CALDERÓN-ZYGMUND OPERATORS ON HARDY SPACES
WITHOUT THE DOUBLING CONDITION

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Abstract. Let \( \mu \) be a non-negative Radon measure on \( \mathbb{R}^d \) which only satisfies some growth condition. In this paper, the authors obtain the boundedness of Calderón-Zygmund operators in the Hardy space \( H^1(\mu) \).

1. Introduction

In recent years, many papers focus on the analysis on \( \mathbb{R}^d \) with non-doubling measure; see [2, 5, 6, 8, 3, 4, 1] and their references. Moreover, the analysis on such \( \mathbb{R}^d \) was proved to play a striking role in solving the long open Painlevé’s problem by Tolsa in [9]; see also [10] for more background of this. Throughout this paper, the Euclidean space \( \mathbb{R}^d \) is endowed with a non-negative Radon measure \( \mu \) which only satisfies the following growth condition that there exists \( C_0 > 0 \) such that

\[
|B(x, r)| \leq C_0 r^n
\]

for all \( x \in \mathbb{R}^d \) and \( r > 0 \), where \( B(x, r) = \{ y \in \mathbb{R}^d : |y - x| < r \} \), \( n \) is a fixed number and \( 0 < n \leq d \). Such a measure \( \mu \) is not necessary to be doubling. We recall that \( \mu \) is said to satisfy the doubling condition if there exists \( C > 0 \) such that \( \mu(B(x, 2r)) \leq C \mu(B(x, r)) \) for all \( x \in \text{supp}(\mu) \) and \( r > 0 \). It is well known that the doubling condition in the analysis on spaces of homogeneous type is a key assumption. However, some research has now indicated that the doubling condition is superfluous for most of the classical Calderón-Zygmund theory.

Let \( K \) be a function on \( \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, y) : x = y\} \) satisfying that for \( x \neq y \),

\[
|K(x, y)| \leq C|x - y|^{-n},
\]

and for \( |x - y| \geq 2|x - x'| \),

\[
|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq C \frac{|x - x'|^\delta}{|x - y|^{n+\delta}},
\]

where \( \delta \in (0, 1] \) and \( C > 0 \) is a constant. The Calderón-Zygmund operator associated to the above kernel \( K \) and the measure \( \mu \) is formally defined by

\[
Tf(x) = \int_{\mathbb{R}^d} K(x, y)f(y)d\mu(y).
\]

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This integral may not be convergent for many functions. Thus we consider the truncated operators $T_\varepsilon$ for $\varepsilon > 0$ defined by

$$T_\varepsilon f(x) = \int_{|x-y| > \varepsilon} K(x, y) f(y) \, d\mu(y).$$

We say that $T$ is bounded on $L^2(\mu)$ if the operators $\{T_\varepsilon\}_{\varepsilon > 0}$ are bounded on $L^2(\mu)$ uniformly on $\varepsilon > 0$. In this case, there is an operator $\tilde{T}$ which is the weak limit as $\varepsilon \to 0$ of some subsequence of operators $\{T_\varepsilon\}_{\varepsilon > 0}$; see \cite{5}. It is easy to see that $\tilde{T}$ is still bounded on $L^2(\mu)$; moreover, for $f \in L^2(\mu)$ with compact support and a. e. $x \in \mathbb{R}^d \setminus \text{supp} (f)$,

$$\tilde{T} f(x) = \int_{\mathbb{R}^d} K(x, y) f(y) \, d\mu(y)$$

with the same $K$ as in (1.2) and (1.3). By the same argument of Tolsa as in \cite{5,7}, we see that $\tilde{T}$ is also bounded from $L^1(\mu)$ into weak-$L^1(\mu)$ and from $H^1(\mu)$ into $L^1(\mu)$.

In this paper, we will prove that $\tilde{T}$ is bounded on the Hardy space $H^1(\mu)$ if $\tilde{T}^* 1 = 0$. Here, by $\tilde{T}^* 1 = 0$, we mean that for any bounded function $b$ with compact support and $\int_{\mathbb{R}^d} b \, d\mu = 0$,

$$\int_{\mathbb{R}^d} \tilde{T} b(x) \, d\mu(x) = 0.$$

We remark that for such a function $b, b \in H^1(\mu)$ and therefore, $\tilde{T} b \in L^1(\mu)$. Also, if $\tilde{T} b \in H^1(\mu)$, then $\tilde{T} b$ should satisfy (1.5) by the definition of the Hardy space $H^1(\mu)$; see \cite{5,8} or Definition 2 below. Thus, in some sense, the condition (1.5) is also necessary.

If $\mu$ is the $d$-dimensional Lebesgue measure on $\mathbb{R}^d$, this result is well known and it was proved by verifying that $\tilde{T}$ maps any atom of the Hardy space $H^1(\mathbb{R}^d)$ into some molecule. However, if $\mu$ only satisfies (1.1), it is still unknown if there is a molecular characterization for the Hardy space $H^1(\mu)$. We will prove that $\tilde{T}$ is bounded on the Hardy space $H^1(\mu)$ via its “grand” maximal function characterization of Tolsa in \cite{8} and its new atomic characterization of the authors in \cite{1}.

**Definition 1.** Given $f \in L^1_{\text{loc}}(\mu)$, we set

$$M_{\Phi} f(x) = \sup_{\varphi \sim x} \left| \int_{\mathbb{R}^d} f \varphi \, d\mu \right|,$$

where the notation $\varphi \sim x$ means that $\varphi \in L^1(\mu) \cap C^1(\mathbb{R}^d)$ and satisfies

(i) $\|\varphi\|_{L^1(\mu)} \leq 1$,

(ii) $0 \leq \varphi(y) \leq \frac{1}{|y-x|^n}$ for all $y \in \mathbb{R}^d$, and

(iii) $|\nabla \varphi(y)| \leq \frac{1}{|y-x|^{n+1}}$ for all $y \in \mathbb{R}^d$, where $\nabla = (\partial/\partial x_1, \cdots, \partial/\partial x_d)$.

Based on Theorem 1.2 of Tolsa in \cite{8}, we define the Hardy space $H^1(\mu)$ as follows.

**Definition 2.** The Hardy space $H^1(\mu)$ is the set of all functions $f \in L^1(\mu)$ satisfying that $\int_{\mathbb{R}^d} f \, d\mu = 0$ and $M_{\Phi} f \in L^1(\mu)$. Moreover, we define the norm of $f \in H^1(\mu)$ by

$$\|f\|_{H^1(\mu)} = \|f\|_{L^1(\mu)} + \|M_{\Phi} f\|_{L^1(\mu)}.$$
Theorem 1. Let $K$ be the function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, y) : x = y\}$ satisfying (1.2) and (1.3). Suppose that the operator $\widetilde{T}$ in (1.4) is bounded on $L^2(\mu)$ and $\widetilde{T}^* 1 = 0$ as in (1.5). Then $\widetilde{T}$ is bounded on $H^1(\mu)$.

It is known that the dual space of $H^1(\mu)$ is the space $RBMO(\mu)$, which was introduced by Tolsa in [5]. From Theorem 1, the fact that $RBMO(\mu) = (H^1(\mu))^*$ (see [5]) and a standard dual argument, it is easy to deduce the boundedness of the transpose operator of $\widetilde{T}$ in $RBMO(\mu)$ as below.

Corollary 1. Let $\widetilde{T}$ be the same as in Theorem 1. Then $\widetilde{T}^*$, the transpose operator of $\widetilde{T}$, is bounded on $RBMO(\mu)$.

Remark 1. Obviously, from different subsequences of operators $\{T_\varepsilon\}_{\varepsilon > 0}$ which are bounded on $L^2(\mu)$ uniformly on $\varepsilon > 0$, one may deduce different $\tilde{T}$'s. However, they are all bounded on $L^2(\mu)$ and satisfy (1.4). But, the relation between these different $\tilde{T}$'s is still open.

In what follows, $C$ denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line.

2. Proof of Theorem 1

We begin with some necessary notation and definitions. Throughout this paper, we only consider the closed cubes with sides parallel to the coordinate axes. For any cube $Q$ and any $\alpha > 0$, $\alpha Q$ denotes the cube with the same center as $Q$ and $l(\alpha Q) = \alpha l(Q)$, where $l(Q)$ denotes the side length of the cube $Q$.

Given two cubes $Q \subset R$ in $\mathbb{R}^d$, set

$$K_{Q, R} = 1 + \sum_{k=1}^{N_{Q, R}} \frac{\mu(2^k Q)}{l(2^k Q)^n},$$

where $N_{Q, R}$ is the smallest positive integer $k$ such that $l(2^k Q) \geq l(R)$; see [5] for some properties of $K_{Q, R}$.

To prove Theorem 1, we need to recall the atomic characterization of the Hardy space $H^1(\mu)$ as follows.

Definition 3. Let $\rho > 1$, $1 < p \leq \infty$ and $\gamma \in \mathbb{N}$. A function $b \in L^1_{\text{loc}}(\mu)$ is called a $(p, \gamma)$-atomic block if

1. there exists some cube $R$ such that $\text{supp}(b) \subset R$,
2. $\int_{\mathbb{R}^d} b d\mu = 0$,
3. for $j = 1, 2$, there are functions $a_j$ supported on cube $Q_j \subset R$ and numbers $\lambda_j \in \mathbb{R}$ such that $b = \lambda_1 a_1 + \lambda_2 a_2$, and

$$\|a_j\|_{L^p(\mu)} \leq [\mu(\rho Q_j)]^{1/p-1} \left[ K_{Q_j, R} \right]^{-\gamma}.$$

Then we define

$$|b|_{H^{1, p}_{\text{at}}, \gamma}(\mu) = |\lambda_1| + |\lambda_2|.$$
We say that $f \in H^{1, p}_{\text{atb}, \gamma}(\mu)$ if there are $(p, \gamma)$-atomic blocks $\{b_i\}_{i \in \mathbb{N}}$ such that

$$f = \sum_{i=1}^{\infty} b_i$$

with $\sum_{i=1}^{\infty} |b_i|_{H^{1, p}_{\text{atb}, \gamma}(\mu)} < \infty$. The $H^{1, p}_{\text{atb}, \gamma}(\mu)$ norm of $f$ is defined by

$$\|f\|_{H^{1, p}_{\text{atb}, \gamma}(\mu)} = \inf \left\{ \sum_{i} |b_i|_{H^{1, p}_{\text{atb}, \gamma}(\mu)} \right\},$$

where the infimum is taken over all the possible decompositions of $f$ into $(p, \gamma)$-atomic blocks.

The above definition when $\gamma = 1$ was introduced by Tolsa in [5] and when $\gamma > 1$ by the authors in [11]. It was proved in [5, 11] that the definition of $H^{1, p}_{\text{atb}, \gamma}(\mu)$ is independent of the chosen constant $\rho > 1$, and for any integer $\gamma \geq 1$ and $1 < p \leq \infty$, all the atomic Hardy spaces $H^{1, p}_{\text{atb}, \gamma}(\mu)$ are just the Hardy space $H^1(\mu)$ with equivalent norms. We remark that in the proof of Theorem 1 below, we need to choose $\gamma > 1$, especially, $\gamma = 2$.

The following lemma will be used in the proof of Theorem 1.

**Lemma 1.** Let $M_\Phi$ be as in Definition 1 and $1 < p < \infty$. Then $M_\Phi$ is bounded on $L^p(\mu)$.

In fact, Tolsa proved that $M_\Phi$ is bounded from $H^1(\mu)$ into $L^1(\mu)$; see Lemma 3.1 in [8]. On the other hand, it is obvious that $M_\Phi$ is bounded on $L^\infty(\mu)$. By Theorem 7.2 in [5], we obtain that $M_\Phi$ is bounded on $L^p(\mu)$ for $1 < p < \infty$.

Now we turn to the proof of Theorem 1.

**Proof of Theorem 1.** By a standard argument, it suffices to verify that for any atomic block $b$ as in Definition 3 with $\rho = 4$, $p = \infty$ and $\gamma = 2$, $\tilde{T}b$ is in $H^1(\mu)$ with norm $C|b|_{H^{1, \infty}_{\text{atb}}(\mu)}$, where $C$ is independent of $b$. Let all the notation be the same as in Definition 3. By our choices, $a_j$ now satisfies the following size condition that

$$\|a_j\|_{L^\infty(\mu)} \leq \left[ \mu(4Q_j)K_{Q_j}^2, R \right]^{-1},$$

where $j = 1, 2$.

The assumption that $\tilde{T^*}1 = 0$ tells us that $\int_{\mathbb{R}^d} \tilde{T}b d\mu = 0$. Recalling that $\tilde{T}$ is bounded from $H^1(\mu)$ into $L^1(\mu)$ (see [5]), we obtain

$$\|\tilde{T}b\|_{L^1(\mu)} \leq C|b|_{H^{1, \infty}_{\text{atb}}(\mu)}.$$ 

By this and Definition 2, we deduce that the proof of Theorem 1 can be reduced to proving that

$$\|M_\Phi(\tilde{T}b)\|_{L^1(\mu)} \leq C|b|_{H^{1, \infty}_{\text{atb}}(\mu)}.$$ 

Write

$$
\|M_\Phi(\tilde{T}b)\|_{L^1(\mu)} = \int_{\mathbb{R}} M_\Phi(\tilde{T}b)(x) d\mu(x) + \int_{\mathbb{R}^d \setminus \mathbb{R}} M_\Phi(\tilde{T}b)(x) d\mu(x) = I + II.
$$
Noting that $M_{\Phi}$ is sublinear, we can control $I$ by

$$I \leq \int_{4R} M_{\Phi} \left[ (\widetilde{T}b)\chi_{8R} \right] (x) \, d\mu(x) + \int_{4R} M_{\Phi} \left[ (\widetilde{T}b)\chi_{R^{k+1}8R} \right] (x) \, d\mu(x) = I_1 + I_2.$$ 

From the fact that for $j = 1, 2$, $Q_j \subset R$, it follows that for any $z \in Q_j$ and any $y \in 2^{k+1}R \setminus 2^k R$, $k \geq 3$, $|y - z| \geq l(2^{k-2}R)$. By this fact, (ii) of Definition 1, (1.2) and (2.1), we obtain

$$I_2 \leq \sum_{j=1}^{2} |\lambda_j| \int_{4R} \sup_{\varphi \sim 2} \left[ \int_{\mathbb{R}^n \setminus 8R} |\tilde{T}b(y)| \varphi(y) \, d\mu(y) \right] \, d\mu(x)
\leq \sum_{j=1}^{2} |\lambda_j| \int_{4R} \sum_{k=1}^{\infty} \left[ \int_{2^{k+1}R \setminus 2^k R} K(y, z) a_j(z) \, d\mu(z) \right] \frac{1}{|x - y|} \, d\mu(y) \, d\mu(x)
\leq C \sum_{j=1}^{2} |\lambda_j| \sum_{k=3}^{\infty} \|a_j\|_{L^\infty(\mu)} \mu(Q_j) \frac{(2^{k+1}R)}{l(2^{k-2}R)^n} \frac{\mu(4R)}{l(2^{k-2}R)^n}
\leq C \sum_{j=1}^{2} |\lambda_j|.$$

To estimate $I_1$, we write

$$I_1 \leq \sum_{j=1}^{2} |\lambda_j| \int_{4Q_j} M_{\Phi} \left[ (\widetilde{T}a_j)\chi_{8R} \right] (x) \, d\mu(x)
+ \sum_{j=1}^{2} |\lambda_j| \int_{4R \setminus 4Q_j} M_{\Phi} \left[ (\widetilde{T}a_j)\chi_{2Q_j} \right] (x) \, d\mu(x)
+ \sum_{j=1}^{2} |\lambda_j| \int_{4R \setminus 4Q_j} M_{\Phi} \left[ (\widetilde{T}a_j)\chi_{R^{k+1}2Q_j} \right] (x) \, d\mu(x)
= I_{11} + I_{12} + I_{13}.$$

The Hölder inequality, Lemma 1, the boundedness of $\widetilde{T}$ in $L^2(\mu)$ and (2.1) lead to

$$I_{11} \leq \sum_{j=1}^{2} |\lambda_j| \mu(4Q_j)^{1/2} \left\| M_{\Phi}[(\widetilde{T}a_j)\chi_{8R}] \right\|_{L^2(\mu)}
\leq C \sum_{j=1}^{2} |\lambda_j| \mu(4Q_j)^{1/2} \|\widetilde{T}a_j\|_{L^2(\mu)}
\leq C \sum_{j=1}^{2} |\lambda_j| \mu(4Q_j)^{1/2} \|a_j\|_{L^2(\mu)}
\leq C \sum_{j=1}^{2} |\lambda_j| \mu(4Q_j) \|a_j\|_{L^\infty(\mu)}
\leq C \sum_{j=1}^{2} |\lambda_j|.$$
For $j = 1, 2$, denote $N_{Q_j, 4R}$ simply by $N_j$. By (ii) of Definition 1, the Hölder inequality, the boundedness of $\widetilde{T}$ in $L^2(\mu)$ and (2.1), we have

\[
I_{12} \leq \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \int_{2^{k+1}Q_j \setminus 2^kQ_j} \sup_{\varphi \sim x} \int_{2Q_j} |\widetilde{T}a_j(y)\varphi(y)| \, d\mu(y) \, d\mu(x)
\leq \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \int_{2^{k+1}Q_j \setminus 2^kQ_j} \frac{1}{l(2^{k-2}Q_j)^n} \, d\mu(x) \int_{2Q_j} |\widetilde{T}a_j(y)| \, d\mu(y)
\leq \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \frac{\mu(2^{k+1}Q_j)}{l(2^{k-2}Q_j)^n} \|T a_j\|_{L^2(\mu)} \mu(2Q_j)^{1/2}
\leq C \sum_{j=1}^{2} |\lambda_j| |K_{Q_j, R}| \mu(2Q_j)^{1/2} \|a_j\|_{L^2(\mu)}
\leq C \sum_{j=1}^{2} |\lambda_j|,
\]

where we have used the fact that

\[
(2.3) \quad K_{Q_j, 4R} \leq C K_{Q_j, R}.
\]

For $I_{13}$, we further decompose it into

\[
I_{13} = \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \int_{2^{k+1}Q_j \setminus 2^kQ_j} M_{\Phi} \left[ \left( \widetilde{T}a_j \right) \chi_{8R \setminus 2Q_j} \right] (x) \, d\mu(x)
\leq \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \int_{2^{k+1}Q_j \setminus 2^kQ_j} M_{\Phi} \left[ \left| \widetilde{T}a_j \right| \chi_{2^{k+2}Q_j \setminus 2^{k-1}Q_j} \right] (x) \, d\mu(x)
\leq \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \int_{2^{k+1}Q_j \setminus 2^kQ_j} M_{\Phi} \left[ \left| \widetilde{T}a_j \right| \chi_{\max\{2^{k+2}Q_j, 8R\} \setminus 2^{k+2}Q_j} \right] (x) \, d\mu(x)
\leq E + F + G.
\]

Lemma 1, (1.2) and (2.1) tell us that

\[
E \leq \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \mu(2^{k+1}Q_j)^{1/2} \left\| M_{\Phi} \left[ \left| \widetilde{T}a_j \right| \chi_{2^{k+2}Q_j \setminus 2^{k-1}Q_j} \right] \right\|_{L^2(\mu)}
\leq C \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \mu(2^{k+1}Q_j)^{1/2}
\times \left\{ \int_{2^{k+2}Q_j \setminus 2^{k-1}Q_j} \left| \int_{Q_j} K(y, z) a_j(z) \, d\mu(z) \right|^2 \, d\mu(y) \right\}^{1/2}
\leq C \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \frac{\mu(2^{k+2}Q_j)}{l(2^{k-3}Q_j)^n} \|a_j\|_{L^2(\mu)} \mu(Q_j)
\leq C \sum_{j=1}^{2} |\lambda_j|.\]
By (ii) of Definition 1, (1.2), (2.3) and (2.1), we easily see that

\[ G \leq \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \int_{2^{k-1}Q_j}^{2^k Q_j} \sup_{\varphi \sim x} \left[ \int_{2^{k-1}Q_j}^{2^k Q_j} |\mathcal{T}a_j(y)| \varphi(y) \, d\mu(y) \right] \, d\mu(x) \]
\[ \leq \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_j} \sum_{l=1}^{k-2} \mu(2^l+1)Q) \mu(2^{l+1}Q) \|a_j\|_{L^\infty(\mu)} \|\mu(\mu(Q_j) \]
\[ \leq C \sum_{j=1}^{2} |\lambda_j| \left[ K_{Q_j, R} \right]^2 \|a_j\|_{L^\infty(\mu)} \mu(\mu(Q_j) \]
\[ \leq C \sum_{j=1}^{2} |\lambda_j|. \]

An argument similar to the estimate for G leads to

\[ F \leq C \sum_{j=1}^{2} |\lambda_j|. \]

The estimates for E, F and G give the desired estimate for I_{13}. Combining the estimates for I_{11}, I_{12}, I_{13} and I_2 yields

\[ I = \int_{4R} M_b(x, y) \, d\mu(x) \leq C \sum_{j=1}^{2} |\lambda_j| = C[b]_{H^{1, \infty}} \cdot \]

Now we turn to the estimate for II. Let \( x_R \) be the center of the cube R. Invoking that \( T^* 1 = 0 \), we obtain

\[ II = \int_{R^d \setminus 4R} \sup_{\varphi \sim x} \left[ \int_{R^d} \mathcal{T}b(y) [\varphi(y) - \varphi(x_R)] \, d\mu(y) \right] \, d\mu(x) \]
\[ \leq \int_{R^d \setminus 4R} \sup_{\varphi \sim x} \left[ \int_{2R} \mathcal{T}b(y) [\varphi(y) - \varphi(x_R)] \, d\mu(y) \right] \, d\mu(x) \]
\[ + \int_{R^d \setminus 4R} \sup_{\varphi \sim x} \left[ \int_{2R} \mathcal{T}b(y) [\varphi(y) - \varphi(x_R)] \, d\mu(y) \right] \, d\mu(x) \]
\[ = II_1 + II_2. \]

Note that for any \( z \in 2R, x \in 2^{k+1} R \setminus 2^k R, \) and \( k \geq 2, \) we have \( |x - z| \geq l(2k-2R) \). This together with (iii) of Definition 1 and the mean value theorem leads to

\[ |\varphi(y) - \varphi(x_R)| \leq C \frac{l(R)}{l(2k-2R)^{n+1}} \]
for \( y \in 2R \). By (2.5), (1.2), the Hölder inequality, the boundedness of \( \tilde{T} \) in \( L^2(\mu) \) and (2.1), we have

\[
\Pi_1 \leq \sum_{j=1}^{2} |\lambda_j| \int_{2^{k+1}R \setminus 2^kR} \sup_{x \in \mathbb{R}^d \setminus 2R} \left| \tilde{T}a_j(y) \right| |\varphi(y) - \varphi(x_R)| \, d\mu(y) \, d\mu(x) + \sum_{j=1}^{2} |\lambda_j| \int_{2^{k+1}R \setminus 2^kR} \sup_{x \in \mathbb{R}^d \setminus 2R} \left| \tilde{T}a_j(y) \right| |\varphi(y) - \varphi(x_R)| \, d\mu(y) \, d\mu(x)
\]

\[
\leq C \sum_{j=1}^{2} |\lambda_j| \int_{2^{k+1}R \setminus 2^kR} \frac{l(R)}{l(2^{k-2}R)^{n+1}} \times \sum_{l=1}^{N_j-1} \int_{2^{l+1}Q_j \setminus 2^lQ_j} |a_j(z)| \, d\mu(z) \, d\mu(y) \, d\mu(x) + C \sum_{j=1}^{2} |\lambda_j| \int_{2^{k+1}R \setminus 2^kR} \frac{l(R)}{l(2^{k-2}R)^{n+1}} \left\| (\tilde{T}a_j) \chi_{2^kQ_j} \right\|_{L^2(\mu)} \, d\mu(x)
\]

\[
\leq C \sum_{j=1}^{2} |\lambda_j| K_{Q_j, R} \left| a_j \right|_{L^\infty(\mu)} + C \sum_{j=1}^{2} |\lambda_j| \left| a_j \right|_{L^2(\mu)}(Q_j)
\]

\[
\leq C \sum_{j=1}^{2} |\lambda_j|
\]

We further estimate \( \Pi_2 \) by

\[
\Pi_2 = \sum_{k=2}^{\infty} \int_{2^{k+1}R \setminus 2^kR} \sup_{x \in \mathbb{R}^d \setminus 2R} \left| \tilde{T}b(y) \right| |\varphi(y) - \varphi(x_R)| \, d\mu(y) \, d\mu(x)
\]

\[
\leq \sum_{k=2}^{\infty} \int_{2^{k+1}R \setminus 2^kR} M_{\Phi} \left[ \tilde{T}b \right]_{2^{k+1}R \setminus 2^kR}(x) \, d\mu(x)
\]

\[
+ \sum_{k=2}^{\infty} \int_{2^{k+1}R \setminus 2^kR} \sup_{x \in \mathbb{R}^d \setminus 2R} \left| \tilde{T}b(y) \right| |\varphi(y) - \varphi(x_R)| \, d\mu(y) \, d\mu(x)
\]

\[
+ \sum_{k=2}^{\infty} \int_{2^{k+1}R \setminus 2^kR} \sup_{x \in \mathbb{R}^d \setminus 2R} \left| \tilde{T}b(y) \right| |\varphi(y) - \varphi(x_R)| \, d\mu(y) \, d\mu(x)
\]

\[
= \Pi_{21} + \Pi_{22} + \Pi_{23} + \Pi_{24}
\]
From Lemma 1, the fact that $\int_{\mathbb{R}^d} b d\mu = 0$ and (1.3), we can deduce that

$$\Pi_{21} \leq \sum_{k=2}^{\infty} \mu(2^{k+1}R)^{1/2} \left\| M_\Phi \left( \tilde{T}b \chi_{2^k+2R, 2^{k-1}R} \right) \right\|_{L^2(\mu)}$$

$$\leq C \sum_{k=2}^{\infty} \mu(2^{k+1}R)^{1/2}$$

$$\times \left\{ \int_{2^{k+2}R \setminus 2^{k-1}R} \left| \int_{\mathbb{R}^d} [K(y, z) - K(y, xR)] b(z) d\mu(z) \right|^2 d\mu(y) \right\}^{1/2}$$

$$\leq C \sum_{k=2}^{\infty} \mu(2^{k+1}R) \frac{l(R)^\delta}{l(2^kR)^{n+\delta}} \|b\|_{L^1(\mu)}$$

$$\leq C \sum_{j=1}^{2} |\lambda_j|,$$

where we have used the fact that

$$\|b\|_{L^1(\mu)} \leq \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^1(\mu)} \leq C \sum_{j=1}^{2} |\lambda_j|.$$

An argument similar to the estimate for $\Pi_{21}$ tells us that

$$\Pi_{22} \leq C \sum_{j=1}^{2} |\lambda_j|.$$

Finally, we estimate $\Pi_{23}$. By the fact that $\int_{\mathbb{R}^d} b d\mu = 0$, (ii) of Definition 1 and (1.3), we obtain

$$\Pi_{23} \leq \sum_{k=2}^{\infty} \int_{2^{k+1}R \setminus 2^kR} \int_{2^{l+1}R \setminus 2^lR} \int_{\mathbb{R}^d} [K(y, z) - K(y, xR)] b(z) d\mu(z)$$

$$\times \left[ \frac{1}{|y-x|^n} + \frac{1}{|xR-x|^n} \right] d\mu(y) d\mu(x)$$

$$\leq C \sum_{k=2}^{\infty} \sum_{l=k+2}^{\infty} \mu(2^{k+1}R) \mu(2^{l+1}R) \frac{l(R)^\delta}{l(2^kR)^{n+\delta}} \|b\|_{L^1(\mu)}$$

$$\leq C \sum_{j=1}^{2} |\lambda_j|.$$

An argument similar to the estimate for $\Pi_{23}$ indicates that

$$\Pi_{24} \leq C \sum_{j=1}^{2} |\lambda_j|.$$

Combining the estimates for $\Pi_{21}$, $\Pi_{22}$, $\Pi_{23}$ and $\Pi_{24}$, we obtain the desired estimate for $\Pi_2$. The estimates for $\Pi_1$ and $\Pi_2$ tell us that

$$(2.6) \quad \Pi = \int_{\mathbb{R}^d \setminus 4R} M_{\Phi}(\tilde{T}b)(x) d\mu(x) \leq C|b|_{H^{1, \infty}_{\text{atb}, 2}(\mu)}.$$

The estimates (2.4) and (2.6) lead to (2.2), and this completes the proof of our theorem.
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References


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