_{2,j}, \ldots, x_{p_j,j}\}$ be a basis for $F(L_j)$. Observe that $\sum_{j=1}^{m} p_j = n$. It follows now that there exists $\{y^*_{i,j} \in X^*|i = 1, 2, \ldots, p_j; j = 1, 2, \ldots, m\}$ such that

$$F = \sum_{j=1}^{m}\sum_{i=1}^{p_j} y^*_{i,j} \otimes x_{i,j}.$$

(The proof is completed by demonstrating that each $y^*_{i,j}$ is in $(L_{j-})^{\perp}$.) Suppose that $y^*_{i,j} \notin (L_{j-})^{\perp}$. Thus there exists a vector $v \in L_{j-}$ such that $y^*_{i,j}(v) \neq 0$. Since F leaves L_{j-} invariant, it follows that

$$\sum_{l=1}^{m}\sum_{k=1}^{p_k} y^*_{k,l}(v)x_{k,l} \in L_{j-}.$$

Observe that $L_l \subseteq L_{j-}$, for $l \neq j$. This shows that $x_{k,l} \in L_{j-}$ for $l \neq j$. Thus

$$\sum_{k=1}^{p_j} y^*_{k,j}(v)x_{k,j} \in L_j \cap L_{j-} = (0),$$

and it follows that $y^*_{i,j}(v) = 0$, a contradiction. Since $y^*_{i,j} \otimes x_{i,j} \in Alg\mathcal{L}$ for every i and j, the proof is complete. $\square$

The above results when collected together imply the following theorem.

Theorem 3.3. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be subspace lattices that are also JSL's on the Banach spaces X_1 and X_2, respectively. Then every algebraic isomorphism from $Alg\mathcal{L}_1$ onto $Alg\mathcal{L}_2$ is rank preserving.*

For the remainder of this section, let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two subspace lattices on the Banach spaces X_1 and X_2, respectively. In addition, let $\mathcal{L}_1$ and $\mathcal{L}_2$ be JSL's, and let $\varphi\colon Alg\mathcal{L}_1 \to Alg\mathcal{L}_2$ be an algebraic isomorphism. Theorem 3.3 above shows that φ preserves the rank of every operator in $Alg\mathcal{L}_1$. It is clear from the requirement that φ is an algebraic isomorphism that φ^{-1} also preserves the rank of every operator in $Alg\mathcal{L}_2$. It is proven below that φ is quasi-spatial.

Let $K \in \mathcal{J}(\mathcal{L}_1)$. A simple application of the Hahn-Banach Theorem shows that since $K \cap K_- = (0)$, there exists an element $f \in K$ and an element $e^* \in (K_-)^{\perp}$ such that $e^*(f) = 1$. For every $K \in \mathcal{J}(\mathcal{L}_1)$, let $f_K \in K$ and $e^*_K \in (K_-)^{\perp}$ be chosen such that $e^*_K(f_K) = 1$. Since φ preserves rank one elements, it follows (by Lemma 2.1) that there exists $L \in \mathcal{J}(\mathcal{L}_2)$ and non-zero vectors $a_K^* \in (L_-)^{\perp}$ and $b_K \in L$ such that $\varphi(e_K^* \otimes f_K) = a_K^* \otimes b_K$. (Observe that since each element of $\mathcal{J}(\mathcal{L}_2)$ is an atom, this $L \in \mathcal{J}(\mathcal{L}_2)$ is uniquely determined.) These observations lead to the following proposition.

Proposition 3.1. *The map $S_K \colon x \mapsto \varphi(e_K^* \otimes x)b_K$ is a linear bijection from K onto L. Moreover, $\varphi(e_K^* \otimes x) = a_K^* \otimes S_K x$, for every $x \in K$.*

Proof. The map S_K is clearly a well-defined linear mapping from K into L. Injectivity of S_K follows easily as well. For if $S_K(x) = 0$, then $\varphi(e_K^* \otimes x) = 0$. Since φ is injective it then follows that $x = 0$. To show that S_K maps onto L, let v be a non-zero vector in L. Form the rank one operator $a_K^* \otimes v$. This operator is in $Alg\mathcal{L}_2$ and $\varphi^{-1}(a_K^* \otimes v) = g^* \otimes h$ is a rank one element of $Alg\mathcal{L}_1$. Observe that $a_K^*(b_K) = 1$ (since $(a_K^* \otimes b_K)^2 = a_K^* \otimes b_K$), so

$$a_K^* \otimes v = (a_K^* \otimes v)\,(a_K^* \otimes b_K)\,,$$

which shows that

$$g^* \otimes h = (g^* \otimes h)\,(e_K^* \otimes f_K)\,.$$

This shows that $g^*(f_K) \neq 0$, from which it then follows that $h \in K$. (For $h \in J$ and $g^* \in (J_-)^{\perp}$ with $J \in \mathcal{J}(\mathcal{L}_1)$, if $J \neq K$, then $K \subseteq J_-$; so $(J_-)^{\perp} \subseteq K^{\perp}$. Thus $g^*(f_K) = 0$.) Now put $x = g^*(f_K)h$. Then

$$\begin{aligned} S_K(x) &= \varphi(e_K^* \otimes x)b_K = \varphi\left((g^* \otimes h)\,(e_K^* \otimes f_K)\right)b_K = \varphi\left(g^* \otimes h\right)b_K \\ &= (a_K^* \otimes v)b_K = v. \end{aligned}$$

Finally, let $x \in K$. Then

$$\begin{aligned} a_K^* \otimes S_K x &= a_K^* \otimes \varphi(e^* \otimes x)b_K \\ &= \varphi(e_K^* \otimes x)\,[a_K^* \otimes b_K] \\ &= \varphi(e_K^* \otimes x)\varphi(e_K^* \otimes f_K) \\ &= \varphi\left((e_K^* \otimes x)(e_K^* \otimes f_K)\right) \\ &= \varphi(e_K^* \otimes x). \end{aligned}$$

This completes the proof. □

There is a corresponding 'dual' to the map S_K (for every $K \in \mathcal{J}(\mathcal{L}_1)$), which is described below.

Proposition 3.2. *Let K, L, e_K^*, f_K, a_K^* and b_K be given as Proposition 3.1. Then the map $T_K \colon x^* \mapsto [\varphi(x^* \otimes f_K)]^*\, a_K^*$ is a linear bijection from $(K_-)^{\perp}$ onto $(L_-)^{\perp}$. Moreover, $\varphi(x^* \otimes f_K) = T_K x^* \otimes b_K$, for every $x^* \in (K_-)^{\perp}$.*

Proof. It is clear that T_K is linear, and the injectivity of φ shows that T_K is injective. For every $x^* \in (K_-)^{\perp}$, $\varphi(x^* \otimes f_K)$ leaves L_- invariant. So its adjoint leaves $(L_-)^{\perp}$ invariant, which demonstrates that T_K maps into $(L_-)^{\perp}$. To demonstrate that T_K maps onto $(L_-)^{\perp}$, proceed as follows. Let $u^* \in (L_-)^{\perp}$ be non-zero, and let $\varphi^{-1}(u^* \otimes b_K) = p^* \otimes q$. Then $(a_K^* \otimes b_K)(u^* \otimes b_K) = u^* \otimes b_K$ gives $(e^* \otimes f_K)(p^* \otimes q) = p^* \otimes e_K^*(q)f_K = y^* \otimes f_K$ where $y = e_K^*(q)p^*$. This implies that $e_K^*(q) \neq 0$, whence it follows that $y^* \in (K_-)^{\perp}$. (For, since $q \notin K_-$ it must follow that $q \in K$, and so by Lemma 2.1, $p^* \in (K_-)^\perp$.) It is easily verified that $T_K y^* = u^*$, and so T_K is onto $(L_-)^\perp$. Finally, let $x^* \in (K_-)^\perp$. Then

$$\begin{aligned} T_K x^* \otimes b_K &= [\varphi(x^* \otimes f_K)]^* a_k^* \otimes b_K = (a_K^* \otimes b_K)\varphi(x^* \otimes f_K) \\ &= \varphi(e_K^* \otimes f_K)\varphi(x^* \otimes f_K) = \varphi(x^* \otimes f_K). \end{aligned}$$

This completes the proof. □

Remark. The maps $S_K\colon K \to L$ and $T_K\colon (K_-)^\perp \to (L_-)^\perp$ in Propositions 3.1 and 3.2, respectively, depend of course, on the choices made for e_K^*, f_K, a_K^* and b_K. For any non-zero scalars α, β, if $e_K'^* = \frac{1}{\alpha} e_K^*$, $f_K' = \alpha f_K$, $a_K'^* = \frac{1}{\beta} a_K^*$ and $b_K' = \beta b_K$, then it is still true that $e_K'^* \in (K_-)^\perp$, $f_K' \in K$, $e_K'^*(f_K') = 1$ and $\varphi(e_K'^* \otimes f_K') = a_K'^* \otimes b_K'$. If $S_K'\colon K \to L$ is the map defined by $S_K' x = \varphi(e_K'^* \otimes x)\, b_K'$, it is readily verified that $S_K' = \lambda S_K$ where $\lambda = \frac{\beta}{\alpha}$. (A similar comment applies to T_K.)

Proposition 3.3. *The maps $S_K\colon K \to L$ and $T_K\colon (K_-)^\perp \to (L_-)^\perp$ as defined in Propositions* 3.1 *and* 3.2*, respectively, are continuous.*

Proof. (We demonstrate that the map S_K is continuous. A similar argument applies to the map T_K.) By the Closed Graph Theorem it suffices to show that S_K is closed. Let $\{x_n\}_{n=1}^\infty$ be a sequence of vectors in K and let $x_n \to x$ and $S_K(x_n) \to z$ with $x \in K$ and $z \in L$. Let $u^* \in (K_-)^\perp$. Now $T_K(u^*)(S_K(x_n - x))$ converges to $T_K(u^*)(z - S_K x)$. Observe that

$$\begin{aligned} T_K(u^*)(S_K(x_n - x)) &= \varphi^*\left(\hat{f}_K \otimes u^*\right) a_K^*(S_K(x_n - x)) \\ &= a_K^*(\varphi(u^* \otimes f_K)\, S_K(x_n - x)) \\ &= a_K^*(\varphi(u^* \otimes f_K)\, \varphi(e_K^* \otimes (x_n - x))\, b_K) \\ &= u^*(x_n - x)\, a_K^*(\varphi(e_K^* \otimes f_K)\, b_K). \end{aligned}$$

The last expression converges to 0. Hence it has been deduced that

$$T_K(u^*)(z - S_K x) = 0,$$

for every $u^* \in (K_-)^\perp$. Now T_K is a bijection of $(K_-)^\perp$ onto $(L_-)^\perp$. Thus $z - S_K x \in L_-$. But $z - S_K x \in L$; therefore, $z - S_K x \in L \cap L_- = (0)$. This shows that $z = S_K x$ and that S_K is a closed linear transformation. □

Denote by $\mathcal{J}(\mathcal{L}_i)^+$ the (not necessarily closed) linear span of all the elements of $\mathcal{J}(\mathcal{L}_i)$, for $i = 1, 2$. That is, the set of vectors of the form $x = \sum_{j=1}^n x_j$ where each x_j is in some $K_j \in \mathcal{J}(\mathcal{L}_i)$. It is clear that $X_i = \overline{\mathcal{J}(\mathcal{L}_i)^+}$. For such a vector x put

$$(*) \qquad Sx = \sum_{j=1}^n S_{K_j} x_j,$$

where $K_j \in \mathcal{J}(\mathcal{L}_1)$ containing x_j $(j = 1, 2, \ldots, n)$. That this gives a well-defined linear transformation of $\mathcal{J}(\mathcal{L}_1)^+$ into $\mathcal{J}(\mathcal{L}_2)^+$ follows from the following lemma.

Lemma 3.3. *Let $K_1, K_2, \ldots, K_n$ be distinct elements of $\mathcal{J}(\mathcal{L}_1)$ and $\sum_{j=1}^n x_j = 0$ with $x_j \in K_j$ $(j = 1, 2, \ldots, n)$. Then $x_j = 0$, for every $j = 1, 2, \ldots, n$.*

Proof. If $\sum_{j=1}^{n} x_j = 0$, then for each j it is seen that $x_j \in K_j \cap \bigvee_{\substack{i=1\\ i\neq j}}^{n} K_i \subseteq K_j \cap K_{j-} = (0)$, so $x_j = 0$. □

It is clear that S is densely defined. In fact S has dense range. To see this, let $L \in \mathcal{J}(\mathcal{L}_2)$ and $a_L^* \in (L_-)^\perp$ and $b_L \in L$ such that $a_L^*(b_L) = 1$. Let $\varphi^{-1}(a_L^* \otimes b_L) = e_L^* \otimes f_L$. Then Proposition 3.1 (applied to φ^{-1}) shows that the map

$$\tilde{S}_L \colon y \mapsto \varphi^{-1}(a_L^* \otimes y) f_L$$

is a linear bijection of L onto K say, of $\mathcal{J}(\mathcal{L}_1)$. The remarks immediately following that proposition imply that $S_K(K) = L$ and that $\tilde{S}_L = (S_K)^{-1}$. This shows that $\mathcal{J}(\mathcal{L}_2)^+ \subseteq \mathcal{R}(S)$. Since $\mathcal{J}(\mathcal{L}_2)^+$ is dense, so is the range of S.

Proposition 3.4. *Let $S\colon \mathcal{J}(\mathcal{L}_1)^+ \to \mathcal{J}(\mathcal{L}_2)^+$ be defined as in equation $(*)$ above. Then S has a closed extension.*

Proof. First it is shown that, for every pair of elements M, K of $\mathcal{J}(\mathcal{L}_1)$, we have $(T_M u^*)(S_K x) = u^*(x)$, for every $u^* \in (M_-)^\perp$ and every $x \in K$, where $S_K\colon K \to L$ and $T_M\colon (M_-)^\perp \to (N_-)^\perp$ are the maps defined in Propositions 3.1 and 3.2, respectively. Observe that

$$\begin{aligned}
(T_M u^*)(S_K x) &= \left(\varphi^*\left(\hat{f}_M \otimes u^*\right) a_M^*\right)(S_K x)\\
&= a_M^*\left(\varphi(u^* \otimes f_M)(S_K x)\right)\\
&= a_M^*\left(\varphi(u^* \otimes f_M)\,\varphi(e_K^* \otimes x)\, b_K\right)\\
&= a_M^*\left(\varphi\left(u^*(x)\,(e_K^* \otimes f_M)\right) b_K\right).
\end{aligned}$$

If $K \neq M$, then $K \subseteq M_-$, so $(T_M u^*)(S_K x) = u^*(x) = 0$. If $K = M$, then

$$(T_M u^*)(S_K x) = u^*(x) a_K^*\left(\varphi(e_K^* \otimes f_K)\, b_K\right) = u^*(x) a_K^*(b_K) = u^*(x).$$

It follows now by linearity that $(T_M u^*)(Sx) = u^*(x)$, for every $u^* \in (M_-)^\perp$ and every $x \in \mathcal{J}(\mathcal{L}_1)^+$.

Let $(0, y)$ be an element of the closure of the graph of S. Thus there exists a sequence of vectors in $\mathcal{J}(\mathcal{L}_1)^+$, $\{x_n\}_{n=1}^\infty$ say, such that (x_n, Sx_n) converges in norm to $(0, y)$. (To show that S has a closed extension, it is enough to verify that $y = 0$.) Let $v^* \in$ linear span $\left\{(L_-)^\perp \colon L \in \mathcal{J}(\mathcal{L}_2)\right\}$. It follows from Proposition 3.2 that $v^* = \sum_{j=1}^{p} T_{M_j} u_j^*$, where $M_j \in \mathcal{J}(\mathcal{L}_1)$ and $u_j^* \in (M_-)^\perp$. From what has just been demonstrated it follows that $v^*(Sx_n) = \sum_{j=1}^{p} u_j^*(x_n)$. But Sx_n converges to y, and x_n converges to 0. Therefore, we have demonstrated that $v^*(y) = 0$, for every vector in the linear span $\left\{(L_-)^\perp \colon L \in \mathcal{J}(\mathcal{L}_2)\right\}$. The weak star density of the linear span $\left\{(L_-)^\perp \colon L \in \mathcal{J}(\mathcal{L}_2)\right\}$ in X_2^* shows that $y = 0$ as required. □

By the preceding proposition, $\overline{G(S)} = G(\bar{S})$ for some closed linear transformation $\bar{S}$. The theorem below now shows that φ is quasi-spatially implemented by $\bar{S}$.

Theorem 3.4. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be JSL's on the Banach spaces X_1 and X_2, respectively. Let φ be an algebraic isomorphism of $Alg\mathcal{L}_1$ onto $Alg\mathcal{L}_2$. Then φ is quasi-spatial.*

Proof. The first requirement of quasi-spatiality, namely the existence of a closed densely defined linear transformation with dense range, is obviously satisfied by $\bar{S}$. Next it is demonstrated that $\mathcal{D}(\bar{S})$, the domain of $\bar{S}$, is an invariant linear manifold of $Alg\mathcal{L}_1$ and that $\varphi(A)\bar{S}z = \bar{S}Az$, for every $z \in \mathcal{D}(\bar{S})$ and for every $A \in Alg\mathcal{L}_1$. Notice that $S_K Ax = \varphi(A)S_K x$, for every $K \in \mathcal{J}(\mathcal{L}_1)$ and every $x \in K$, since

$$S_K Ax = \varphi\left(e_K^* \otimes Ax\right) b_K = \varphi(A)\varphi\left(e_K^* \otimes x\right) b_K = \varphi(A)S_K x.$$

By linearity it follows that $SAx = \varphi(A)Sx$, for every $x \in \mathcal{J}(\mathcal{L}_1)^+$. Let $z \in \mathcal{D}(\bar{S})$ and $A \in Alg\mathcal{L}_1$. Then $\left(z, \bar{S}z\right) \in \overline{G(S)}$ so there is a sequence $\{z_n\}_{n=1}^{\infty}$ with $z_n \in \mathcal{J}_+^{(1)}$, for every $n \geq 1$, such that $z_n \to z$ and $Sz_n \to \bar{S}z$. Clearly $Az_n \to Az$. Since $\bar{S}Az_n = \varphi(A)Sz_n$, for every n, it is observed that $\bar{S}Az_n \to \varphi(A)\bar{S}z$. Since $\bar{S}$ is closed, it follows that $Az \in \mathcal{D}(\bar{S})$ and $\varphi(A)\bar{S}z = \bar{S}Az$.

It remains to show that $\bar{S}$ is injective. With z, z_n as above, it was shown in the proof of Proposition 3.4 that $\left(T_M u^*\right)\left(Sz_n\right) = u^*(z_n)$, for every $n \geq 1$, every $M \in \mathcal{J}(\mathcal{L}_1)$ and every $u^* \in (M_-)^{\perp}$. Taking limits and using the facts that $z_n \to z$ and $Sz_n \to \bar{S}z$, it follows that $\left(T_M u^*\right)\left(\bar{S}z\right) = u^*(z)$. Hence, if $\bar{S}z = 0$, then $z \in M_-$, for every $M \in \mathcal{J}(\mathcal{L}_1)$, so $z = 0$ (since $\bigcap\{M_- | M \in \mathcal{J}(\mathcal{L}_1)\} = (0)$). $\square$

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