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ON THE PROJECTIVITY OF THREEFOLDS

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Abstract. Let X be a smooth complete three-dimensional algebraic variety (defined over an algebraically closed field k). We show that X is projective if it contains a divisor which is positive on the cone of effective curves.

1. Introduction

Let X be a smooth complete algebraic variety defined over an algebraically closed field k (of arbitrary characteristic). A 1-cycle is a formal linear combination of irreducible, reduced and complete curves $C = \sum a_i C_i$. Two 1-cycles C, C' are called numerically equivalent if $C.D = C'.D$ for any Cartier divisor D. The class of a 1-cycle C is denoted by $[C]$. 1-cycles with real coefficients modulo numerical equivalence form an $\mathbb{R}$-vector space; it is denoted by $N_1(X)$. Let $NS(X)$ denote the Néron-Severi group of X (cf. [2]). The intersection of curves and divisors gives a perfect pairing

$$N_1(X) \times (NS(X) \otimes_{\mathbb{Z}} \mathbb{R}) \to \mathbb{R}.$$

The Theorem of the Base of Néron-Severi asserts that $N_1(X)$ is finite dimensional. Let

$$NE(X) = \{\sum a_i[C_i] : \ C_i \subset X, \ a_i \geq 0\} \subset N_1(X)$$

be a cone of effective curves. In general this cone is not a closed set, and we consider its closure $\overline{NE(X)} \subset N_1(X)$. In 1966 Steven Kleiman proved the following (see [4]):

Kleiman Ampleness Criterion. *Let X be a smooth complete algebraic variety. Assume that a Cartier divisor D is positive on the set $\overline{NE(X)} \setminus \{0\}$. Then D is an ample divisor.*

In particular Kleiman's result implies:

Kleiman Projectivity Criterion. *Let X be a smooth complete algebraic variety. Then X is projective if and only if there is a Cartier divisor D, which is positive on the set $\overline{NE(X)} \setminus \{0\}$.*

However, since the set $\overline{NE(X)}$ has no clear geometrical meaning, this Projectivity Criterion is not as natural as we would wish. It is interesting to ask whether it

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is possible to prove a stronger and geometrically more clear result, namely whether it is enough to test the divisor D on the cone $NE(X) \setminus \{0\}$ only?

Since there is a smooth complete threefold X and an effective divisor D on X such that $D.C > 0$ for every effective curve $C \subset X$ but D is not ample (such an example is constructed e.g. in [1], Example 10.8, p. 57), we cannot expect that the divisor D, which is positive on the set $NE(X) \setminus \{0\}$, must be ample.

However, we show (in the third part of this note) the following:

Projectivity Criterion for Threefolds. *Let X be a smooth complete algebraic threefold. Then X is projective if and only if there is a Cartier divisor D such that $D.C > 0$ for every (non-zero) effective curve $C \subset X$.*

Moreover, in the second part of this note we describe maximal quasi-projective subsets of a smooth, complete algebraic threefold X.

2. Maximal quasi-projective subsets

In this section we describe the geometry of maximal quasi-projective subsets of a non-projective threefold. We start with the following basic fact:

Proposition 2.1. *Let X be a normal complete variety and let $U \subset X$ be an open quasi-projective subset of X. Then there exists a normal projective variety Z and a birational morphism $f : Z \to X$, such that the inverse mapping f^{-1} is defined on an open quasi-projective subset V which contains U. Moreover, codim $X \setminus V \geq 2$.*

Proof. By the Nagata Theorem ([5], Theorem 3.2) there is a projective variety Z containing U and dominating X. Indeed, take any projective variety X' which contains U as an open dense subset. Then Z can be obtained as a join of some blowing-ups of X' (with centers disjoint from U). We can take the normalization of Z, and hence we can assume that Z is normal. Let $f : Z \to X$ be a birational morphism, which is an isomorphism on U. Then f^{-1} defines an open embedding into Z outside the exceptional locus S, which by normality of X and the Zariski Main Theorem, is of codimension ≥ 2 in X. Now it is enough to take $V = X \setminus S$. $\square$

We also have:

Proposition 2.2. *Let X be a smooth complete non-projective threefold and let $U \subset X$ be a maximal quasi-projective subset of X. Then the set $G := X \setminus U$ has a pure dimension* 1.

Proof. Let $f : Z \to X$ be a birational morphism as in the proof of Proposition 2.1. Assume that G has a point a as an isolated component. Let $V = U \cup \{a\}$ and let $W = f^{-1}(X \setminus \{a\})$. We can glue V and W along $U \cong f^{-1}(U)$ to obtain a new algebraic complete variety Z'. Note that we have a birational morphism $f' : Z \to Z'$, which is induced by f and which is an isomorphism outside the set $E = f^{-1}(a)$. Let $p = (f')^{-1}$ and let $D = p^*(O(1))$.

For an integral curve C and a point $P \in C$ we denote by $m_P(C)$ the multiplicity of the point P on a curve C. Let $m(C) = \mathrm{Sup}_{P \in C} m_P(C)$.

Since the linear system $|D|$ given by D has at most one base point a and this system is very ample outside a, it is easy to check that for an integral curve $C \subset Z'$ we have $D.C \geq m(C)$. Indeed, if $P \neq a$, then $D.C \geq m_P(C)$, because the system $|D|$ is very ample outside the point a. Now let $P = a$. By the construction of the system $|D|$, we obtain that for every point $z \in Z' \setminus \{a\}$ there is an effective divisor

$D_z \in |D|$, such that $a \in Supp(D_z)$ but $z \notin Supp(D_z)$. If we take $z \in C \setminus \{a\}$, then $D_z.C \geq m_P(C)$.

Hence by the Seshadri theorem ([1], p. 37) we have that D is ample. Consequently Z' is projective and V is quasi-projective, a contradiction with the maximality of U. □

Remark 2.3. Let X be a smooth complete non-projective threefold (e.g., Hironaka threefold, see [3]). Let $F \subset X$ be a finite set. Then (by Proposition 2.2) the variety $X' = X \setminus F$ is a not-complete variety which is not quasi-projective.

3. Criterion for projectivity

In this section we prove that a smooth complete algebraic threefold is projective if it contains a divisor which is positive on the cone of effective curves.

Theorem 3.1. *Let X be a smooth complete algebraic threefold. Then X is projective if and only if there is a Cartier divisor D such that $D.C > 0$ for every (non-zero) effective curve $C \subset X$.*

Proof. If X is a projective variety and $X \subset \mathbb{P}^n$ is a suitable embedding, then it is enough to take $D = a\ hyperplane$.

Conversely, assume that a divisor D is positive on effective curves $C \subset X$. By Proposition 2.1 we can cover X by open subsets $U_i, i = 1, ..., m$, such that:

1) $X \setminus U_i$ is a curve, which we will denote by G_i;

2) there is a normal projective variety Z_i and a birational morphism $f_i : Z_i \to X$, which induces an isomorphism $Z_i \setminus f^{-1}(G_i) \to X \setminus G_i$.

For an integral curve C and a point $P \in C$ we denote by $m_P(C)$ the multiplicity of the point P on a curve C. For a curve C let $m(C) = \mathrm{Sup}_{P \in C} m_P(C)$.

Let us take a hyperplane section divisor H_i on Z_i and let $D_i = f(H_i)$. By the construction it is ample on U_i. Now take $H = \sum_{i=1}^{m} D_i$. Let C be a curve which is not a component of any $G_i, i = 1, ..., m$. Then clearly $D_i.C \geq 0$ for every $i = 1, ..., m$. Moreover, for a point $P \in C$ there is an index j such that $P \in U_j$. Since D_j is very ample in U_j we have $D_j.C \geq m_P(C)$ and consequently $H.C \geq m(C)$.

Since there is only a finite number of curves which are components of curves $G_i,\ i = 1, ..., m$, and the divisor D is positive on each effective curve, we have that for $s >> 0$ the divisor $H' = H + sD$ satisfies $H'.C \geq m(C)$ for every integral curve C. Hence by the Seshadri theorem ([1], p. 37) we have that H' is ample and consequently the variety X is projective. This finishes the proof. □

Remark 3.2. In the same way we can prove a stronger theorem:

Let X be a smooth complete threefold and let $U \subset X$ be an open quasi-projective subset of X. Assume that there is a numerically-effective Cartier divisor D such that $D.C > 0$ for every effective curve which is disjoint from U. Then X is projective.

Remark 3.3. The author does not know whether Theorem 3.1 is true in higher dimensions.

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