EVERY NONTRIVIAL KNOT IN $S^3$
HAS NONTRIVIAL A-POLYNOMIAL

STEVEN BOYER AND XINGRU ZHANG

(Communicated by Ronald A. Fintushel)

Abstract. We show that every nontrivial knot in the 3-sphere has a nontrivial A-polynomial.

In Theorem 1 of [3], Kronheimer and Mrowka give a proof of the following remarkable theorem, thereby establishing the truth of the Property P conjecture.

Theorem 0.1 (Kronheimer-Mrowka). Let $K$ be any nontrivial knot in $S^3$ and let $M(r)$ be the manifold obtained by Dehn surgery on $K$ with slope $r$ with respect to the standard meridian-longitude coordinates of $K$. If $|r| \leq 2$, then there is an irreducible homomorphism from $\pi_1(M(r))$ to $SU(2)$.

The purpose of this note is to describe another consequence of Theorem 0.1, answering a question which has been around for about ten years. We show

Theorem 0.2. Every nontrivial knot $K$ in $S^3$ has nontrivial A-polynomial.

The A-polynomial was introduced in [1]. We recall its definition for a knot in $S^3$.

For a compact manifold $W$, we use $R(W)$ and $X(W)$ to denote the $SL_2(\mathbb{C})$ representation variety and character variety of $W$ respectively, and $q : R(W) \to X(W)$ to denote the quotient map sending a representation $\rho$ to its character $\chi_{\rho}$ (see [2] for detailed definitions). Note that $q$ is a regular map between the two varieties defined over the rationals.

Let $K$ be a knot in $S^3$, $M$ its exterior, and $\{\mu, \lambda\}$ the standard meridian-longitude basis for $\pi_1(\partial M)$. Let $i^* : X(M) \to X(\partial M)$ be the restriction map, also regular, induced by the homomorphism $i_* : \pi_1(\partial M) \to \pi_1(M)$, and let $\Lambda$ be the set of diagonal representations of $\pi_1(\partial M)$, i.e.

$$\Lambda = \{\rho \in R(\partial M) \mid \rho(\mu), \rho(\lambda)\text{ are both diagonal matrices}\}.$$ 

Then $\Lambda$ is a subvariety of $R(\partial M)$ and $q|_{\Lambda} : \Lambda \to X(\partial M)$ is a degree 2, surjective, regular map.

We may identify $\Lambda$ with $\mathbb{C}^* \times \mathbb{C}^*$ through the eigenvalue map $E : \Lambda \to \mathbb{C}^* \times \mathbb{C}^*$ which sends $\rho \in \Lambda$ to $(x, y) \in \mathbb{C}^* \times \mathbb{C}^*$ if $\rho(\mu) = (x^0 0)_{-1}$ and $\rho(\lambda) = (y^0 -1)_{-1}$. Let
$X^*(M)$ be the set of components of $X(W_K)$ each of which has a 1-dimensional image in $X(\partial M)$ under $i^*$ and then define

- $V$ to be the Zariski closure of $i^*(X^*(M))$ in $X(\partial M)$;
- $Z$ to be the algebraic curve $(q|_\Lambda)^{-1}(V)$ in $\Lambda$;
- $D$ to be the Zariski closure of $E(Z)$ in $\mathbb{C}^2$.

It can be verified that each of $X^*(M), V, Z,$ and $D$ is defined over the rationals.

The $A$-polynomial of $K$ is the defining polynomial $A_K(x, y)$ of the plane curve $D$ determined up to sign by the requirements that it has no repeated factors, it lies in $\mathbb{Z}[x, y]$, and the greatest common divisor of its coefficients is 1. For every knot $K$ in $S^3$, $X(M)$ has a unique component $Y_0$ consisting of reducible characters. The image $Y_0$ under $i^*$ is 1-dimensional and contributes the factor $y - 1$ to $A_K(x, y)$. Thus the $A$-polynomial of the trivial knot is $y - 1$. For a knot $K$ in $S^3$, its $A$-polynomial is said to be nontrivial if $A_K(x, y) \neq y - 1$. (See [1] for more details.)

We now proceed to the proof of Theorem 0.2. We take $K$ to be a nontrivial knot in $S^3$ with exterior $M$ and we let $M(r)$ denote the manifold obtained by Dehn surgery on $K$ with slope $r$. By Theorem 0.1, for every integer $n \neq 0$, the fundamental group of the surgered manifold $M(1/n)$ has an irreducible representation $\rho_n$ into $SU_2(\mathbb{C}) \subset SL_2(\mathbb{C})$. We shall consider $\rho_n$ as a representation of $\pi_1(M)$ through the composition with the quotient homomorphism $\pi_1(M) \to \pi_1(M(1/n))$. Thus $\rho_n(\mu^\lambda) = I$ and so the irreducibility of $\rho_n$ implies that

$$\rho_n(\mu), \rho_n(\lambda) \neq \pm I \text{ for each } n \neq 0. \tag{1}$$

Moreover, by a result of Thurston (see [2], Proposition 3.2.1), any algebraic component of $X(M)$ which contains the character of $\rho_n$ is at least 1-dimensional.

**Claim.** There is a component $X_0$ of $X(M)$ containing some $\chi_{\rho_n}$ whose restriction to $X(\partial M)$ under $i^*$ is 1-dimensional.

Assuming the claim, we can quickly complete the proof of Theorem 0.2. For suppose that $X_0$ contributes a factor $(y - 1)$ to $A_K(x, y)$. Then every representation in $q^{-1}(X_1)$ sends the longitude $\lambda$ to an element of $SL_2(\mathbb{C})$ of trace 2. In particular, if $n$ is chosen so that $\chi_{\rho_n} \in X_0$ we have $\rho_n(\lambda) = I$ or is a parabolic element of $SL_2(\mathbb{C})$. But the first possibility is prohibited by (1) while the second is prohibited by the fact that $SU_2(\mathbb{C})$ contains no parabolic elements. Thus $X_0$ contributes a factor to the $A$-polynomial different from $(y - 1)$. In particular, $A_K(x, y)$ is nontrivial, so the theorem holds.

**Proof of the Claim.** We shall suppose that each component of $X(M)$ containing some $\chi_{\rho_n}$ restricts to a point in $X(\partial M)$ in order to arrive at a contradiction.

We begin by selecting a component $X_1$ of $X(M)$ which contains $\chi_{\rho_n}$ for at least two distinct $n$, say $n_1, n'_1$. (This is possible since $X(M)$ has only finitely many algebraic components.) Let $R_1 = q^{-1}(X_1)$.

Recall that every element $\gamma \in \pi_1(M)$ defines a regular function $\tau_\gamma : X(M) \to \mathbb{C}$ given by $\tau_\gamma(\chi_{\rho}) = \text{trace}(\rho(\gamma))$. Our assumption that $i_*(X_1)$ is a point is equivalent to the fact that for every element $\beta \in \pi_1(\partial M) \subset \pi_1(M)$, the function $\tau_\beta|X_1$ is constant.

Suppose that $\rho \in R_1$ and $\rho(\pi_1(\partial M))$ contains a parabolic element. Then the commutativity of $\pi_1(\partial M)$ shows that every element of $\rho(\pi_1(\partial M))$ is either parabolic...
or \( \pm I \). Hence \( \tau_\rho(X_{\rho n}) = \tau_\rho(X_\rho) = \pm 2 \), which is impossible as it implies that 
\( \rho_{n1}(\mu) \in SU_2(\mathbb{C}) \) is \( \pm I \) (cf. (1)). Thus for each \( \rho \in R_1, \rho(\pi_1(\partial M)) \) consists of 
diagonalisable elements. Since \( i_*(X_1) \) is a point, it follows that for any such 
\( \rho \) we have that \( \rho|\pi_1(\partial M) \) is conjugate in \( SL_2(\mathbb{C}) \) to \( \rho_{n1}|\pi_1(\partial M) \) and therefore 
\( \rho(\mu\lambda^n) = I \) for each \( \rho \in R_1 \) and \( n \) such that \( \chi_{\rho n} \in X_1 \). For such a \( \rho \) we therefore 
have \( I = \rho(\mu\lambda^{n1})\rho(\mu\lambda^{-n1})^{-1} = \rho((\lambda^{-1})^{n1}) \). Thus \( \rho(\lambda) \) is of a fixed finite order \( d_1 \geq 3 \) 
(cf. (1)) and so for \( \rho \in R_1, \rho(\mu\lambda^n) = I \) if and only if \( n \in S_1 := \{n_1 + d_1k; k \in \mathbb{Z}\} \). 

Note that 
\[
(2) \quad d_1\mathbb{Z} \subset \mathbb{Z} \setminus S_1
\]
as otherwise \( n_1 \equiv 0 \) (mod \( d_1 \)) and therefore \( I = \rho_{n1}(\mu\lambda^{n1}) = \rho_{n1}(\mu) \), which is absurd.

Now repeat the argument to produce a component \( X_2 \) of \( X(M) \) satisfying the 
following conditions:

- there are at least two integers \( n_2, n'_2 \in d_1\mathbb{Z} \setminus \{0\} \) such that \( X_2 \) contains the 
  characters of \( \rho_{n2}, \rho_{n'_2} \);
- \( i^*(X_2) \) is a point in \( X(\partial M) \);
- there is an integer \( d_2 \geq 3 \) such that for any \( \rho \in R_2 = q^{-1}(X_2) \) we have 
  \( \rho(\mu\lambda^n) = I \) if and only if \( n \) belongs to the set \( S_2 = \{n_2 + d_2k; k \in \mathbb{Z}\} \); 
- \( d_1d_2\mathbb{Z} \subset \mathbb{Z} \setminus (S_1 \cup S_2) \).

The first of these conditions combines with (2) to show that \( X_2 \neq X_1 \).

Proceeding inductively, one can find, for each integer \( j \geq 1 \), a component \( X_j \) of 
\( X(M) \) satisfying the following conditions:

- there are at least two integers \( n_j, n'_j \in d_1d_2 \ldots d_{j-1}\mathbb{Z} \setminus \{0\} \) such that \( X_j \) 
  contains the characters of \( \rho_{n_j}, \rho_{n'_j} \);
- \( i^*(X_j) \) is a point in \( X(\partial M) \);
- there is an integer \( d_j \geq 3 \) such that for each \( \rho \in R_j = q^{-1}(X_j) \) we have 
  \( \rho(\mu\lambda^n) = I \) if and only if \( n \) belongs to the set \( S_j = \{n_j + d_jk; k \in \mathbb{Z}\} \);
- \( d_1d_2 \ldots d_{j-1}d_j\mathbb{Z} \subset \mathbb{Z} \setminus (S_1 \cup S_2 \cup \ldots \cup S_j) \).

It is easy to see that these conditions imply that \( X_i \neq X_j \) for \( i \neq j \), which is clearly 
impossible as \( X(M) \) has only finitely many components. Thus there must be a 
component \( X_0 \) of \( X(M) \) containing some \( \chi_{\rho n} \) such that \( i^*(X_0) \) is 1-dimensional. 
This completes the proof of the claim and therefore of Theorem 0.2.

Remark 0.3. Theorem 0.2 has been obtained independently by Nathan Dunfield 
and Stavros Garoufalidis.

REFERENCES

1. D. Cooper, M. Culler, H. Gillet, D. Long and P. Shalen, Plane curves associated to character 
2. M. Culler and P. Shalen, Varieties of group representations and splittings of 3-manifolds, 
3. P. Kronheimer and T. Mrowka, Dehn surgery, the fundamental group and SU(2), preprint 
  (available online at http://front.math.ucdavis.edu).

DéPARTEMENR DE MATHEMATIRES, UNIVERSITÉ DU QUÉBEC À MONTREAL, P.O. BOX 8888, 
CENTRE-VILLE, MONTREAL, QUÉBEC, CANADA H3C 3P8

E-mail address: boyer@math.uqam.ca

DEPARTMENT OF MATHEMATICS, SUNY AT BUFFALO, BUFFALO, NEW YORK, 14260-2900

E-mail address: xinzhang@math.buffalo.edu