ON THE PROJECTIVITY OF THREEFOLDS

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Abstract. Let $X$ be a smooth complete three-dimensional algebraic variety (defined over an algebraically closed field $k$). We show that $X$ is projective if it contains a divisor which is positive on the cone of effective curves.

1. Introduction

Let $X$ be a smooth complete algebraic variety defined over an algebraically closed field $k$ (of arbitrary characteristic). A 1-cycle is a formal linear combination of irreducible, reduced and complete curves $C = \sum a_i C_i$. Two 1-cycles $C, C'$ are called numerically equivalent if $C.D = C'.D$ for any Cartier divisor $D$. The class of a 1-cycle $C$ is denoted by $\left[ C \right]$. 1-cycles with real coefficients modulo numerical equivalence form an $\mathbb{R}$-vector space; it is denoted by $N_1(X)$. Let $NS(X)$ denote the Néron-Severi group of $X$ (cf. \cite{2}). The intersection of curves and divisors gives a perfect pairing

$$ N_1(X) \times (NS(X) \otimes \mathbb{Z} \mathbb{R}) \to \mathbb{R}. $$

The Theorem of the Base of Néron-Severi asserts that $N_1(X)$ is finite dimensional.

Let

$$ NE(X) = \{ \sum a_i[C_i] : C_i \subset X, \ a_i \geq 0 \} \subset N_1(X) $$

be a cone of effective curves. In general this cone is not a closed set, and we consider its closure $\overline{NE(X)} \subset N_1(X)$. In 1966 Steven Kleiman proved the following (see \cite{4}):

**Kleiman Ampleness Criterion.** Let $X$ be a smooth complete algebraic variety. Assume that a Cartier divisor $D$ is positive on the set $\overline{NE(X)} \setminus \{0\}$. Then $D$ is an ample divisor.

In particular Kleiman’s result implies:

**Kleiman Projectivity Criterion.** Let $X$ be a smooth complete algebraic variety. Then $X$ is projective if and only if there is a Cartier divisor $D$, which is positive on the set $\overline{NE(X)} \setminus \{0\}$.

However, since the set $\overline{NE(X)}$ has no clear geometrical meaning, this Projectivity Criterion is not as natural as we would wish. It is interesting to ask whether it

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is possible to prove a stronger and geometrically more clear result, namely whether it is enough to test the divisor \( D \) on the cone \( NE(X) \setminus \{0\} \) only?

Since there is a smooth complete threefold \( X \) and an effective divisor \( D \) on \( X \) such that \( D.C > 0 \) for every effective curve \( C \subset X \) but \( D \) is not ample (such an example is constructed e.g. in \([1]\), Example 10.8, p. 57), we cannot expect that the divisor \( D \), which is positive on the set \( NE(X) \setminus \{0\} \), must be ample.

However, we show (in the third part of this note) the following:

**Projectivity Criterion for Threefolds.** Let \( X \) be a smooth complete algebraic threefold. Then \( X \) is projective if and only if there is a Cartier divisor \( D \) such that \( D.C > 0 \) for every (non-zero) effective curve \( C \subset X \).

Moreover, in the second part of this note we describe maximal quasi-projective subsets of a smooth, complete algebraic threefold \( X \).

### 2. Maximal Quasi-Projective Subsets

In this section we describe the geometry of maximal quasi-projective subsets of a non-projective threefold. We start with the following basic fact:

**Proposition 2.1.** Let \( X \) be a normal complete variety and let \( U \subset X \) be an open quasi-projective subset of \( X \). Then there exists a normal projective variety \( Z \) and a birational morphism \( f : Z \to X \), such that the inverse mapping \( f^{-1} \) is defined on an open quasi-projective subset \( V \) which contains \( U \). Moreover, \( \text{codim} X \setminus V \geq 2 \).

**Proof.** By the Nagata Theorem ([5], Theorem 3.2) there is a projective variety \( Z \) containing \( U \) and dominating \( X \). Indeed, take any projective variety \( X' \) which contains \( U \) as an open dense subset. Then \( Z \) can be obtained as a join of some blowing-ups of \( X' \) (with centers disjoint from \( U \)). We can take the normalization of \( Z \), and hence we can assume that \( Z \) is normal. Let \( f : Z \to X \) be a birational morphism, which is an isomorphism on \( U \). Then \( f^{-1} \) defines an open embedding into \( Z \) outside the exceptional locus \( S \), which by normality of \( X \) and the Zariski Main Theorem, is of codimension \( \geq 2 \) in \( X \). Now it is enough to take \( V = X \setminus S \). \( \square \)

We also have:

**Proposition 2.2.** Let \( X \) be a smooth complete non-projective threefold and let \( U \subset X \) be a maximal quasi-projective subset of \( X \). Then the set \( G := X \setminus U \) has a pure dimension 1.

**Proof.** Let \( f : Z \to X \) be a birational morphism as in the proof of Proposition 2.1. Assume that \( G \) has a point \( a \) as an isolated component. Let \( V = U \cup \{a\} \) and let \( W = f^{-1}(X \setminus \{a\}) \). We can glue \( V \) and \( W \) along \( U \equiv f^{-1}(U) \) to obtain a new algebraic complete variety \( Z' \). Note that we have a birational morphism \( f' : Z \to Z' \), which is induced by \( f \) and which is an isomorphism outside the set \( E = f^{-1}(a) \). Let \( p = (f')^{-1} \) and let \( D = p^*(O(1)) \).

For an integral curve \( C \) and a point \( P \in C \) we denote by \( m_P(C) \) the multiplicity of the point \( P \) on a curve \( C \). Let \( m(C) = \text{Sup}_{P \in C} m_P(C) \).

Since the linear system \( |D| \) given by \( D \) has at most one base point \( a \) and this system is very ample outside \( a \), it is easy to check that for an integral curve \( C \subset Z' \), we have \( D.C \geq m(C) \). Indeed, if \( P \neq a \), then \( D.C \geq m_P(C) \), because the system \( |D| \) is very ample outside the point \( a \). Now let \( P = a \). By the construction of the system \( |D| \), we obtain that for every point \( z \in Z' \setminus \{a\} \) there is an effective divisor
Remark 2.3 Let $X$ be a smooth complete non-projective threefold (e.g., Hironaka threefold, see [3]). Let $F \subset X$ be a finite set. Then (by Proposition 2.2) the variety $X' = X \setminus F$ is a not-complete variety which is not quasi-projective.

3. Criterion for projectivity

In this section we prove that a smooth complete algebraic threefold is projective if it contains a divisor which is positive on the cone of effective curves.

**Theorem 3.1.** Let $X$ be a smooth complete algebraic threefold. Then $X$ is projective if and only if there is a Cartier divisor $D$ such that $D.C > 0$ for every (non-zero) effective curve $C \subset X$.

**Proof.** If $X$ is a projective variety and $X \subset \mathbb{P}^n$ is a suitable embedding, then it is enough to take $D = a$ hyperplane.

Conversely, assume that a divisor $D$ is positive on effective curves $C \subset X$. By Proposition 2.1 we can cover $X$ by open subsets $U_i, i = 1, ..., m$, such that:

1) $X \setminus U_i$ is a curve, which we will denote by $G_i$;

2) there is a normal projective variety $Z_i$ and a birational morphism $f_i : Z_i \to X$, which induces an isomorphism $Z_i \setminus f_i^{-1}(G_i) \to X \setminus G_i$.

For an integral curve $C$ and a point $P \in C$ we denote by $m_P(C)$ the multiplicity of the point $P$ on a curve $C$. For a curve $C$ let $m(C) = \sup_{P \in C} m_P(C)$.

Let us take a hyperplane section divisor $H_i$ on $Z_i$ and let $D_i = f(H_i)$. By the construction it is ample on $U_i$. Now take $H = \sum_{i=1}^m D_i$. Let $C$ be a curve which is not a component of any $G_i, i = 1, ..., m$. Then clearly $D_i.C \geq 0$ for every $i = 1, ..., m$. Moreover, for a point $P \in C$ there is an index $j$ such that $P \in U_j$. Since $D_j$ is very ample in $U_j$ we have $D_j.C \geq m_P(C)$ and consequently $H.C \geq m(C)$.

Since there is only a finite number of curves which are components of curves $G_i, i = 1, ..., m$, and the divisor $D$ is positive on each effective curve, we have that for $s >> 0$ the divisor $H' = H + sD$ satisfies $H'.C \geq m(C)$ for every integral curve $C$. Hence by the Seshadri theorem ([11], p. 37) we have that $H'$ is ample and consequently the variety $X$ is projective. This finishes the proof. 

**Remark 3.2.** In the same way we can prove a stronger theorem:

Let $X$ be a smooth complete threefold and let $U \subset X$ be an open quasi-projective subset of $X$. Assume that there is a numerically-effective Cartier divisor $D$ such that $D.C > 0$ for every effective curve which is disjoint from $U$. Then $X$ is projective.

**Remark 3.3.** The author does not know whether Theorem 3.1 is true in higher dimensions.

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