FULLY COMMUTATIVE ELEMENTS AND KAZHDAN-LUSZTIG CELLS IN THE FINITE AND AFFINE COXETER GROUPS, II

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Introduction

Let $W = (W, S)$ be a Coxeter group with $S$ the distinguished generator set. For any $J = \{s_1, \ldots, s_r\} \subseteq S$, denote by $w_J$ or $w_{s_1s_2\ldots s_r}$ the longest element in the subgroup $W_J$ of $W$ generated by $J$. The fully commutative elements of $W$ were defined by Stembridge: $w \in W$ is fully commutative, if any two reduced expressions of $w$ can be transformed from each other by only applying the relations $st = ts$ with $s, t \in S$ and $o(st) = 2$ (otherwise, or equivalently, $w$ has no reduced expression of the form $w = xw_{st}y$ with $o(st) > 2$ for some $s \neq t$ in $S$ (see [17, Proposition 2.1]). The fully commutative elements were studied extensively by a number of people (see [2], [4], [6], [7], [16], [17]). Let $W_c$ be the set of all the fully commutative elements in $W$.

In the present paper, we only consider (and always assume) the case where $W$ is an irreducible finite or affine Coxeter group unless otherwise specified. The paper is a continuation of my previous paper [16]; the latter proved that the set $W_c$ is a union of two-sided cells (in the sense of Kazhdan-Lusztig, see [8]) if and only if $W$ has a non-branching Coxeter graph and is not $\tilde{F}_4$. The aim of this paper is to give a necessary and sufficient condition for the set $W_c$ being closed under the Kazhdan–Lusztig preorder $\geq_{LR}$ (see Theorem 2.1). We use the result of [16] mentioned above and the following key observation: If $W$ has a non-branching Coxeter graph and is not $\tilde{F}_4$, then for any $w \notin W_c$, there exists some $y \in M(w)$ (see 1.5 for the notation) such that $\mathcal{L}(y)$ is not fully commutative (see 1.1). Then we get our result by comparing the generalized $\tau$-invariants on the elements in the set $W_c$ and in its complement $W \setminus W_c$ (see [12, Section 4]).
In [7, Section 3.1], Green and Losonczy proved that an irreducible finite Coxeter group \( W \) contains no subgraph of type \( D_4 \) in its Coxeter graph if and only if the set \( W_c \) is closed under \( \geq \) and is a union of two-sided cells. They gave a conceptual (resp., a computer) proof for \( W = B_m, A_n, m \geq 2, n \geq 1 \) (resp., \( W = F_4, H_3, H_4 \)) and referred the proof for the other cases to the papers [3], [5]. Then in [6, Theorem 3.4], Green proved that \( W \) is a union of two-sided cells closed under \( \geq \) for \( W = \tilde{A}_n, n \geq 1 \). The results [6, Theorem 3.4] and [7, Section 3.1] on \( \tilde{A}_n, A_n, n \geq 1 \), may also be obtained from my earlier results [11, Theorem 17.4], [13, Theorem 3.1] and [14, Section 2.9] by [17, Theorem 2.1].

In the proof of our main result (i.e., Theorem 2.1), we use the right cell graphs, rather than a computer, in dealing with the cases of \( W = \tilde{G}_2, F_4, H_3, H_4 \) (see the Appendix and the proof of Lemma 2.2).

The contents of the paper are organized as follows. We collect some notation, terminology and known results for later use.

1. Some results on Coxeter groups

Let \((W, S)\) be a Coxeter system. In the Introduction we defined the set \( W_c \) of all the fully commutative elements of \( W \). In this section, we collect some notation, terminology and known results for later use.

1.1. Let \( \preceq \) be the Bruhat–Chevalley order and \( \ell(w) \) the length function on \( W \). Call a subset \( J \) of \( S \) fully commutative if the element \( w_J \) is so.

For \( w, x, y \in W \), we use the notation \( w = x \cdot y \) to mean \( w = xy \) and \( \ell(w) = \ell(x) + \ell(y) \). If \( w = x \cdot y \in W_c \), then \( x, y \in W_c \). In particular, if \( w \in W_c \) has an expression \( w = x \cdot w_J \cdot y \) with \( x, y \in W \) and \( J \subseteq S \), then \( J \) is fully commutative.

1.2. Let \( \preceq \) (resp., \( \preceq, \preceq \)) be the preorder on \( W \) defined as in [8], and let \( \sim \) (resp., \( \sim, \sim \)) be the equivalence relation on \( W \) determined by \( \preceq \) (resp., \( \preceq, \preceq \)). The corresponding equivalence classes are called left (resp., right, two-sided) cells of \( W \).

The preorder \( \preceq \) (resp., \( \preceq, \preceq \)) on \( W \) induces a partial order on the set of left (resp., right, two-sided) cells of \( W \).

1.3. For any \( w \in W \), let \( \mathcal{L}(w) = \{ s \in S \mid sw < w \} \) and \( \mathcal{R}(w) = \{ s \in S \mid ws < w \} \).

Assume \( m = o(st) > 2 \) for some \( s, t \in S \). A sequence of elements

\[
\underbrace{sy, tsy, stsy, \ldots}_{m - 1 \text{ terms}}
\]

is called a left \( \{s, t\} \)-string if \( y \in W \) satisfies \( \mathcal{L}(y) \cap \{s, t\} = \emptyset \).

We say that \( z \) is obtained from \( w \) by a left \( \{s, t\} \)-star operation, if \( z, w \) are two neighboring terms in a left \( \{s, t\} \)-string. Clearly, a resulting element \( z \) of a left \( \{s, t\} \)-star operation on \( w \), when it exists, need not be unique unless \( w \) is a terminal term of the left \( \{s, t\} \)-string containing it.

The following result follows directly from the definition of the relation \( \sim \) on \( W \).
Lemma. If \( x, y \in W \) can be obtained from each other by successively applying left star operations, then \( x \sim_L y \).

1.4. By the notation \( x \sim y \) in \( W \), we mean that either \( x < y \) or \( y < x \) holds and that \( \max \{ \deg P_{x,y}, \deg P_{y,x} \} = \frac{1}{2}(|\ell(x) - \ell(y)| - 1) \), where \( P_{x,y} \) is the celebrated Kazhdan–Lusztig polynomial associated to \( x, y \in W \) (see [8, Theorem 1.1]).

(a) The relation \( x \sim_L y \) (resp., \( x \sim_R y \)) implies \( R(x) \supseteq R(y) \) (resp., \( L(x) \supseteq L(y) \)). In particular, the relation \( x \sim_L y \) (resp., \( x \sim_R y \)) implies \( R(x) = R(y) \) (resp., \( L(x) = L(y) \)) (see [8, Proposition 2.4]). Hence it makes sense to write \( L(\Gamma) \) (resp., \( R(\Gamma) \)) for any right (resp., left) cell \( \Gamma \) of \( W \), where \( L(\Gamma) = L(z) \) (resp., \( R(\Gamma) = R(z) \)) for any \( z \in \Gamma \).

(b) If \( x, y \in W \) with \( x \sim y \) are in some left \( \{ s, t \} \)-strings (not necessarily in the same left string; see 1.3) for some \( s, t \in S \) with \( st \neq ts \), then there exist some \( x', y' \in W \) which are obtained from \( x, y \) respectively by a left \( \{ s, t \} \)-star operation and satisfy \( x' \sim y' \) (see [9, Section 10.4]).

(c) \( x \sim_{LR} x^{-1} \) for any \( x \in W \) (see [10, Corollary 1.9 (a) and Theorem 1.10] and [1, Corollary 3.2]).

1.5. For any \( w \in W \), let \( M(w) \) be the set of all the elements \( y \) satisfying: there exists a sequence of elements \( z_0 = w, z_1, ..., z_t = y \) in \( W \) with \( t \geq 0 \) such that \( z_i \) is obtained from \( z_{i-1} \) by a left star operation for every \( 1 \leq i \leq t \). We see by Lemma 1.3 that all the elements in \( M(w) \) are in the same left cell of \( W \).

§2. The Condition for \( W_c \) Being Closed Under the Preorder \( \geq_{LR} \)

In this section, assume that \( W \) is an irreducible finite or affine Coxeter group. In [16, Theorem 3.4 and Sections 3.5–3.7], we showed that the set \( W_c \) is a union of two-sided cells of \( W \) if and only if \( W \) has a non-branching Coxeter graph and is not \( F_4 \). We understand that this result was already known in the case where \( W \) is any irreducible finite Coxeter group (see [7]).

A subset \( K \) of \( W \) is closed under the preorder \( \geq_{LR} \) if the conditions \( x \in K, y \in W \) and \( y \geq_{LR} x \) together imply \( y \in K \).

In the present section, we want to give a necessary and sufficient condition for the set \( W_c \) to be closed under \( \geq_{LR} \).

Theorem 2.1. Let \( W \) be an irreducible finite or affine Coxeter group. Then \( W_c \) is closed under \( \geq_{LR} \) if and only if \( W_c \) is a union of two-sided cells of \( W \).

To prove Theorem 2.1, we need to prove some lemmas.

Lemma 2.2. If \( W \) is an irreducible finite or affine Coxeter group such that \( W_c \) is a union of two-sided cells of \( W \), then for any \( w \in W \setminus W_c \), there exists some \( y \in M(w) \) (see 1.5) such that \( L(y) \) is not fully commutative (see 1.1).

Proof. By [16 Theorem 3.4 and 3.5–3.7], we know that \( W_c \) is a union of two-sided cells of \( W \) if and only if \( W \) has a non-branching Coxeter graph and is not \( F_4 \), i.e., \( W \) is one of the following groups: \( A_n, \tilde{A}_n, I_2(m), \tilde{C}_l, B_l, F_l, H_3, H_4, \tilde{G}_2 \), where \( n \geq 1, m \geq 5 \) and \( l \geq 2 \). The result follows by [11] Theorems 17.4, 17.6 and
Propositions 9.3.7, 16.2.4 for the groups \( \tilde{A}_n \) and \( A_n \), and by [10] Corollary 3.3] for the groups \( \tilde{C}_l \). By the fact that \( B_l \) is a standard parabolic subgroup of \( \tilde{C}_l \), we can show the result for the groups \( B_l \) by the same argument as that for [10] Corollary 3.3]. Then the result for the groups \( F_4, H_3, H_4 \) and \( \tilde{G}_2 \) can be checked directly from their right cell graphs (see the Appendix). Finally, the result for the groups \( L_2(3) \) is obvious.

Remark 2.3. The assumption in Lemma 2.2 that \( W_c \) is a union of two-sided cells of \( W \) is necessary. There is a counterexample when such a condition is removed. Let \( W = F_4 \) and \( S = \{ s_0, s_1, s_2, s_3, s_4 \} \) be such that \( o(s_0 s_1) = o(s_1 s_2) = o(s_3 s_4) = 3 \) and \( o(s_3 s_3) = 4 \). Then the element \( w = s_4 s_2 s_3 s_2 s_0 s_1 s_0 \) is not fully commutative. However, \( L(y) \) is fully commutative for any element \( y \) in \( M(w) \) (see [12] Section 5.4).

By Lemma 2.2, we can prove the following.

Lemma 2.4. When it is a union of two-sided cells of \( W \), the set \( W_c \) is closed under the preorder \( \geq \).

Proof. Suppose not. Then there exist some \( x \in W_c \) and some \( w \in W \setminus W_c \) with \( x \not\leq w \). We may assume \( x \) and \( L(x) \not\subseteq L(w) \) without loss of generality. So \( R(x) \supseteq R(w) \) by 1.4 (a). Hence \( L(x^{-1}) \supseteq L(w^{-1}) \). By Lemma 2.2, there exists an element \( y \) in \( M(w^{-1}) \) with \( L(y) \) not fully commutative. Then there exists a sequence of elements \( w_0 = w^{-1}, w_1, ..., w_r = y \) in \( M(w^{-1}) \) such that \( w_i \) is obtained from \( w_{i-1} \) by a left \( \{ s_i, t_i \} \)-star operation for every \( 1 \leq i \leq r \) and some \( s_i, t_i \in S \) with \( s_i t_i \neq t_i s_i \). We may assume \( r \) is minimal with this property. Hence the \( L(w_i) \)'s, \( 0 \leq i < r \), are all fully commutative. Since \( w_1 \) is obtained from \( w^{-1} \) by a left \( \{ s_1, t_1 \} \)-star operation, we have \( \{ s_1, t_1 \} \cap L(w^{-1}) = 1 \). Since \( L(x^{-1}) \) is fully commutative and \( L(x^{-1}) \supseteq L(w^{-1}) \), we have \( \{ s_1, t_1 \} \cap L(x^{-1}) = 1 \) also. So we can apply a left \( \{ s_1, t_1 \} \)-star operation on \( x^{-1} \) to obtain some element \( x_1 \) in \( M(x^{-1}) \) with \( x_1 = w_1 \) by 1.4 (b). Since \( R(x_1) = R(x^{-1}) = L(x) \not\subseteq L(w) = R(w^{-1}) = R(w_1) \), we have \( x_1 \not\leq w_1 \) and hence \( L(x_1) \supseteq L(w_1) \) by 1.4 (a). When \( r > 1 \), we can apply a left \( \{ s_2, t_2 \} \)-star operation on \( x_1 \) to obtain some element \( x_2 \) with \( x_2 = w_2 \) by the same reason as that for getting \( x_1 \) from \( x^{-1} \). Continuing this process, we get a sequence of elements \( x_0 = x^{-1}, x_1, ..., x_r \) in \( M(x^{-1}) \) such that \( x_i \) is obtained from \( x_{i-1} \) by a left \( \{ s_i, t_i \} \)-star operation and \( x_i = w_i \) for \( 1 \leq i \leq r \). By the assumption that \( W_c \) is a union of two-sided cells of \( W \) and by the facts that \( x_r \sim x^{-1} \sim x \) by 1.4 (c) and \( x \in W_c \), we have \( x_r \in W_c \) and hence the set \( L(x_r) \) is fully commutative. Since \( L(w_r) \) is not fully commutative, we have \( L(w_r) \not\subseteq L(x_r) \). Since \( x_r = w_r \), this implies \( w_r \leq x_r \) and hence \( x \not\leq w \sim w^{-1} \sim w_r \leq x_r \sim x^{-1} \sim x \) by 1.4 (c). We get \( x \not\leq w \), contradicting the assumption that \( W_c \) is a union of two-sided cells of \( W \). So our result follows.

2.5. Proof of Theorem 2.1. The implication “\( \iff \)” is just Lemma 2.4. For the implication “\( \Rightarrow \)”, we need only show that \( x \not\leq_L y \) for any \( x \in W_c \) and any \( y \in W \setminus W_c \). Suppose not. Then there exist some \( x \in W_c \) and some \( y \in W \setminus W_c \) with \( x \not\leq_L y \) (and hence \( y \geq_L x \)). But this would imply \( y \in W_c \) by the assumption that \( W_c \) is closed under \( \geq_L \), a contradiction. So Theorem 2.1 follows.
Appendix. A right cell graph associated to an element $x \in W$ (written $\mathcal{M}_R(x)$) is by definition a graph whose vertex set $V(x)$ consists of all the right cells $\Gamma$ of $W$ with $\Gamma \cap M(x) \neq \emptyset$ (each right cell is represented by a box). Two vertices $\Gamma, \Gamma'$ of $\mathcal{M}_R(x)$ are joined by an edge, if there are some $y \in M(x) \cap \Gamma$ and $z \in M(x) \cap \Gamma'$ such that $y, z$ are two neighboring terms of a left string. Each vertex $\Gamma$ of $\mathcal{M}_R(x)$ is labelled by the set $\mathcal{L}(\Gamma)$ (see 1.4 (a))

It is easily seen that the set of the subsets of $S$ occurring as the labels of the vertices in $\mathcal{M}_R(x)$ is equal to the set $\{I \subseteq S \mid I = \mathcal{L}(y) \text{ for some } y \in M(x)\}$.

Two right cell graphs $\mathcal{M}_R(x)$ and $\mathcal{M}_R(y)$ are isomorphic if there exists a bijection $\phi : V(x) \to V(y)$ such that $\mathcal{L}(\Gamma) = \mathcal{L}(\phi(\Gamma))$ for any $\Gamma \in V(x)$ and such that any pair $\Gamma, \Gamma' \in V(x)$ are joined by an edge if and only if $\phi(\Gamma), \phi(\Gamma')$ are so.

Note that the definition of a right cell graph imitates that of a left cell graph; the latter was given in my previous paper [15, Subsection 2.11].

We work out all the right cell graphs in $W \setminus W_\perp$ (resp., a representative set of the isomorphism classes of those graphs) for the groups $W = G_2, F_4$ (resp., $H_4, H_3$) according to the results in [9], [18], [4].

(1) $W = G_2$ with $S = \{s_0, s_1, s_2\}$ satisfying $o(s_0s_2) = 3$ and $o(s_1s_2) = 6$:

\[
\begin{align*}
\text{\textbullet}{03} & \quad \text{\textbullet}{2} & \quad \text{\textbullet}{3} & \quad \text{\textbullet}{12} & \quad 01 & \quad 02 & \quad 0 & \quad 2 & \quad 1 & \\
\text{\textbullet}{02} & \quad \text{\textbullet}{2} & \quad 1 & \quad \text{\textbullet}{01} & \quad 02 & \quad 01 & \quad 0 & \quad 2
\end{align*}
\]

$\mathcal{M}_R(w_{02}), \mathcal{M}_R(w_{12}), \mathcal{M}_R(s_0s_2s_1s_2s_0w_{12})$

Here and later the boldfaced numbers in a box $\Gamma$ represent the elements in $\mathcal{L}(\Gamma)$. The box of $\mathcal{M}_R(x)$ with inside numbers underlined represents the right cell $\Gamma_x$ containing $x$. For example, the box $[02]$ in $\mathcal{M}_R(s_0s_2s_1s_2s_0w_{12})$ represents the right cell $\Gamma = \Gamma_{s_0s_2s_1s_2s_0w_{12}}$ with $\mathcal{L}(\Gamma) = \{s_0, s_2\}$; while two boxes $[01]$ in $\mathcal{M}_R(w_{12})$ represent respectively two right cells $\Gamma, \Gamma' \in V(w_{12})$ with $\mathcal{L}(\Gamma) = \mathcal{L}(\Gamma') = \{s_0, s_1\}$. The notation $w_{ij...}$ stands for the element $w_{s_is_j...}$ (see the first paragraph in the Introduction).

(2) $W = F_4$ with $S = \{s_1, s_2, s_3, s_4\}$ satisfying $o(s_1s_2) = o(s_3s_4) = 3$ and $o(s_2s_3) = 4$.

\[
\begin{align*}
\text{\textbullet}{4} & \quad \text{\textbullet}{3} & \quad \text{\textbullet}{2} & \quad \text{\textbullet}{1} & \quad \text{\textbullet}{24} & \quad 14 & \\
\text{\textbullet}{1} & \quad \text{\textbullet}{2} & \quad \text{\textbullet}{3} & \quad \text{\textbullet}{24} & \quad 14 & \\
\text{\textbullet}{3} & \quad \text{\textbullet}{2} & \\
\text{\textbullet}{12} & \quad 13 & \quad 25 & \\
\text{\textbullet}{12} & \quad 13 & \quad 25 & \quad 124 & \quad 134 & \quad 24 & \quad 24 & \quad 23 & \quad 134 & \\
\text{\textbullet}{124} & \quad 134 & \quad 24 & \quad 24 & \quad 23 & \quad 134 &
\end{align*}
\]

$\mathcal{M}_R(w_{34}), \mathcal{M}_R(w_{12}), \mathcal{M}_R(w_{123}), \mathcal{M}_R(w_{234})$
(3) $W = H_4$ with $S = \{s_1, s_2, s_3, s_4\}$ satisfying $o(s_1s_2) = o(s_2s_3) = 3$ and $o(s_3s_4) = 5$.

(4) $W = H_3$ with $S = \{s_1, s_2, s_3\}$ satisfying $o(s_1s_2) = 3$ and $o(s_2s_3) = 5$.

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References


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