PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 9, Pages 2795–2803
S 0002-9939(05)07831-7
Article electronically published on March 22, 2005

SEIBERG-WITTEN INVARIANTS AND BRANCHED COVERS ALONG TORI

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(Communicated by Ronald A. Fintushel)

Abstract. We compute the Seiberg-Witten invariants of double covers of smooth four-manifolds branched along tori of self-intersection zero.

1. Introduction

This short paper is a modest generalization of [RW]. In that paper Ruan and Wang computed (mod 2) the Seiberg-Witten invariants of double covers of 4-manifolds branched along a connected orientable Riemann surface of genus greater than one. In what follows, we shall present a similar (mod 2) computation when the branched locus is a torus. Our shared philosophy is to express the Seiberg-Witten invariants of the closed 4-manifolds in terms of the relative invariants of the complements of the branch locus. The computation depends essentially on the gluing formulae for the Seiberg-Witten invariant found in [P2] or [T]. In the last section, we answer in the affirmative a conjecture made by Ruan and Wang in [RW]. More examples of our formulae will appear in [P3].

2. Preliminary setup and definitions

We shall try to be faithful to the notation in [RW] and [P2]. Let $\widetilde{X}, X$ be closed smooth 4-manifolds and $\Sigma \subset X$ be a smoothly embedded connected orientable surface. We say that a smooth map $p : \widetilde{X} \to X$ is a cyclic m-fold cover branched along Σ when the following holds: if $\widetilde{\Sigma} = p^{-1}(\Sigma)$, then the restriction $p : \widetilde{X} \setminus \widetilde{\Sigma} \to X \setminus \Sigma$ is an unbranched m-fold cover, and p has the form $z \mapsto z^m$ locally on the normal complex planes of $\widetilde{\Sigma}$ and Σ.

$$\begin{array}{ccccc} \widetilde{X} & = & \widetilde{Y}_0 & \cup & \nu(\widetilde{\Sigma}) \\ \Big\downarrow p & & \Big\downarrow m:1 & & \Big\downarrow z\mapsto z^m \\ X & = & Y_0 & \cup & \nu(\Sigma) \end{array}$$

Let $\nu(\Sigma)$ denote a tubular neighborhood of Σ and let $Y_0 := X \setminus \nu(\Sigma)$, $\widetilde{Y}_0 := p^{-1}(Y_0)$. Again, p restricts to an unbranched cover $p : \widetilde{Y}_0 \to Y_0$. When $m = 2$, let $\sigma : \widetilde{Y}_0 \to \widetilde{Y}_0$ be the covering involution. We shall reserve the $\sim$ symbol for indicating some pull-back object by p. As in [RW] we will always assume that $[\Sigma]^2 \geq 0$.

Received by the editors February 5, 2004 and, in revised form, May 3, 2004.
2000 *Mathematics Subject Classification.* Primary 57M12, 57R57.

Definition 2.1. (i) Given any complex line bundle $L \to X$, the virtual dimension with respect to L is defined to be

$$d_L := \frac{1}{4}[c_1(L)^2 - (2e_X + 3s_X)]\,,$$

where e_X and s_X are respectively the Euler characteristic and the signature of X.

(ii) The adjunction term of Σ with respect to L is defined to be

$$J_L(\Sigma) := |c_1(L)\cdot PD[\Sigma]| + [\Sigma]\cdot[\Sigma] + e_\Sigma\,,$$

where $PD : H_2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z})$ is the Poincaré duality isomorphism.

By changing $[\Sigma]$ to $-[\Sigma]$ if necessary, we can always assume that $c_1(L)\cdot PD[\Sigma] \leq 0$. With that understood, we have the following

Lemma 2.2. (i) $[\widetilde{\Sigma}]^2 = \frac{1}{m}[\Sigma]^2$.
(ii) $e_{\widetilde{X}} = me_X - (m-1)e_\Sigma$, *and* $s_{\widetilde{X}} = ms_X - \frac{m^2-1}{3}[\widetilde{\Sigma}]^2$.
(iii) *For any complex line bundle* $L \to X$, *we have* $c_1(p^*L)^2 = mc_1(L)^2$.
(iv) *Let* $\widehat{L}$ *be a line bundle such that* $c_1(\widehat{L}) = c_1(p^*L) - (m-1)PD[\widetilde{\Sigma}]$. *Then* $J_{\widehat{L}}(\widetilde{\Sigma}) = J_L(\Sigma)$.
(v) $d_{\widehat{L}} = md_L + \frac{1}{2}(m-1)J_L(\Sigma)$.

Proof. We refer to the excellent book [GS] for some of the proofs and [RW] for the special case when $m = 2$. The signature formula in part (ii) can be found in [H]. The rest is an easy exercise, which we omit. □

From now on we will restrict our attention to the case when the genus of Σ is one and $[\Sigma]^2 = 0$. We now briefly recall the setup in [P2]. Suppose we are given a compact 4-manifold Y_0 with boundary $\partial Y_0 \cong T^3$ and a fixed factorization $\partial Y_0 = \Sigma \times S^1$, where $\Sigma = T^2$. Let $\gamma = [\{\mathrm{pt}\}\times S^1] \in H_1(\partial Y_0)$, and let $\gamma^* \in H^1(\partial Y_0)$ be the Poincaré dual of $[\Sigma] \in H_2(\partial Y_0)$. Let X be the closed 4-manifold $Y_0 \cup_{\partial Y_0} (\Sigma \times D^2)$, where the boundaries are identified using the above factorization.

Definition 2.3. Let γ denote the homology class of the meridian of Σ in ∂Y_0 as above, and let $i : \partial Y_0 \hookrightarrow Y_0$ be the inclusion map. We shall say that the pair (Y_0, Σ) is *admissible* if the following three conditions are satisfied:

(i) $n\gamma \in \mathrm{Ker}\,[\, i_* : H_1(\partial Y_0;\mathbb{Z}) \to H_1(Y_0;\mathbb{Z})\,]$ for some positive integer n.
(ii) $\mathrm{Coker}\,[\, i^* : H^1(Y_0;\mathbb{Z}) \to H^1(\partial Y_0;\mathbb{Z})\,]$ is torsion-free.
(iii) The intersection form of Y_0 is not negative definite.

For an admissible pair (Y_0, Σ), we define

$$n_\gamma := \min\{n > 0 \mid n\gamma \in \mathrm{Ker}\,[\, i_* : H_1(\partial Y_0;\mathbb{Z}) \to H_1(Y_0;\mathbb{Z})\,]\},$$
$$n_\Sigma := \max\{n > 0 \mid [\Sigma] = n\alpha \ \text{ for some } \ \alpha \in H_2(X;\mathbb{Z})\}.$$

We shall say that the pair (Y_0, Σ) is *strongly admissible* if it is admissible and $n_\gamma = n_\Sigma$.

Note that condition (i) above implies the existence of an embedded surface $\Gamma_0 \subset Y_0$ such that $[\partial\Gamma_0] = n_\gamma \gamma$. Let Γ denote the closed connected surface

$$\Gamma_0 \cup \Big(\coprod_{i=1}^{n_\gamma} \{\mathrm{pt}_i\} \times D^2\Big) \ \subset\ X. \tag{2.1}$$

It follows that $[\Gamma] \in H_2(X;\mathbb{Z})$ is primitive and $[\Gamma]\cdot[\Sigma] = n_\gamma$. Γ is not unique, but we may choose Γ so that its genus is minimal among all such surfaces.

Note that for an m-fold branched cover, m divides n_Σ. Let Θ denote a smoothly embedded surface in X such that $[\Sigma] = n_\Sigma[\Theta]$. Let N denote the regular neighborhood of the union $\Theta \cup \Gamma$ inside X. If (Y_0, Σ) is strongly admissible, then $[\Gamma]\cdot[\Theta] = 1$, and hence there is an orthogonal decomposition

$$H_2(X;\mathbb{Z}) = \langle [\Theta], [\Gamma] \rangle \oplus H_2(X \setminus N;\mathbb{Z})$$

and the corresponding splitting of the intersection form:

$$Q_X = \begin{bmatrix} 0 & 1 \\ 1 & c \end{bmatrix} \oplus Q_{X\setminus N}\,, \tag{2.2}$$

where $c = [\Gamma]\cdot[\Gamma]$.

If (Y_0, Σ) is admissible, then we can define a relative Seiberg-Witten invariant of Y_0 as follows. Let $\mathcal{S}(Y_0)$ denote the set of isomorphism classes of Spin^c structures on Y_0 that restricts to the trivial (canonical) Spin^c structure on the boundary ∂Y_0. Let $\mathcal{H}_{Y_0}$ denote the cokernel of the homomorphism $i^* : H^1(Y_0;\mathbb{Z}) \to H^1(\partial Y_0;\mathbb{Z})$. In [P2], we defined a function

$$SW_{Y_0} : \mathcal{S}(Y_0) \times \mathcal{H}_{Y_0} \longrightarrow \mathbb{Z}\,,$$

which algebraically counts the number of solutions to the twice-perturbed Seiberg-Witten equations for a Spin^c structure $\mathcal{L} \in \mathcal{S}(Y_0)$ on the cylindrical end 4-manifold, $Y := Y_0 \cup_{\partial Y_0} (\partial Y_0 \times [0,\infty))$:

$$\begin{cases} \not\partial_A \phi = 0\,, \\ \rho(F_A + \eta) = q(\phi)\,, \\ \eta = f\cdot(ih^*_{(A,\phi)}(\omega) - ir\pi_1^*\pi^*\mu_\Sigma)\,. \end{cases} \tag{2.3}$$

We refer the reader to [P2] for the exact definition of the terms in (2.3). The cylindrical end moduli space $\mathcal{M}^r_Y(\mathcal{L}, g, \omega)$ is defined by dividing the space of finite energy solutions to (2.3) by the action of the $L^2_{5,\mathrm{loc}}$ gauge group $\mathcal{G}(Y)$. Note that every solution to (2.3) is irreducible, i.e. $\phi \not\equiv 0$. Recall that there exists a continuous map

$$\partial_\infty : \mathcal{M}^r_Y(\mathcal{L}, g, \omega) \longrightarrow \mathcal{H}_{Y_0}\,.$$

If $\mathcal{M}^r_Y(\mathcal{L}, g, \omega)$ is 0-dimensional, then we define

$$SW_{Y_0}(\mathcal{L}, x) := \#[\mathcal{M}^r_Y(\mathcal{L}, g, \omega) \cap \partial_\infty^{-1}(x)].$$

Definition 2.4. (i) Let $\mathcal{K}_{Y_0}$ be the set of isomorphism classes of complex line bundles on Y_0 that pull back to a trivial bundle on $p^{-1}(Y_0)$. Let $\mathcal{K}^*_{Y_0} = \mathcal{K}_{Y_0} \setminus \{\underline{\mathbb{C}}\}$ be the subset of non-trivial line bundles.

(ii) Suppose (Y_0, Σ) is strongly admissible. For any Spin^c structure ξ on X with $\det \xi = L$ and $J_L(\Sigma) = -c_1(L)\cdot PD[\Sigma] = 0$, we let

$$m_L := c_1(L)\cdot PD[\Gamma]\,,$$

where $\Gamma \subset X$ is defined as in (2.1). We also define

$$k_\xi(X, \Sigma) := \sum_{\kappa \in \mathcal{K}^*_{Y_0}} \sum_{n=1}^{\infty} SW_{Y_0}(\xi|_{Y_0} \otimes \kappa\,, (\llbracket m_L/n_\gamma \rrbracket - n)\gamma^*)\,,$$

where $\llbracket m_L/n_\gamma \rrbracket$ denotes the greatest integer less than or equal to m_L/n_γ, and γ^* is the non-zero element of $\mathcal{H}_{Y_0}$ coming from $PD[\Sigma] \in H^1(\partial Y_0)$.

Note that $\mathcal{K}_{Y_0}$ is isomorphic to the kernel of the homomorphism $p^* : H^2(Y_0;\mathbb{Z}) \to H^2(\widetilde{Y}_0;\mathbb{Z})$. Also recall that $c_1(\kappa)$ is m-torsion for every line bundle $\kappa \in \mathcal{K}_{Y_0}$. (See the proof of Theorem 3.8 in [RW] for the case of double covers.) For any complex line bundle $L \to X$, let us write $c_1(L) = a\,PD[\Theta] + b\,PD[\Gamma] + \beta$, with $\beta \in H^2(X \setminus N;\mathbb{Z})$ according to the decomposition (2.2). Then $c_1(L)\cdot PD[\Sigma] = 0$ implies that $b = 0$ and $a = m_L$. When $b_2^+(X) > 1$, one can show that the sum defining $k_\xi(X,\Sigma)$ is finite as in [P1].

3. Main formula

We now restrict ourselves to the case when $m = 2$. The following is a direct generalization of Theorem 6.8 in [RW].

Theorem 3.1. *Let $p : \widetilde{X} \to X$ be a double cover branched along a homologically non-trivial torus $\Sigma \subset X$. Let $Y_0 = X \setminus \nu(\Sigma)$ be the complement of the tubular neighborhood of Σ, and let $\widetilde{Y}_0 = p^{-1}(Y_0)$, $\widetilde{\Sigma} = p^{-1}(\Sigma)$. Assume that $b_2^+(X), b_2^+(\widetilde{X}) > 1$ and $[\Sigma]^2 = 0$. Suppose that ξ is a* Spin^c *structure on X whose determinant line bundle L satisfies $c_1(L)\cdot PD[\Sigma] = 0$ and $d_L = 0$. Let $\mathcal{L}$ denote the restriction $\xi|_{Y_0}$. If both $(\widetilde{Y}_0, \widetilde{\Sigma})$ and (Y_0,Σ) are admissible, then we have*

$$SW_{\widetilde{Y}_0}(p^*(\mathcal{L}), p^*(x)) \;\equiv\; SW_{Y_0}(\mathcal{L}, x) \;+ \sum_{\kappa \in \mathcal{K}^*_{Y_0}} SW_{Y_0}(\mathcal{L}\otimes\kappa, x) \mod 2.$$

Moreover let $\widehat{\xi}$ be a Spin^c *structure on $\widetilde{X}$ whose determinant bundle is $\widehat{L} = p^*L \otimes PD[\widetilde{\Sigma}]^{-1}$ and whose restriction to $\widetilde{Y}_0$ is $p^*(\mathcal{L})$. Also define*

$$\Lambda_L(\Sigma) := \frac{m_L}{n_\Sigma} - \left[\!\!\left[\frac{m_L}{n_\Sigma} \right]\!\!\right] .$$

Suppose both $(\widetilde{Y}_0, \widetilde{\Sigma})$ and (Y_0,Σ) are strongly admissible. If $\Lambda_L(\Sigma) \neq \frac{1}{2}$, then we have

$$SW_{\widetilde{X}}(\widehat{\xi}) \;\equiv\; SW_X(\xi) \;+\; k_\xi(X,\Sigma) \mod 2. \tag{3.1}$$

If $\Lambda_L(\Sigma) = \frac{1}{2}$, then we have

$$SW_X(\xi) \;+\; k_\xi(X,\Sigma) \;\equiv\; 0 \mod 2.$$

Proof. Let $\widetilde{Y} := \widetilde{Y}_0 \cup_{\partial\widetilde{Y}_0} (\partial\widetilde{Y}_0 \times [0,\infty))$. Consider the following system of equations for the Spin^c structure $p^*(\mathcal{L})$ on $\widetilde{Y}$:

$$\begin{cases} \widetilde{\partial\!\!\!/}_A \phi = 0\,, \\ \rho(F_A + \widetilde{\eta}) = q(\phi)\,, \\ \widetilde{\eta} = (f\circ p)\cdot(ih^*_{(A,\phi)}(p^*\omega) - ir\,p^*\pi_1^*\pi^*\mu_\Sigma)\,. \end{cases} \tag{3.2}$$

Here, $\widetilde{\partial\!\!\!/}_A$ is a twisted Dirac operator constructed from the Levi-Civita connection of the pull-back metric p^*g on $\widetilde{Y}$. (g is a generic cylindrical end metric on Y used to define (2.3).) Let $\widetilde{\mathcal{M}}^r_{\widetilde{Y}}(p^*\mathcal{L}, p^*g, p^*\omega)$ denote the solution space of (3.2) divided by the action of a suitable gauge group $\mathcal{G}(\widetilde{Y})$. We can choose ω such that both ω and $p^*\omega$ are supported away from the critical values of the Chern-Simons-Dirac functional of [P2]. Hence we can prove the compactness the same way and obtain a continuous map

$$\partial_\infty : \widetilde{\mathcal{M}}^r_{\widetilde{Y}}(p^*\mathcal{L}, p^*g, p^*\omega) \longrightarrow \mathcal{H}_{\widetilde{Y}_0}\,.$$

Note that $d_L = 0$ implies that the virtual dimension of $\mathcal{M}_Y^r(\mathcal{L}, g, \omega)$ is zero. Since we have also assumed that $J_L(\Sigma) = 0$, it follows that $d_{\widehat{L}} = 0$, and hence the virtual dimension of $\widetilde{\mathcal{M}}_{\widetilde{Y}}^r(p^*\mathcal{L}, p^*g, p^*\omega)$ is also zero. We introduce the notation

$$\begin{aligned}\widetilde{\mathcal{N}}_{\widetilde{Y}_0}(p^*\mathcal{L}, p^*x\,; r, p^*g, p^*\omega) &:= \widetilde{\mathcal{M}}_{\widetilde{Y}}^r(p^*\mathcal{L}, p^*g, p^*\omega) \cap \partial_\infty^{-1}(p^*x)\,,\\ \mathcal{N}_{Y_0}(\mathcal{L}, x\,; r, g, \omega) &:= \mathcal{M}_Y^r(\mathcal{L}, g, \omega) \cap \partial_\infty^{-1}(x)\,.\end{aligned}$$

Note that there is a well-defined homomorphism $p^* : \mathcal{H}_{Y_0} \to \mathcal{H}_{\widetilde{Y}_0}$.

Next let $P_{SO(4)}$ denote the frame bundle of $\widetilde{Y}_0$, and let $\widetilde{P}$ be the principal Spin$^c(4)$ bundle corresponding to $p^*\mathcal{L}$. As in [RW] we choose a lifting τ of the covering involution σ:

$$\begin{array}{ccc}\widetilde{P} & \xrightarrow{\ \tau\ } & \widetilde{P}\\ \downarrow & & \downarrow\\ P_{SO(4)} & \xrightarrow{\ \sigma_*\ } & P_{SO(4)}\end{array}$$

Let τ^* denote the induced action on the Seiberg-Witten configuration space $\mathcal{B}(p^*\mathcal{L})$. Given any subset $S \subset \mathcal{B}(p^*\mathcal{L})$, we let $S^\tau \subset S$ be the fixed point set of $\tau^* : \mathcal{B}(p^*\mathcal{L}) \to \mathcal{B}(p^*\mathcal{L})$. Since every solution to (3.2) is irreducible, Theorem 3.8 of [RW] gives us a homeomorphism

$$\begin{aligned}&\widetilde{\mathcal{N}}_{\widetilde{Y}_0}(p^*\mathcal{L}, p^*x\,; r, p^*g, p^*\omega)^\tau\\ &\qquad \approx \mathcal{N}_{Y_0}(\mathcal{L}, x\,; r, g, \omega) \coprod \left(\coprod_{\kappa \in \mathcal{K}_{Y_0}^*} \mathcal{N}_{Y_0}(\mathcal{L} \otimes \kappa, x\,; r, g, \omega) \right).\end{aligned}$$

Now let g' be a generic cylindrical end metric on $\widetilde{Y}$. We need to compare the moduli spaces of (2.3) and (3.2) on $\widetilde{Y}$. Since $p^*\pi_1^*\pi^*\mu_\Sigma = 2\pi_1^*\pi^*\mu_{\widetilde{\Sigma}}$ and we are free to vary the parameter $r \in \mathbb{R} \setminus \{0\}$ in (2.3) (as long as $|r|$ is very small and the sign of r remains the same), we easily see that the only difference between $\mathcal{M}_{\widetilde{Y}}^{2r}(p^*\mathcal{L}, g', p^*\omega)$ and $\widetilde{\mathcal{M}}_{\widetilde{Y}}^r(p^*\mathcal{L}, p^*g, p^*\omega)$ is in the non-generic choice of the metric p^*g in the definition of (3.2). Hence Theorem 2.2 of [RW] implies that

$$\begin{aligned}SW_{\widetilde{Y}_0}(p^*\mathcal{L}, p^*x) \;&=\; \#[\mathcal{N}_{\widetilde{Y}_0}(p^*\mathcal{L}, p^*x\,; 2r, g', p^*\omega)]\\ &\equiv\; \#[\widetilde{\mathcal{N}}_{\widetilde{Y}_0}(p^*\mathcal{L}, p^*x\,; r, p^*g, p^*\omega)^\tau] \quad \bmod 2.\end{aligned}$$

This proves our first congruence. To prove the second, recall from [P2] (Corollary 20) that

$$\overline{SW}_X = \overline{SW}_{Y_0} \cdot ([\Sigma] + [\Sigma]^3 + [\Sigma]^5 + \cdots)\,, \tag{3.3}$$

where $\overline{SW}$ denotes the Seiberg-Witten series. Hence it follows from the product formula in [P2] that

$$SW_X(\xi) \;= \sum_{n=1}^{\infty} SW_{Y_0}(\mathcal{L}\,, (\llbracket m_L/n_\Sigma \rrbracket - 2n + 1)\gamma^*)\,. \tag{3.4}$$

(Note that $PD[\Theta] \in H^2(Y_0; \mathbb{Z}) \cong H_2(Y_0, \partial Y_0; \mathbb{Z})$ is n_Σ-torsion.) γ^* is indivisible and since the difference (modulo torsion) of any two characteristic elements of $H^2(Y_0, \partial Y_0; \mathbb{Z})$ is divisible by two (cf. [MMS], [P1]), we must have by default

$$SW_{Y_0}(\mathcal{L}\,, (\llbracket m_L/n_\Sigma \rrbracket - 2n)\gamma^*) \;=\; 0 \tag{3.5}$$

for every integer n. Thus we may rewrite (3.4) as

$$SW_X(\xi) = \sum_{n=1}^{\infty} SW_{Y_0}(\mathcal{L}\,,(\llbracket m_L/n_\Sigma \rrbracket - n)\gamma^*)\,.$$

Summing our first congruence over n, we conclude that

$$\begin{aligned} &SW_X(\xi) + k_\xi(X,\Sigma) \\ &\quad = \sum_{n=1}^{\infty}\left(SW_{Y_0}(\mathcal{L}\,,(\left\llbracket \frac{m_L}{n_\Sigma}\right\rrbracket - n)\gamma^*) + \sum_{\kappa\in\mathcal{K}^*_{Y_0}} SW_{Y_0}(\mathcal{L}\otimes\kappa\,,(\left\llbracket \frac{m_L}{n_\Sigma}\right\rrbracket - n)\gamma^*)\right) \\ &\quad \equiv \sum_{n=1}^{\infty} SW_{\widetilde{Y}_0}(p^*\mathcal{L},\, p^*(\llbracket m_L/n_\Sigma \rrbracket - n)\gamma^*) \mod 2. \end{aligned}$$

Let $\delta \in H_1(\partial\widetilde{Y}_0)$ denote the homology class of the meridian of $\widetilde{\Sigma}$, and let $\delta^* := PD[\widetilde{\Sigma}] \in H^1(\partial\widetilde{Y}_0)$. As in (2.1), let $\Delta \subset \widetilde{X}$ be a minimal genus surface such that $[\widetilde{\Sigma}]\cdot[\Delta] = n_\delta = n_{\widetilde{\Sigma}}$. Note that $[\Sigma]\cdot p_*[\Delta] = 2n_\delta = n_\gamma$. Since $p^*(\gamma^*) = 2\delta^*$, we get

$$SW_X(\xi) + k_\xi(X,\Sigma) \equiv \sum_{n=1}^{\infty} SW_{\widetilde{Y}_0}(p^*\mathcal{L}\,,\, 2(\llbracket m_L/n_\Sigma \rrbracket - n)\delta^*) \mod 2. \tag{3.6}$$

As in (3.4), we have

$$SW_{\widetilde{X}}(\widehat{\xi}) = \sum_{n=1}^{\infty} SW_{\widetilde{Y}_0}(p^*\mathcal{L}\,,(\llbracket m_{\widehat{L}}/n_{\widetilde{\Sigma}} \rrbracket - 2n + 1)\delta^*)\,.$$

But now note that

$$m_{\widehat{L}} = c_1(\widehat{L})\cdot PD[\Delta] = -n_\delta + c_1(p^*L)\cdot PD[\Delta] = -n_\delta + m_L\,.$$

Since $n_\Sigma = 2n_{\widetilde{\Sigma}}$ (cf. [GS]), we must have

$$\frac{m_{\widehat{L}}}{n_{\widetilde{\Sigma}}} = \frac{-n_\delta + m_L}{n_\Sigma/2} = -1 + \frac{2m_L}{n_\Sigma}\,.$$

Hence it follows that

$$SW_{\widetilde{X}}(\widehat{\xi}) = \sum_{n=1}^{\infty} SW_{\widetilde{Y}_0}(p^*\mathcal{L}\,,(\llbracket 2m_L/n_\Sigma \rrbracket - 2n)\delta^*)\,. \tag{3.7}$$

Let $\overline{\xi}$ be the complex conjugate Spin^c structure on X with $\det\overline{\xi} = -L$. We have $SW_X(\overline{\xi}) = (-1)^{(e_X+s_X)/4}SW_X(\xi)$. Hence by changing ξ to $\overline{\xi}$ if necessary, we may assume that

$$0 \le \frac{m_L}{n_\Sigma} - \left\llbracket \frac{m_L}{n_\Sigma} \right\rrbracket \le \frac{1}{2}\,.$$

Now if $m_L/n_\Sigma - \llbracket m_L/n_\Sigma \rrbracket \neq 1/2$, then we get

$$\llbracket 2m_L/n_\Sigma \rrbracket = 2\llbracket m_L/n_\Sigma \rrbracket.$$

Thus the combination of (3.6) and (3.7) proves our second congruence, provided that $m_L/n_\Sigma - \llbracket m_L/n_\Sigma \rrbracket \neq 1/2$. Finally suppose that

$$m_L = \left\llbracket \frac{m_L}{n_\Sigma} \right\rrbracket n_\Sigma + \frac{n_\Sigma}{2}\,. \tag{3.8}$$

Then we have $[\![2m_L/n_\Sigma]\!] = 2[\![m_L/n_\Sigma]\!] + 1$, and thus

$$SW_{\widetilde{X}}(\widehat{\xi}) \;=\; \sum_{n=1}^{\infty} SW_{\widetilde{Y}_0}(p^*\mathcal{L}\,,(2[\![m_L/n_\Sigma]\!] + 1 - 2n)\delta^*)\,.$$

As in (3.5), it follows that

$$SW_{\widetilde{Y}_0}(p^*\mathcal{L}\,,(2[\![m_L/n_\Sigma]\!] - 2n)\delta^*) \;=\; 0$$

for every integer n. Hence (3.6) implies that

$$SW_X(\xi) + k_\xi(X,\Sigma) \equiv 0 \mod 2. \qquad \square$$

Remark 3.2. (i) If $b_2^+(X) > 1$, then by the adjunction inequality (cf. [OS]), $c_1(L) \cdot PD[\Sigma] \neq 0$ implies that $SW_X(\xi) = 0$. Since $c_1(\widehat{L}) \cdot PD[\widetilde{\Sigma}] = c_1(L) \cdot PD[\Sigma]$, it also implies that if $b_2^+(\widetilde{X}) > 1$, then we must have $SW_{\widetilde{X}}(\widehat{\xi}) = 0$ as well.

(ii) If $c_1(L) \cdot PD[\Sigma] = 0$ but $d_L \neq 0$, then by Lemma 2.2 (v) we have $d_{\widehat{L}} \neq 0$. Hence if $\widetilde{X}$ is of Seiberg-Witten simple-type (e.g. if $\widetilde{X}$ is a symplectic manifold), then $SW_{\widetilde{X}}(\widehat{\xi}) = 0$.

(iii) Let $\lambda \to X$ be the complex line bundle with $c_1(\lambda) = \frac{1}{2}PD[\Sigma]$ determined by the branched cover $p : \widetilde{X} \to X$. If (Y_0, Σ) is admissible, then $[\Sigma] \in H_2(X;\mathbb{Z})$ is not torsion. Hence Proposition 5.3 of [RW] implies that $\lambda|_{Y_0} \in \mathcal{K}^*_{Y_0}$, and if $H_1(X;\mathbb{Z})$ contains no 2-torsion, then $\mathcal{K}^*_{Y_0} = \{\lambda|_{Y_0}\}$.

4. Examples

We start out with the following conjecture made by Ruan and Wang:

Conjecture 4.1 (cf. [RW], p. 502)**.** *The difference of the SW-invariants,*

$$SW_{\widetilde{X}}(\widehat{\xi}) - SW_X(\xi) \mod 2,$$

is not always zero.

We prove the conjecture (at least when the genus of Σ is one) by providing examples with $k_\xi(X,\Sigma) \equiv 1 \mod 2$.

For each integer $n > 0$, let $E(n)$ be a simply-connected elliptic surface with no multiple fibers and with geometric genus $p_g = n - 1$. Let F denote a generic torus fiber of $E(n)$, and let $E(n)_q$ be the result of a logarithmic transformation of multiplicity $q > 1$ along F. Let $F_q \subset E(n)_q$ denote the multiple fiber. Recall that $[F] = q[F_q]$ in $H_2(E(n)_q;\mathbb{Z})$. If $q = 2\ell$ is even, then let $\widetilde{E}(n)_q$ be the double cover of $E(n)_q$ branched along the torus $\Sigma = F$.

We may take Γ to be the union of a normal disk of F_q and q punctured sections in $E(n) \setminus \nu(F)$, and set $\Theta = F_q$. It is not very hard to check that the pairs $(\widetilde{E}(n)_q \setminus \nu(\widetilde{F}), \widetilde{F})$ and $(E(n)_q \setminus \nu(F), F)$ are strongly admissible. (Note that $\mathcal{H}_{\widetilde{E}(n)_q \setminus \nu(\widetilde{F})} \cong \mathcal{H}_{E(n)_q \setminus \nu(F)} \cong \mathbb{Z}^3$.)

Lemma 4.2. *Let $t = [F]$ and $t_q = [F_q]$ in $H_2(E(n)_q;\mathbb{Z})$. Then we have*

$$\overline{SW}_{E(n)_q} = (t^{-1} - t)^{n-2} \cdot (t_q^{q-1} + t_q^{q-3} + \cdots + t_q^{-(q-3)} + t_q^{-(q-1)})\,,$$
$$\overline{SW}_{E(n)_q \setminus \nu(F)} = (t^{-1} - t)^{n-1} \cdot (t_q^{q-1} + t_q^{q-3} + \cdots + t_q^{-(q-3)} + t_q^{-(q-1)})\,.$$

Proof. $\overline{SW}_{E(n)_q}$ was computed in [FS]. Apply Corollary 20 of [P2] to it. (See formula (3.3) above.) $\square$

Theorem 4.3. *Let ξ_j denote the* Spin^c *structure on $E(2)_{2\ell}$ with $c_1(\det \xi_j) = PD(j[F_{2\ell}])$. If j is odd and satisfies $1 \le j \le 2\ell - 1$, then*

$$k_{\xi_j}(E(2)_{2\ell}, F) = 1 .$$

Moreover, if $j \neq \ell$, then

$$SW_{\widetilde{E}(2)_{2\ell}}(\widehat{\xi}_j) - SW_{E(2)_{2\ell}}(\xi_j) \equiv 1 \mod 2.$$

Therefore Conjecture 4.1 *is true.*

Proof. As before let $Y_0 = E(2)_{2\ell} \setminus \nu(F)$ and let $L_j = \det \xi_j$. Let $t = [F]$ and $t_q = [F_{2\ell}]$ with $q = 2\ell$. Note that $n_F = 2\ell$ and $m_{L_j} = j$. Thus $[\![m_{L_j}/n_F]\!] = 0$. Since $E(n)_q$ is simply-connected, $\mathcal{K}^*_{Y_0} = \{\lambda|_{Y_0}\}$, where $c_1(\lambda) = \ell PD(t_q)$. Thus if $1 \le j \le 2\ell - 1$, we have

$$k_{\xi_j}(E(2)_{2\ell}, F) = \sum_{n=1}^{\infty} SW_{Y_0}(\xi_j|_{Y_0} \otimes \lambda|_{Y_0}, (-n)\gamma^*) .$$

Now recall from [P2] that

$$SW_{Y_0}(\xi_j|_{Y_0} \otimes \lambda|_{Y_0}, m\gamma^*) = SW_{Y_0}(\xi_j|_{Y_0}, (m-1)\gamma^*)$$

for every integer m. Hence we conclude that

$$k_{\xi_j}(E(2)_{2\ell}, F) = \sum_{n=1}^{\infty} SW_{Y_0}(\xi_j|_{Y_0}, (-n-1)\gamma^*) .$$

From Lemma 4.2, we have

$$\overline{SW}_{Y_0} = (t^{-1} - t) \cdot (t_q^{q-1} + t_q^{q-3} + \cdots + t_q^{-(q-3)} + t_q^{-(q-1)}) .$$

Let $a_{n,j}$ denote the coefficient of $t^n t_q^j$ in $\overline{SW}_{Y_0}$ with $|j| \le q-1$. From the definition of a Seiberg-Witten series in [P2], we deduce that if $|j| \le q-1$, then

$$a_{n,j} = SW_{Y_0}(\xi_j|_{Y_0}, n\gamma^*) .$$

It follows that if j is odd and $1 \le j \le 2\ell - 1$, then

$$k_{\xi_j}(E(2)_{2\ell}, F) = \sum_{n=1}^{\infty} a_{-n-1,j} = a_{-2,j} = a_{-1,j-2\ell} = 1 .$$

For the last statement note that if $1 \le j \le 2\ell - 1$ and $j \neq \ell$, then

$$\Lambda_{L_j}(F) = \frac{m_{L_j}}{n_F} - \left[\!\!\left[\frac{m_{L_j}}{n_F} \right]\!\!\right] = \frac{j}{2\ell} - \left[\!\!\left[\frac{j}{2\ell} \right]\!\!\right] = \frac{j}{2\ell} \neq \frac{1}{2} .$$

Now apply congruence (3.1) of Theorem 3.1. □

Remark 4.4. (i) For $n > 2$, let $\xi_{i,j}$ be the Spin^c structure on $E(n)_{2\ell}$ such that $c_1(\det \xi_{i,j}) = PD(it + j t_q)$ with $|j| < 2\ell$. As in the last proof, we can express $k_{\xi_{i,j}}(E(n)_{2\ell}, F)$ as a sum of coefficients of $\overline{SW}_{E(n)_{2\ell} \setminus \nu(F)}$. In this way we can find a plethora of instances where $k_{\xi_{i,j}}(E(n)_{2\ell}, F) \equiv SW_{\widetilde{E}(n)_{2\ell}}(\widehat{\xi}_{i,j}) - SW_{E(n)_{2\ell}}(\xi_{i,j}) \equiv 1$ mod 2.

(ii) If ℓ is odd and $j = \ell$, then $\Lambda_{L_j}(F) = \frac{1}{2}$ and $SW_{E(2)_{2\ell}}(\xi_j) = 1$. Thus we have

$$SW_{E(2)_{2\ell}}(\xi_j) + k_{\xi_j}(E(2)_{2\ell}, F) = 2 \equiv 0 \mod 2,$$

which is the final congruence of Theorem 3.1.

(iii) Many more examples with $k_\xi(X, \Sigma) \equiv 1 \mod 2$ can be found in [P3].

Acknowledgments

The author would like to thank Ian Hambleton for introducing him to the field of equivariant gauge theory and providing inspiration. He would also like to thank Tian-Jun Li and Michael McCooey for very helpful discussions.

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