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BRANGESIAN SPACES IN $H^p(\mathbf{T}^2)$

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ABSTRACT. In this note, we characterize certain algebraic subspaces of $H^p(\mathbf{T}^2)$ extending D. Singh's $H^2(\mathbf{T}^2)$ result.

1. Introduction

The so-called Brangesian spaces (Hilbert spaces contained in Banach spaces) were first introduced by L. de Branges in his proof of the famous Bieberbach conjecture. L. de Branges characterized the contractively contained Hilbert spaces in $H^2(\mathbf{T})$ (see [5]). Since then, many authors have examined this notation in various settings (see [4], [5] for some results in this area). In [4], D. Singh characterized certain algebraic subspaces of $H^2(\mathbf{T}^2)$. His main result follows.

Singh's Theorem. *$\mathcal{N}$ is a Hilbert space contained in $H^2(\mathbf{T}^2)$ such that $\mathcal{N}$ is invariant under S_1 and S_2 and for which S_1 and S_2 are doubly commuting isometries on $\mathcal{N}$ if and only if there exists g in $H^\infty(\mathbf{T}^2)$ unique up to a factor of modulus one such that $\mathcal{N} = gH^2(\mathbf{T}^2)$ with norm $\|gf\|_{\mathcal{N}} = \|f\|_2$ for all f in $H^2(\mathbf{T}^2)$.*

This is a generalization of Mandrekar's main result in [1], since every subspace (closed linear manifold) of $H^2(\mathbf{T}^2)$ is a Hilbert space in the $H^2(\mathbf{T}^2)$-norm.

Mandrekar's Theorem. *$\mathcal{N}$ is a subspace in $H^2(\mathbf{T}^2)$ such that $\mathcal{N}$ is invariant under S_1 and S_2 and for which S_1 and S_2 are doubly commuting on $\mathcal{N}$ if and only if there exists g in $H^\infty(\mathbf{T}^2)$ which is inner and unique up to a constant factor of modulus one such that $\mathcal{N} = gH^2(\mathbf{T}^2)$.*

Both results weigh heavily on the doubly commuting condition. Rudin in [3] constructed an invariant subspace in $H^2(\mathbf{T}^2)$ that contains no bounded elements and an invariant subspace that is not generated by a single function; in fact, it is not even finitely generated. Of course, on these subspaces S_1 and S_2 are not doubly commuting. These examples show that even subspaces of $H^2(\mathbf{T}^2)$ can get pretty complicated if the doubly commuting condition is removed. In fact, it is still an open question to describe all of the invariant subspaces of $H^2(\mathbf{T}^2)$. In this note we do not concern ourselves with subspaces, but rather Brangesian spaces. Our main result is to extend Singh's result to $H^p(\mathbf{T}^2)$. As a corollary, we prove a result about certain algebraic subspaces of $BMOA(\mathbf{T}^2)$.

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2. Notation and terminology

We let $\mathbf{C}^2$ denote the cartesian product of two copies of $\mathbf{C}$. The unit bidisc in $\mathbf{C}^2$ is denoted by $\mathbf{U}^2$ and the distinguished boundary by $\mathbf{T}^2$, where $\mathbf{U}$ and $\mathbf{T}$ are the unit disc and unit circle in the complex plane, respectively.

The Hardy space $H^p(\mathbf{U}^2)$ $(1 \leq p < \infty)$ is the Banach space of holomorphic functions over $\mathbf{U}^2$ which satisfy the inequality

$$\sup_{0 \leq r < 1} \int_{\mathbf{T}^2} |f(r\xi_1, r\xi_2)|^p \, dm_2(\xi_1, \xi_2) < \infty$$

where m_2 denotes normalized Lebesgue measure on $\mathbf{T}^2$. Note that holomorphic here means holomorphic in each variable. The norm $\|f\|_p$ of a function f in $H^p(\mathbf{U}^2)$ is defined by

$$\|f\|_p = \sup_{0 \leq r < 1} \left(\int_{\mathbf{T}^2} |f(r\xi_1, r\xi_2)|^p \, dm_2(\xi_1, \xi_2) \right)^{1/p}.$$

The Hardy space $H^\infty(\mathbf{U}^2)$ is the Banach space of holomorphic functions over $\mathbf{U}^2$ which satisfy the inequality

$$\sup_{(z_1, z_2) \in \mathbf{U}^2} |f(z_1, z_2)| < \infty.$$

The norm $\|f\|_\infty$ of a function f in $H^\infty(\mathbf{U}^2)$ is defined by

$$\|f\|_\infty = \sup_{(z_1, z_2) \in \mathbf{U}^2} |f(z_1, z_2)|.$$

It is well known (see [3]) that every function in $H^p(\mathbf{U}^2)$ $(1 \leq p \leq \infty)$ has a nontangential limit at $[m_2]$ almost every point of $\mathbf{T}^2$. Let f^* denote the boundary function of an f in $H^p(\mathbf{U}^2)$. Then

$$f^* \in H^p(\mathbf{T}^2) \equiv \overline{span}^{L^p(\mathbf{T}^2, m_2)} \left\{ \xi_1^n \xi_2^m : n, m \geq 0 \right\}.$$

It is also known (see [3]) that f can be reconstructed by the Poisson integral as well as the Cauchy integral of f^*. Further,

$$\|f\|_p = \|f^*\|_p$$

where the second norm is the $L^p(\mathbf{T}^2, m_2)$ norm. For this reason, we identify $H^p(\mathbf{U}^2)$ and $H^p(\mathbf{T}^2)$ and no longer distinguish between f and f^*. Therefore, these Banach spaces of holomorphic functions $H^p(\mathbf{U}^2)$ may be viewed as a subspace of $L^p(\mathbf{T}^2, m_2)$.

For f in $L^p(\mathbf{T}^2) = L^p(\mathbf{T}^2, m_2)$, S_1 and S_2 will denote the operators of multiplication by the first and second coordinate functions, respectively. That is,

$$S_1(f)(\xi_1, \xi_2) = \xi_1 f(\xi_1, \xi_2)$$

and

$$S_2(f)(\xi_1, \xi_2) = \xi_2 f(\xi_1, \xi_2).$$

When S_1 commutes with S_2 and S_1 commutes with S_2^* (S_1 commuting with S_2^* is equivalent to S_1^* commuting with S_2) we say that S_1 and S_2 are **doubly commuting**. This concept is used throughout this paper. We finally recall two other concepts from operator theory used in this note. An operator S from a Hilbert space $\mathcal{H}$ into itself is called an **isometry** if

$$\|Sx\|_{\mathcal{H}} = \|x\|_{\mathcal{H}}$$

for all x in $\mathcal{H}$ and a **shift** if

$$\bigcap_{n\geq 0} S^n(\mathcal{H}) = \{0\}.$$

3. Main result

In the following theorem, when we say a Hilbert space is contained in a Banach space we mean the Hilbert space sits inside of the Banach space as an algebraic subspace.

Theorem. *If $\mathcal{M}$ is a Hilbert space contained in $H^p(\mathbf{T}^2)$, invariant under S_1 and S_2 and if S_1 and S_2 are doubly commuting isometries on $\mathcal{M}$, then*

$$\mathcal{M} = bH^2(\mathbf{T}^2)$$

for a unique b (unique up to a factor of modulus one):

(1) *If $1 \leq p \leq 2$, $b \in H^{\frac{2p}{2-p}}(\mathbf{T}^2)$. When $p = 2$, we mean $H^\infty(\mathbf{T}^2)$.*
(2) *If $p > 2$, $b = 0$.*

Further, $\|bf\|_{\mathcal{M}} = \|f\|_2$ for all f in $H^2(\mathbf{T}^2)$ $(1 \leq p \leq 2)$.

When $p = 2$, we get D. Singh's main result in [4]. We also point out that the converse of this theorem is true. This theorem was motivated by D. Singh and S. Agrawal's work in [5]. Before we prove this theorem, we give several lemmas. The first two lemmas are due to Slocinski [6].

Lemma 1 (Slocinski [6])**.** *Suppose that V_1 and V_2 are commuting isometries on a Hilbert space $\mathcal{H}$ and write $R_i^\perp = \mathcal{H} \ominus V_i(\mathcal{H})$ $(i = 1, 2)$. Then the following are equivalent:*

(1) *There is a wandering subspace $\mathcal{L}$ for the semigroup $\left\{V_1^n V_2^m\right\}_{n,m\geq 0}$ such that*

$$\mathcal{H} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \oplus V_1^n V_2^m(\mathcal{L}).$$

(2) *V_1 and V_2 are doubly commuting shifts.*
(3) *$R_1^\perp \cap R_2^\perp$ is a wandering subspace for the semigroup $\left\{V_1^n V_2^m\right\}_{n,m\geq 0}$ and*

$$\mathcal{H} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \oplus V_1^n V_2^m(R_1^\perp \cap R_2^\perp).$$

Lemma 2 (Slocinski [6])**.** *Suppose V_1 and V_2 are commuting isometries on the Hilbert space $\mathcal{H} \neq \{0\}$. If $R_1^\perp \cap R_2^\perp = \{0\}$ where $R_i^\perp = \mathcal{H} \ominus V_i(\mathcal{H})$ $(i = 1, 2)$, then V_1 and V_2 are not doubly commuting shifts.*

Lemma 3. *If ψ is positive and lower semicontinuous (l.s.c.) on $\mathbf{T}^2$ and $\psi \in L^p(\mathbf{T}^2)$, then $\psi = |f|$ a.e. for some $f \in H^p(\mathbf{T}^2)$.*

We point out that this lemma was proved for the case $p = \infty$ in [3].

Proof. Since *l.s.c.* functions attain their minimum on compact sets, we may assume without loss of generality that $\psi > 1$. Applying Theorem 2.4.2 from [3] to $\log\psi$ asserts the existence of a singular measure $\sigma \geq 0$ and a holomorphic function g in

$\mathbf{U}^2$ such that $Re(g) = P[\log\psi - d\sigma]$. Put $f = \exp(g)$. Then f is holomorphic in $\mathbf{U}^2$ and

$$|f| = \exp(P[\log\psi - d\sigma]) \leq \exp(P[\log\psi]) \leq P[\psi]$$

in $\mathbf{U}^2$. Since $\psi \in L^p(\mathbf{T}^2)$, $f \in H^p(\mathbf{T}^2)$ and

$$|f| = |\exp(g)| = \exp(\log\psi) = \psi$$

a.e. on $\mathbf{T}^2$ as desired. $\square$

In the above proof, $P[\log\psi - d\sigma]$ means the Poisson integral of $\log\psi - d\sigma$.

Lemma 4. *For all $h \in L^p(\mathbf{T}^2)$ with $1 \leq p < \infty$, there exists a positive l.s.c. $\phi \in L^p(\mathbf{T}^2)$ such that $\phi \geq |h|$ a.e. on $\mathbf{T}^2$.*

Proof. If $h \in L^p(\mathbf{T}^2)$, then $|h| \in L^p(\mathbf{T}^2)$ and is real-valued. So, by Lemma 1 of [2], there exist two positive *l.s.c.* functions ϕ and ψ in $L^p(\mathbf{T}^2)$ such that

$$|h| = \phi - \psi \quad \textit{a.e.} \text{ on } \mathbf{T}^2.$$

So

$$|h| \leq \phi \quad \textit{a.e.} \text{ on } \mathbf{T}^2.$$

$\square$

Lemma 5. *Let f be an element of $H^p(\mathbf{T}^2)$ that multiplies $H^2(\mathbf{T}^2)$ into $H^p(\mathbf{T}^2)$. Then f multiplies $L^2(\mathbf{T}^2)$ into $L^p(\mathbf{T}^2)$.*

Proof. Let g be an element of $L^2(\mathbf{T}^2)$. Then by Lemma 4 there exists a positive *l.s.c.* function ϕ in $L^2(\mathbf{T}^2)$ such that $|g| \leq \phi$ *a.e.* on $\mathbf{T}^2$. Then by Lemma 3 there exists an h in $H^2(\mathbf{T}^2)$ such that $|h| = \phi$ *a.e.* on $\mathbf{T}^2$. Now consider

$$\begin{aligned}
\int_{\mathbf{T}^2} |fg|^p \, dm_2 &= \int_{\mathbf{T}^2} |f|^p |g|^p \, dm_2 \\
&\leq \int_{\mathbf{T}^2} |f|^p |h|^p \, dm_2 \quad \text{since } |g| \leq \phi = |h| \ \textit{a.e.} \text{ on } \mathbf{T}^2 \\
&= \int_{\mathbf{T}^2} |fh|^p \, dm_2 \ < \ \infty \quad \text{by hypothesis.}
\end{aligned}$$

$\square$

This next lemma is a straightforward calculation found in [5].

Lemma 6. *If g is a measurable function on $\mathbf{T}^2$ that multiplies $L^2(\mathbf{T}^2)$ into $L^q(\mathbf{T}^2)$ where $1 \leq q \leq 2$, then $g \in L^{\frac{2q}{2-q}}(\mathbf{T}^2)$. When $q = 2$, we mean $L^\infty(\mathbf{T}^2)$.*

We now prove our theorem.

Proof. We first consider the case $1 \leq p \leq 2$. Observe that $\bigcap_{n=0}^\infty S_i^n(\mathcal{M}) = \{0\}$ $(i = 1, 2)$. This observation and our doubly commuting hypothesis give us that

$$\mathcal{M} = \sum_{n=0}^\infty \sum_{m=0}^\infty \oplus V_1^n V_2^m (R_1^\perp \cap R_2^\perp) \quad \text{by Lemma 1}$$

and that

$$R_1^\perp \cap R_2^\perp \neq \{0\} \quad \text{by Lemma 2}$$

where $R_i^\perp = \mathcal{H} \ominus V_i(\mathcal{H})$ $(i = 1, 2)$. So we may take g from $R_1^\perp \cap R_2^\perp$, with $\|g\|_{\mathcal{M}} = 1$. Then $\left\{ g e^{in\theta_1} e^{im\theta_2} \right\}_{n,\ m \geq 0}$ is an orthonormal sequence in $\mathcal{M}$. Let f be an arbitrary element of $H^2(\mathbf{T}^2)$. Then $f(e^{i\theta_1}, e^{i\theta_2}) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\hat{f}(n,\ m)e^{in\theta_1}e^{im\theta_2}$. Let $f_{nm}(e^{i\theta_1}, e^{i\theta_2}) = \sum_{k=0}^{n}\sum_{l=0}^{m}\hat{f}(k,l)e^{ik\theta_1}e^{il\theta_2}$. Then f_{nm} converges to f in $L^2(\mathbf{T}^2)$ and *a.e.* along rectangles. We make the following computation:

$$\begin{aligned}
\|f_{nm}\|_2^2 &= \sum_{k=0}^{n}\sum_{l=0}^{m}|\hat{f}(k,l)|^2 \\
&= \sum_{k=0}^{n}\sum_{l=0}^{m}|\hat{f}(k,l)|^2\|ge^{ik\theta_1}e^{il\theta_2}\|_{\mathcal{M}}^2 \\
&= \sum_{k=0}^{n}\sum_{l=0}^{m}\|\hat{f}(k,l)ge^{ik\theta_1}e^{il\theta_2}\|_{\mathcal{M}}^2 \\
&= \left\|\sum_{k=0}^{n}\sum_{l=0}^{m}\hat{f}(k,l)ge^{ik\theta_1}e^{il\theta_2}\right\|_{\mathcal{M}}^2 .
\end{aligned} \tag{1}$$

Since $\Big(f_{nm}\Big)_{(n,m)}$ is Cauchy in $L^2(\mathbf{T}^2)$, $\Big(\sum_{k=0}^{n}\sum_{l=0}^{m}\hat{f}(k,l)ge^{ik\theta_1}e^{il\theta_2}\Big)_{(n,m)}$ is Cauchy in $\mathcal{M}$. Since $\mathcal{M}$ is a Hilbert space, there exists an h in $\mathcal{M}$ such that

$$\left\|\sum_{k=0}^{n}\sum_{l=0}^{m}\hat{f}(k,l)ge^{ik\theta_1}e^{il\theta_2} - h\right\|_{\mathcal{M}} \longrightarrow 0 \quad \text{as } (n,m)\longrightarrow\infty \text{ along rectangles.}$$

Thus,

$$h = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\hat{f}(k,l)ge^{ik\theta_1}e^{il\theta_2}$$

and since

$$g = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\hat{g}(k,l)e^{ik\theta_1}e^{il\theta_2},$$

we have for fixed m and n,

$$\begin{aligned}
h &= \hat{f}(0,0)g + \hat{f}(0,1)ge^{i\theta_2} + \hat{f}(1,0)ge^{i\theta_1} \\
&\quad + \cdots + \hat{f}(m,n)ge^{im\theta_1}e^{in\theta_2} + h_1e^{i(m+1)\theta_1} + h_2e^{i(n+1)\theta_2}
\end{aligned} \tag{2}$$

where

$$h_1 = \hat{f}(m+1,0)g + \hat{f}(m+1,1)ge^{i\theta_2} + \hat{f}(m+2,0)ge^{i\theta_1} + \cdots$$

and

$$h_2 = \hat{f}(0,n+1)g + \hat{f}(0,n+2)ge^{i\theta_2} + \hat{f}(1,n+1)ge^{i\theta_1} + \cdots .$$

It is clear that h_1 and h_2 are in $\mathcal{M}$ and hence in $H^p(\mathbf{T}^2)$. Thus from equation (2), we see that the (m,n)-th Fourier coefficients of h are the same as the (m,n)-th Fourier coefficients of the formal product of the series of g and f. This means that $h = gf$ in $H^p(\mathbf{T}^2)$ and hence in $\mathcal{M}$. This observation along with equation (1) gives us that

$$\|gf\|_{\mathcal{M}} = \|f\|_2.$$

Since f was an arbitrary element of $H^2(\mathbf{T}^2)$, we see that g multiplies $H^2(\mathbf{T}^2)$ into $\mathcal{M} \subseteq H^p(\mathbf{T}^2)$. By Lemma 5 we conclude that g multiplies $L^2(\mathbf{T}^2)$ into $L^p(\mathbf{T}^2)$. Lemma 6 shows us that any d that multiplies $L^2(\mathbf{T}^2)$ into $L^p(\mathbf{T}^2)$ must be a member of $L^{\frac{2p}{2-p}}(\mathbf{T}^2)$. Thus g must be in $H^{\frac{2p}{2-p}}(\mathbf{T}^2)$.

It is left to show that $R_1^{\perp}\cap R_2^{\perp}$ is one dimensional. Note that $\frac{2p}{2-p} \geq 2$ when $1 \leq p \leq 2$, so g is in $H^2(\mathbf{T}^2)$. Suppose there is a g_1 in $R_1^{\perp}\cap R_2^{\perp}$ with unit norm and

$g \perp g_1$ in $\mathcal{M}$. Then by the same computations as above we get that $g_1 H^2(\mathbf{T}^2)$ is also contained in $\mathcal{M}$ and by our decomposition we get that $gH^2(\mathbf{T}^2) \perp g_1 H^2(\mathbf{T}^2)$ in $\mathcal{M}$. Further, $gg_1 = g_1 g$ is in $gH^2(\mathbf{T}^2)$ as well as $g_1 H^2(\mathbf{T}^2)$. So, $gg_1 = 0$. As g and g_1 do not vanish on a set of positive Lebesgue measure unless they are identically zero we get a contradiction. Hence $R_1^\perp \cap R_2^\perp$ is one dimensional as desired.

Now we consider the case $p > 2$. Suppose $\mathcal{M} \neq \{0\}$. Proceeding as in the previous case we get that g multiples $L^2(\mathbf{T}^2)$ into $L^p(\mathbf{T}^2) \subset L^2(\mathbf{T}^2)$ and hence g is in $H^\infty(\mathbf{T}^2)$. Choosing an appropriate $\epsilon > 0$ such that

$$E = \left\{(e^{i\theta_1}, e^{i\theta_2}) : |g(e^{i\theta_1}, e^{i\theta_2})| > \epsilon\right\}$$

has positive measure, let b be a function that vanishes on the complement of E which is in $L^2(\mathbf{T}^2)$ but not $L^p(\mathbf{T}^2)$. But then, gb is in $L^p(\mathbf{T}^2)$ and so b will lie in $L^p(\mathbf{T}^2)$ since g is invertible on E, hence a contradiction. So our supposition must be incorrect. So, $\mathcal{M} = \{0\}$. □

Before we give a corollary we recall some definitions. Let $BMO(\mathbf{T}^2)$ be the class of all $L^1(\mathbf{T}^2)$ functions f such that

$$\|f\|_* = \sup \frac{1}{|I|} \int_I |f - \frac{1}{|I|} \int_I f| < \infty$$

where the supremum is taken over all squares of $\mathbf{T}^2$ and $|I|$ denotes the normalized Lebesgue measure of I.

$BMO(\mathbf{T}^2)$ is a Banach space under the norm

$$\|f\| = \|f\|_* + |\hat{f}(0,0)|.$$

$VMO(\mathbf{T}^2)$ is the closure of the continuous functions in $BMO(\mathbf{T}^2)$. $BMOA(\mathbf{T}^2) = BMO(\mathbf{T}^2) \cap H^1(\mathbf{T}^2)$ and $VMOA(\mathbf{T}^2) = VMO(\mathbf{T}^2) \cap H^1(\mathbf{T}^2)$. By the John-Nirenberg theorem [7], we get that $BMOA(\mathbf{T}^2) \subset H^p(\mathbf{T}^2)$ for $p < \infty$. We are now ready to state our corollary.

Corollary. *If $\mathcal{M}$ is a Hilbert space contained in $BMOA(\mathbf{T}^2)$ ($VMOA(\mathbf{T}^2)$), invariant under S_1 and S_2 and if S_1 and S_2 are doubly commuting isometries on $\mathcal{M}$, then $\mathcal{M} = \{0\}$.*

Proof. By the John-Nirenberg Theorem mentioned above we get that $BMOA(\mathbf{T}^2) \subset H^p(\mathbf{T}^2)$ for $p < \infty$. So in particular, $BMOA(\mathbf{T}^2) \subset H^p(\mathbf{T}^2)$ for $p > 2$. So by our theorem, $\mathcal{M} = \{0\}$. □

Remark. We point out that the above corollary holds for $\mathcal{M}$ in any vector space of analytic functions contained in $H^p(\mathbf{T}^2)$ for any $p > 2$. We specified $BMOA(\mathbf{T}^2)$ only to parallel the results in [5].

References

[1] V. Mandrekar, *The validity of Beurling theorems in polydiscs*, Proc. Amer. Math. Soc., **103** (1988), 145–148. MR0938659 (90c:32008)

[2] D. A. Redett, *"Beurling Type" Subspaces of $L^p(\mathbf{T}^2)$ and $H^p(\mathbf{T}^2)$*, Proc. Amer. Math. Soc., **133** no. 4 (2005), 1151–1156.

[3] W. Rudin, *Function theory in polydiscs*, Benjamin, New York, 1969. MR0255841 (41:501)

[4] Dinesh Singh, *Brangesian spaces in the polydisk*, Proc. Amer. Math. Soc., **110** no.4 (1990), 971–977. MR1028289 (91c:46038)

[5] Dinesh Singh and Sanjeev Agrawal, *DeBranges Spaces Contained in some Banach Spaces of Analytic Functions*, Illinois Journal of Math., **39** no. 3(1995), 351–357. MR1339831 (96d:46021)
[6] M. Slocinski, *On Wold-type decomposition of a pair of commuting isometries*, Ann. Polon. Math., **37** (1980), 255–262. MR0587496 (83e:47031)
[7] Alberto Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Academic Press, 1986. MR0869816 (88e:42001)

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