PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 9, Pages 2711–2720
S 0002-9939(05)07857-3
Article electronically published on March 22, 2005

WEIERSTRASS FUNCTIONS IN ZYGMUND'S CLASS

YANICK HEURTEAUX

(Communicated by David Preiss)

ABSTRACT. Consider the function

$$f(x) = \sum_{n=0}^{+\infty} b^{-n} g(b^n x)$$

where $b > 1$ and g is an almost periodic $C^{1,\varepsilon}$ function. It is well known that the function f lives in the so-called Zygmund class. We prove that f is generically nowhere differentiable. This is the case in particular if the elementary condition $g'(0) \neq 0$ is satisfied. We also give a sufficient condition on the Fourier coefficients of g which ensures that f is nowhere differentiable.

1. Introduction

The function

$$w(x) = \sum_{n=0}^{+\infty} b^{-n\alpha} \cos(2\pi b^n x)$$

where $b > 1$ and $\alpha \leq 1$ is probably one of the most famous continuous nowhere differentiable functions. This function was introduced by Weierstrass. He proved that w is nowhere differentiable for some of these values b and α. A few years later, Hardy gave the proof for every $b > 1$ and every $\alpha \leq 1$ (see [5]). The critical case $\alpha = 1$ is especially interesting. In that case, the function w is in the Zygmund class Λ_1, that is, it satisfies

$$|w(x+h) + w(x-h) - 2w(x)| \leq C\,|h| \tag{1.1}$$

for some $C > 0$, but is nowhere differentiable. In particular, w is not Lipschitz.

More generally, one can consider the function

$$f(x) = \sum_{n=0}^{+\infty} b^{-n\alpha} g(b^n x),$$

where g is almost periodic and $1 < b < +\infty$. Such a function is often called a Weierstrass-type function.

If $\alpha < 1$ and g is Lipschitz, it is easy to prove that f is of class C^α (see for example [4]). Moreover, studying the oscillations of f, numerous works give the conclusion that in good cases the function f is nowhere differentiable (see for example [1, 2,

Received by the editors April 29, 2004.
2000 *Mathematics Subject Classification.* Primary 26A27, 28A80.
Key words and phrases. Weierstrass functions, almost periodic functions, second-order oscillations, Zygmund class.

6, 7, 8, 11]). In particular, it is proved in [2] and [1] that, as soon as f is not Lipschitz, there exists a constant $C > 0$ such that for every interval I of length $|I| \leq 1$, $\operatorname{osc}(f, I) = \sup_I(f) - \inf_I(f) \geq C\,|I|^{\alpha}$.

The method proposed in [2] and [1] does not work any more in the limit case $\alpha = 1$. Moreover, the inequality $\operatorname{osc}(f, I) \geq C\,|I|$ would not imply that f is nowhere differentiable. For example, the sawtooth function $\varphi(x) = \operatorname{dist}(x, \mathbb{Z})$ is Lipschitz and satisfies $\operatorname{osc}(\varphi, I) \geq \frac{1}{2}\,|I|$ if $|I| \leq 1$.

The purpose of this note is to study the regularity properties of the function

$$f(x) = \sum_{n=0}^{+\infty} b^{-n} g(b^n x)\ . \tag{1.2}$$

We suppose that g is almost periodic and of class $C^{1,\varepsilon}$. A natural question is then to ask if f is nowhere differentiable. We will answer that it is generically the case when the set of functions g is endowed with the natural norm

$$\|g\|_{1,\varepsilon} = \|g\|_\infty + N_{1+\varepsilon}(g)$$

with

$$N_{1+\varepsilon}(g) = \sup\left(\frac{|g(x+h) + g(x-h) - 2g(x)|}{|h|^{1+\varepsilon}},\quad x \in \mathbb{R},\ h > 0\right)\ .$$

As a preliminary result concerning the regularity properties of the function f, let us recall the following classical proposition (see for example [10] and the references therein).

Proposition 1.1. *Suppose that g is a bounded function of class $C^{1,\varepsilon}$ for some $\varepsilon > 0$. Let $1 < b < +\infty$ and define f by (1.2). Then, f is in the Zygmund class Λ_1. That is,*

$$|f(x+h) + f(x-h) - 2f(x)| \leq C\,|h| \quad \text{for all } x \in \mathbb{R} \text{ and } |h| \leq 1\ .$$

More precisely, we can choose

$$C = \frac{N_{1+\varepsilon}(g)}{1 - b^{-\varepsilon}} + \frac{2\operatorname{osc}(g)}{1 - b^{-1}}, \qquad \text{where} \quad \operatorname{osc}(g) = \sup(|g(x) - g(y)|\,,\ x, y \in \mathbb{R})\ .$$

Proof. This result is well known. Let us briefly sketch the proof. Suppose that $b^{-(n_0+1)} < |h| \leq b^{-n_0}$. We can write

$$\begin{aligned}
|f(x+h) + f(x-h) - 2f(x)| &\leq \sum_{n=0}^{+\infty} b^{-n}\,|g(b^n(x+h)) + g(b^n(x-h)) - 2g(b^n x)| \\
&\leq \sum_{n=0}^{n_0} b^{-n} N_{1+\varepsilon}(g)\,|b^n h|^{1+\varepsilon} + \sum_{n=n_0+1}^{+\infty} 2b^{-n}\operatorname{osc}(g) \\
&= \frac{N_{1+\varepsilon}(g)\,|h|^{1+\varepsilon}\,(b^{(n_0+1)\varepsilon} - 1)}{b^\varepsilon - 1} + \frac{2b^{-n_0-1}\operatorname{osc}(g)}{1 - b^{-1}} \\
&\leq \left[\frac{N_{1+\varepsilon}(g)}{1 - b^{-\varepsilon}} + \frac{2\operatorname{osc}(g)}{1 - b^{-1}}\right]|h|\ .
\end{aligned}$$

□

The previous proposition suggests that it is easier to estimate the second-order oscillations of the function f defined by (1.2). Let us introduce the following notation:

$$\operatorname{osc}_2(f, I) = \sup(|f(x+h) + f(x-h) - 2f(x)|,\ [x-h, x+h] \subset I) \tag{1.3}$$

where I is a bounded closed interval.

Finally, let us denote by $\Delta_h^2 f(x)$ the second-order difference

$$\Delta_h^2 f(x) = f(x+h) + f(x-h) - 2f(x)\ .$$

The main result of this note states that for most functions f, the second-order oscillations $\operatorname{osc}_2(f, I)$ are uniformly bounded from below by $C\,|I|$. This implies that the function f is nowhere differentiable. More precisely, we have the following results.

Theorem 1.2. *Let g be an almost periodic function of class $C^{1,\varepsilon}$ and $1 < b < +\infty$. Define f using formula* (1.2)*. There are only two mutually exclusive possible cases:*

(i) $N_{1+\varepsilon}(f) \le \dfrac{N_{1+\varepsilon}(g)}{b^\varepsilon - 1}$

or

(ii) *there exists a constant $C > 0$ such that for every closed interval I of length $|I| \le 1$,*

$$\operatorname{osc}_2(f, I) \ge C|I|\ .$$

We can then deduce the following corollary.

Corollary 1.3. *The hypotheses are the same as in Theorem* 1.2*. There are only two mutually exclusive possible cases:*

(i) *f is of class $C^{1,\varepsilon}$*

or

(ii) *f is nowhere differentiable.*

Case (i) is obviously a possible case. If γ is an almost periodic function of class $C^{1,\varepsilon}$ and if $g(x) = \gamma(x) - b^{-1}\gamma(bx)$, then $f = \gamma$ and f is of class $C^{1,\varepsilon}$. In fact, this is the exceptional case (see Theorems 3.1, 3.3 and 4.3). In the generic case, (ii) is satisfied and the function f is nowhere differentiable.

The paper is organised as follows. In the next section, we give the proof of Theorem 1.2 and Corollary 1.3. In Section 3, we prove that the set of functions g such that f is nowhere differentiable is a dense open subset of the set of almost periodic functions of class $C^{1,\varepsilon}$. In particular it contains the set of functions g such that $g'(0) \neq 0$. In the last section, we give a sufficient condition (in terms of the Fourier coefficients of g) which ensures that the function f is nowhere differentiable. We can then construct a lot of Weierstrass-type functions in the Zygmund class which are nowhere differentiable.

2. Proof of Theorem 1.2 and Corollary 1.3

Throughout this section g is an almost periodic function of class $C^{1,\varepsilon}$ and f is defined by formula (1.2). The first lemma gives a minoration of second-order differences at some initial scale ℓ.

Lemma 2.1. *Suppose that hypothesis* (i) *of Theorem* 1.2 *is not satisfied. There are three real numbers $\ell \geq 1$, $v > 0$ and $h > 0$ such that every closed interval I of length $|I| = \ell$ contains a point x_I satisfying*

$$x_I - h \in I, \quad x_I + h \in I \quad \textit{and} \quad \left|\Delta_h^2 f(x_I)\right| \geq \frac{N_{1+\varepsilon}(g) + v}{b^\varepsilon - 1} |h|^{1+\varepsilon} \ .$$

Proof. Suppose that (i) is not satisfied. We can find $x_0 \in \mathbb{R}$, $v > 0$ and $h > 0$ such that

$$\left|\Delta_h^2 f(x_0)\right| \geq \frac{N_{1+\varepsilon}(g) + 5v}{b^\varepsilon - 1} |h|^{1+\varepsilon} \ .$$

Set

$$\varepsilon_0 = \frac{v\,|h|^{1+\varepsilon}}{b^\varepsilon - 1} \ .$$

Since f is the uniform limit of almost periodic functions, it is also almost periodic. We can then find a real $\ell > 0$ (which we can assume is greater than 1 and greater than $3h$) such that

$$\forall \gamma \in \mathbb{R}, \quad \exists \delta \in [\gamma, \gamma + \ell/3[\ ; \quad \|\tau_\delta f - f\|_\infty \leq \varepsilon_0 \ ,$$

where $\tau_\delta f(x) = f(x + \delta)$.

Let $I = [a, a + \ell]$ be a closed interval of length ℓ and take $\gamma = a - x_0 + \ell/3$. We can find $\delta \in [\gamma, \gamma + \ell/3)$ such that

$$\|\tau_\delta f - f\|_\infty \leq \varepsilon_0 \ .$$

Set $x_I = x_0 + \delta$. We have

$$x_I \in [a + \ell/3, a + 2\ell/3) \quad \text{and} \quad a \leq x_I - h < x_I < x_I + h \leq a + \ell \ .$$

Moreover,

$$\begin{cases} |f(x_I - h) - f(x_0 - h)| \leq \varepsilon_0, \\ |f(x_I) - f(x_0)| \leq \varepsilon_0, \\ |f(x_I + h) - f(x_0 + h)| \leq \varepsilon_0 \ . \end{cases}$$

Finally,

$$\left|\Delta_h^2 f(x_I)\right| \geq \left|\Delta_h^2 f(x_0)\right| - 4\varepsilon_0 \geq \frac{N_{1+\varepsilon}(g) + v}{b^\varepsilon - 1} |h|^{1+\varepsilon}$$

and the proof is done. □

Let us now remark that the Weierstrass function f satisfies the functional equation

$$f(x) = g(x) + b^{-1} f(bx) \ . \tag{2.1}$$

This equation allows us to deduce minorations of second-order differences at scale ℓ/b from similar ones at scale ℓ. This is the object of the following elementary lemma.

Lemma 2.2. *Let $x, h \in \mathbb{R}$ satisfy*

$$\left|\Delta_h^2 f(x)\right| \geq \frac{(N_{1+\varepsilon}(g) + v)}{b^\varepsilon - 1} |h|^{1+\varepsilon} \quad \textit{for some } v > 0 \ .$$

Then

$$\left|\Delta_{b^{-1}h}^2 f(b^{-1}x)\right| \geq \frac{(N_{1+\varepsilon}(g) + v b^\varepsilon)}{b^\varepsilon - 1} \left|b^{-1}h\right|^{1+\varepsilon} \ .$$

Proof. The above lemma is an easy consequence of the functional equation (2.1). We have

$$\Delta^2_{b^{-1}h} f(b^{-1}x) = \Delta^2_{b^{-1}h} g(b^{-1}x) + b^{-1}\Delta^2_h f(x) \ .$$

Suppose that x, h and v satisfy the hypothesis of the lemma. Then

$$\begin{aligned} \left|\Delta^2_{b^{-1}h} f(b^{-1}x)\right| &\geq b^{-1}\left|\Delta^2_h f(x)\right| - \left|\Delta^2_{b^{-1}h} g(b^{-1}x)\right| \\ &\geq b^{-1}\frac{(N_{1+\varepsilon}(g)+v)}{b^\varepsilon - 1}\left|h\right|^{1+\varepsilon} - N_{1+\varepsilon}(g)\left|b^{-1}h\right|^{1+\varepsilon} \\ &= \frac{(N_{1+\varepsilon}(g)+vb^\varepsilon)}{b^\varepsilon - 1}\left|b^{-1}h\right|^{1+\varepsilon} \ . \end{aligned}$$

□

We can now finish the proof of Theorem 1.2. Suppose that property (i) is not satisfied and consider $\ell \geq 1$, $v > 0$ and $h > 0$ given by Lemma 2.1. Let I be an interval of length $|I| \leq 1$. Define k as the unique positive integer such that $\ell b^{-k} \leq |I| < \ell b^{-k+1}$. Finally, let $J \subset I$ be an interval of length $|J| = \ell b^{-k}$ and $\tilde{J} = b^k J$. According to Lemma 2.1 we can find $x_{\tilde{J}} \in \tilde{J}$ such that

$$x_{\tilde{J}} - h,\ x_{\tilde{J}} + h \in \tilde{J} \quad \text{and} \quad \left|\Delta^2_h f(x_{\tilde{J}})\right| \geq \frac{N_{1+\varepsilon}(g)+v}{b^\varepsilon - 1}\left|h\right|^{1+\varepsilon} \ .$$

Iterating Lemma 2.2, we obtain

$$\left|\Delta^2_{b^{-k}h} f(b^{-k}x_{\tilde{J}})\right| \geq \frac{N_{1+\varepsilon}(g)+vb^{k\varepsilon}}{b^\varepsilon - 1}\left|b^{-k}h\right|^{1+\varepsilon} \geq \frac{v\left|h\right|^{1+\varepsilon} b^{-k}}{b^\varepsilon - 1} \ .$$

Set

$$C = \frac{v\left|h\right|^{1+\varepsilon}}{(b^\varepsilon - 1)\ell b} \ .$$

We get

$$\operatorname{osc}_2(f,I) \geq \operatorname{osc}_2(f,J) \geq \frac{v\left|h\right|^{1+\varepsilon} b^{-k}}{b^\varepsilon - 1} \geq C\left|I\right| \ .$$

□

We can now prove Corollary 1.3. Let us first recall that a continuous bounded function φ is of class $C^{1,\varepsilon}$ if and only if the quantity $N_{1+\varepsilon}(\varphi)$ is finite (we can refer to [9], Lemma 5.4, p. 207, for an easy proof of this fact).

More precisely, suppose that the Weierstrass function f is of class $C^{1,\varepsilon}$. The functional equation (2.1) gives

$$b^{-1}\Delta^2_{bh} f(bx) = \Delta^2_h f(x) - \Delta^2_h g(x) \ .$$

It follows that

$$b^\varepsilon N_{1+\varepsilon}(f) \leq N_{1+\varepsilon}(g) + N_{1+\varepsilon}(f) \ .$$

Finally,

$$N_{1+\varepsilon}(f) \leq \frac{N_{1+\varepsilon}(g)}{b^\varepsilon - 1} \ .$$

We have just proved that f is of class $C^{1+\varepsilon}$ if and only if $N_{1+\varepsilon}(f) \leq \frac{N_{1+\varepsilon}(g)}{b^\varepsilon - 1}$. Corollary 1.3 is then an immediate consequence of the following lemma.

Lemma 2.3. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real function and let $x_0 \in \mathbb{R}$. Suppose that φ is differentiable at x_0. Then*

$$\lim_{I \searrow \{x_0\}} \frac{\operatorname{osc}_2(\varphi, I)}{|I|} = 0 \ .$$

Proof. Let $\varepsilon > 0$. We can choose $\eta > 0$ such that

$$\varphi(x_0 + t) = \varphi(x_0) + t\varphi'(x_0) + t\varepsilon(t)$$

with $|\varepsilon(t)| < \varepsilon$ if $|t| < \eta$. Let I be an interval of length $|I| < \eta$ and such that $x_0 \in I$. Finally, suppose that $y \in \mathbb{R}$ and $h > 0$ are such that $[y-h, y+h] \subset I$. We have

$$\begin{aligned} \left|\Delta_h^2 \varphi(y)\right| &= |\varphi(x_0 + (y+h-x_0)) + \varphi(x_0 + (y-h-x_0)) - 2\varphi(x_0 + (y-x_0))| \\ &= |(y+h-x_0)\varepsilon(y+h-x_0) + \\ &\qquad (y-h-x_0)\varepsilon(y-h-x_0) - 2(y-x_0)\varepsilon(y-x_0)| \\ &\le 4\varepsilon\,|I|\,, \end{aligned}$$

and the conclusion follows. $\square$

3. On the set of functions g such that f is nowhere differentiable

We begin this section by describing two elementary sufficient conditions on the function g which ensure that the associated Weierstrass function f is nowhere differentiable.

Theorem 3.1. *Let g be an almost periodic function of class $C^{1,\varepsilon}$ and $1 < b < +\infty$. Suppose that g satisfies one of the following two conditions:*

(1) $g'(0) \neq 0$

or

(2) $g'(0) = 0$, *g is not constant and* 0 *is a global extremum of g.*

Then, the function f defined by formula (1.2) *is nowhere differentiable.*

Remark 3.2. In particular, this theorem can be used to prove that the classical Weierstrass functions

$$w(x) = \sum_{n=0}^{+\infty} b^{-n} \cos(2\pi b^n x) \quad \text{and} \quad \tilde{w}(x) = \sum_{n=0}^{+\infty} b^{-n} \sin(2\pi b^n x)$$

are nowhere differentiable.

Proof of Theorem 3.1. We are arguing by contradiction. Assume that f is differentiable at some point. By Corollary 1.3, f must be differentiable everywhere. The functional equation (2.1) which is satisfied by the function f gives

$$f'(x) = g'(x) + f'(bx) \ .$$

In particular, $g'(0) = 0$.

Suppose now that $g'(0) = 0$ and $g(x) \le g(0)$ for all $x \in \mathbb{R}$. Suppose once more that f is differentiable at some point. By Corollary 1.3, $f'(0)$ must exist. Observe that $f(x) \le f(0)$ for all $x \in \mathbb{R}$. So, $f'(0) = 0$. On the other hand, let $x \neq 0$. For every $n \ge 0$, we have

$$f(b^{-n}x) - f(0) = \sum_{k=0}^{+\infty} b^{-k}(g(b^k b^{-n} x) - g(0)) \le b^{-n}(g(x) - g(0)) \ .$$

It follows that

$$\frac{f(b^{-n}x) - f(0)}{b^{-n}} \leq g(x) - g(0) \ .$$

Taking the limit, we get

$$0 \leq g(x) - g(0) \ .$$

Therefore, g is constant. The case where 0 is a global minimum of g is similar. □

As a consequence of Theorem 1.2, we can also prove the following result about the set of functions g such that the associated Weierstrass function is irregular.

Theorem 3.3 (Function f generically nowhere differentiable)**.** *Denote by E the set of almost periodic functions of class $C^{1,\varepsilon}$ endowed with its standard norm. The set of functions $g \in E$ for which the associated Weierstrass function $f(x) = \sum_{n=0}^{+\infty} b^{-n} g(b^n x)$ is nowhere differentiable is an open dense subset of E. Its complement is a closed vector subspace of E of infinite codimension.*

Proof. First remark that the two norms $\|g\|_{1,\varepsilon} = \|g\|_\infty + N_{1+\varepsilon}(g)$ and $|g|_{1,\varepsilon} = \|g\|_\infty + \|g'\|_\infty + \sup(\frac{|g'(x+h)-g'(x)|}{|h|^\varepsilon})$ are equivalent on the vector space E. This is an easy consequence of Lemma 5.4 on page 207 in [9]. These two norms define the standard topology in E, and the vector space E is a Banach space. Denote by F the set of functions $g \in E$ such that f is regular. It is clear that F is a vector subspace of E. Suppose that $g_n \in F$ and converges to $g \in E$. Denote by f_n and f the associated Weierstrass functions. Theorem 1.2 ensures that

$$N_{1+\varepsilon}(f_n - f_p) \leq \frac{N_{1+\varepsilon}(g_n - g_p)}{b^\varepsilon - 1} \ .$$

Moreover, it is clear that

$$\|f_n - f_p\|_\infty \leq \frac{\|g_n - g_p\|_\infty}{1 - b^{-1}} \ .$$

It follows that the sequence f_n is a Cauchy sequence in E. In particular, according to the previous remark, the sequence f_n' converges uniformly. We can conclude that f is differentiable. Finally, $g \in F$ and F is a closed vector subspace. It follows that $E \setminus F$ is dense in E. We will see in Corollary 4.6 that F is a subspace of infinite codimension. □

4. On the Fourier analysis of functions g such that f is regular

In this section we give a sufficient condition (in terms of Fourier coefficients of g) which ensures that f is nowhere differentiable. We begin with the following elementary proposition.

Proposition 4.1. *Let g be an almost periodic function of class $C^{1,\varepsilon}$ and $1 < b < +\infty$. Suppose that the associated Weierstrass function f is differentiable. Then, for all $x \in \mathbb{R}$,*

$$f(x) - f(0) = f'(0)x - \sum_{n=1}^{+\infty} b^n (g(b^{-n}x) - g(0)) \ . \tag{4.1}$$

Remark 4.2. Suppose for simplicity that $g(0) = 0$. If $a \in \mathbb{R}$ and if $g'(0) = 0$, it is always possible to define $\tilde{f}(x)$ by the formula

$$\tilde{f}(x) = ax - \sum_{n=1}^{+\infty} b^n g(b^{-n}x) \ .$$

The function $\tilde{f}$ is of class C^1 (the derivative series is normally convergent in every compact set). Moreover it is easy to see that for every $x \in \mathbb{R}$, $\tilde{f}(x) = g(x) + b^{-1}\tilde{f}(bx)$. On the other hand, the Weierstrass function associated to g is the unique bounded solution of the functional equation (2.1). Finally, we can conclude that the Weierstrass function f is regular if and only if we can choose $a \in \mathbb{R}$ such that $\tilde{f}$ is bounded.

Proof of Proposition 4.1. Remember that $f(x) = g(x) + b^{-1}f(bx)$. We can then write

$$f(x) - f(0) = b(f(b^{-1}x) - f(0)) - b(g(b^{-1}x) - g(0)) \ .$$

Iterating this relation, we get for every $N \geq 1$,

$$f(x) - f(0) = b^N(f(b^{-N}x) - f(0)) - \sum_{n=1}^{N} b^n(g(b^{-n}x) - g(0)) \ . \tag{4.2}$$

Recall that $g'(0) = 0$ (see Theorem 3.1). The estimate

$$\left|b^n(g(b^{-n}x) - g(0))\right| = \left|x\left(\frac{g(b^{-n}x) - g(0)}{b^{-n}x} - g'(0)\right)\right| \leq |x|\, C\left|b^{-n}x\right|^{\varepsilon}$$

ensures that the series $\sum b^n(g(b^{-n}x) - g(0))$ converges. On the other hand, using that f is regular, we get

$$\lim_{N\to\infty} b^N(f(b^{-N}x) - f(0)) = xf'(0) \ .$$

Finally, we can take the limit in (4.2) and we obtain (4.1). $\square$

The following theorem gives a sufficient condition on the Fourier coefficients of g which ensures that f is nowhere differentiable. A similar condition was previously obtained by Kaplan, Mallet-Paret and Yorke in a slighly different context (see [8]). Recall that the Fourier coefficients of a continuous almost periodic function φ are defined by

$$\widehat{\varphi}(\lambda) = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \varphi(x)e^{-i\lambda x}dx, \qquad \lambda \in \mathbb{R} \ .$$

Moreover, the set of reals λ such that $\widehat{\varphi}(\lambda) \neq 0$ is countable (see for example [3]).

Theorem 4.3. *Let g be an almost periodic function of class C^2 and $1 < b < +\infty$. Suppose that*

$$\sum_{n\in\mathbb{Z}} b^n \widehat{g}(b^n\lambda) \neq 0 \quad \textit{for some } \lambda \in \mathbb{R}^* \ .$$

Then, the associated Weierstrass function f is nowhere differentiable.

Remark 4.4. If g is of class C^2, we have $|\widehat{g}(\mu)| = \left|-\mu^{-2}\widehat{g''}(\mu)\right| \leq |\mu|^{-2}\left\|g''\right\|_\infty$. It follows that the series $\sum_{n\in\mathbb{Z}} b^n\widehat{g}(b^n\lambda)$ is convergent.

Proof. Suppose that f is regular. Theorem 3.1 ensures that $g'(0) = 0$. Moreover, formula (4.1) is true. The estimate

$$\left|g'(b^{-n}x)\right| = \left|g'(b^{-n}x) - g'(0)\right| \leq \|g''\|_\infty \left|b^{-n}x\right| \leq \|g''\|_\infty \left|b^{-n}R\right|,$$

which is valid on the compact set $[-R, R]$, allows us to differentiate (4.1) and to get

$$f'(x) = f'(0) - \sum_{n=1}^{+\infty} g'(b^{-n}x) \ . \tag{4.3}$$

The series $\sum g'(b^{-n}x)$ is not uniformly convergent in $\mathbb{R}$. That is why we cannot compute the Fourier coefficients in relation (4.3). Nevertheless, we observe that the second derivative series is normally convergent. We deduce that the function f is of class C^2 and that

$$f''(x) = -\sum_{n=1}^{+\infty} b^{-n} g''(b^{-n}x) \ .$$

Now, we can calculate the Fourier coefficients and we obtain

$$-\lambda^2 \widehat{f}(\lambda) = \widehat{f''}(\lambda) = \sum_{n=1}^{+\infty} b^{-n} \widehat{g''}(b^n\lambda) = \sum_{n=1}^{+\infty} b^n \lambda^2 \widehat{g}(b^n\lambda) \ .$$

In particular, if $\lambda \neq 0$,

$$\widehat{f}(\lambda) = -\sum_{n=1}^{+\infty} b^n \widehat{g}(b^n\lambda) \ . \tag{4.4}$$

On the other hand, $f(x) = \sum_{n=0}^{+\infty} b^{-n} g(b^n x)$ and the convergence is uniform. Taking the Fourier coefficents, we obtain

$$\widehat{f}(\lambda) = \sum_{n=0}^{+\infty} b^{-n} \widehat{g}(b^{-n}\lambda) \ . \tag{4.5}$$

Relations (4.4) and (4.5) give the conclusion of Theorem 4.3. $\square$

Theorem 4.3 allows us to construct a lot of functions g such that the associated Weierstrass function is nowhere differentiable. In particular, we have the following corollaries.

Corollary 4.5. *Let g be a nonconstant 2π-periodic function of class C^2 and $1 < b < +\infty$. Suppose that for every integer $n \geq 1$, b^n is an irrational number. Then, the associated Weierstrass function f is nowhere differentiable.*

Corollary 4.6. *Let $b > 1$ and $B = \{b^n,\ n \in \mathbb{Z} \setminus \{0\}\}$. Suppose that $(a_k)_{k\geq 1}$ is a sequence of positive real numbers satisfying*

$$\forall k \neq \ell, \quad \frac{a_k}{a_\ell} \notin B \ .$$

Then, the only function g in the vector space $V = \mathrm{vect}\{\cos(a_k x),\ k \geq 1\}$ such that the associated Weierstrass function f is regular is the null function.

Proof of Corollary 4.5. The function g is supposed to be 2π-periodic. Its spectrum $S(g) = \{\lambda \in \mathbb{R}\ ;\ \widehat{g}(\lambda) \neq 0\}$ is included in $\mathbb{Z}$. Let $k \in \mathbb{Z}$ with $k \neq 0$ satisfying $\widehat{g}(k) \neq 0$. The hypothesis on b ensures that if $n \in \mathbb{Z}$ and $n \neq 0$, then $b^n k \notin \mathbb{Z}$.

Finally, $\sum_{n\in\mathbb{Z}} b^n\widehat{g}(b^nk) = \widehat{g}(k) \neq 0$ and the associated Weierstrass function f is nowhere differentiable. $\square$

Proof of Corollary 4.6. The proof of Corollary 4.6 is quite similar. Let $g \in V$ with $g \neq 0$. Its spectrum $S(g)$ is not empty. Moreover $S(g) \subset \{\pm a_k,\ k \geq 1\}$. The hypothesis on the sequence (a_k) ensures that if $\lambda \in S(g)$ and if $n \in \mathbb{Z} \setminus \{0\}$, then $b^n\lambda \notin S(g)$. We can conclude with the same argument as in Corollary 4.5. $\square$

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Laboratoire de Mathématiques, UMR 6620, Université Blaise Pascal, F-63177 Aubière cedex, France

E-mail address: Yanick.Heurteaux@math.univ-bpclermont.fr