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ON NATURAL HOMOMORPHISMS OF WITT RINGS

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ABSTRACT. We prove that the kernel of the ring homomorphism between the Witt ring of any order of a global field and the Witt ring of the field is a nilideal.

1. INTRODUCTION

For a commutative ring A, let $W(A)$ denote the Witt ring of nondegenerate symmetric bilinear forms on finitely generated projective modules (bilinear spaces) over A, as defined by Knebusch in [5]. For an integral domain A and its field of fractions K we consider the ring homomorphism

$$\varphi : W(A) \to W(K)$$

induced by the inclusion $A \hookrightarrow K$. If A is a Dedekind domain it is known that $\ker\varphi$ is zero. This was first proved by Knebusch ([5, Satz 11.1.1]). A more general case was considered by Craven, Rosenberg and Ware in [3]. They proved that when A is a regular noetherian domain of an arbitrary Krull dimension, then $\ker\varphi$ is a nilideal, that is, every element belonging to the kernel is a nilpotent element of the Witt ring $W(A)$. They also produced examples emphasizing the importance of the regularity condition. However, it turns out that the class of domains A with $\ker\varphi$ a nilideal does not only include regular domains. We prove that if A is an order in a global field K, then the kernel of φ is a nilideal. Since nonmaximal orders are not regular rings, this result shows that regularity of the domain A is not a necessary condition for the kernel of φ to be a nilideal. When the field K is not formally real we find a necessary and sufficient condition for $\ker\varphi$ to be a nilideal (Theorem 2.3) immediately applicable to global fields. When K is a formally real number field we use the theory of signatures on Witt rings.

For a commutative ring A we write A^* for the group of invertible elements in A. If $\mathfrak{p}$ is a prime ideal in A we write $A_{\mathfrak{p}}$ for the localization of A with respect to $\mathfrak{p}$. The symbol $\langle E\rangle$ denotes the element of the Witt ring $W(A)$ determined by the bilinear space E over A.

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If B is a commutative ring and A is a subring of B, then by the *natural homomorphism* induced by the inclusion of A into B we mean the map

$$\varphi : W(A) \to W(B), \qquad \varphi\langle E\rangle = \langle E \otimes_A B\rangle.$$

We also use the symbol $\langle E\rangle_B$ for $\varphi\langle E\rangle$.

$W(A)_t$ denotes the torsion part of the Witt ring $W(A)$. If A is a local ring, then each element of $W(A)$ is represented by a free bilinear module. Then $W(A)_0$ denotes the ideal of classes of even ranks. We say that a ring A has finite level if $-1 \in A$ can be represented as a sum of squares of elements of A. Otherwise A has infinite level. A field with infinite level is said to be formally real, and nonreal otherwise.

A global field K is a finite extension of the rational number field $\mathbb{Q}$ or a finite extension of a rational function field in one variable over a finite field. The ring of integers of a number field K is denoted by R. In the function field case R denotes the integral closure in K of the polynomial ring of the underlying rational function field. In any case R is a Dedekind domain. We adopt the following definition (see [4]).

Definition 1.1. An order $\mathcal{O}$ in K is a subring of R such that $R/\mathcal{O}$ is a finitely generated torsion $\mathcal{O}-$module.

The ring R is the maximal order in K. When K is a number field, an order $\mathcal{O}$ in K is a subring of R containing an integral basis of length $[K : \mathbb{Q}]$. Every order is a one-dimensional noetherian domain ([8, p. 73]).

We recall that a nonmaximal order $\mathcal{O}$ is not a regular ring, i.e., there exists a prime ideal $\mathfrak{p}$ such that the localization $\mathcal{O}_\mathfrak{p}$ is not a regular local ring and this is equivalent to the fact that $\mathcal{O}_\mathfrak{p}$ is not a discrete valuation ring. This follows from [8, Theorem 12.10] where it is shown that $\mathcal{O}_\mathfrak{p}$ is regular if and only if $\mathfrak{f} \not\subseteq \mathfrak{p}$, where $\mathfrak{f}$ is the conductor of the ring extension $R \supset \mathcal{O}$. Since for a nonmaximal order $\mathcal{O}$ we have $\mathfrak{f} \neq (0)$ and $\mathfrak{f} \neq \mathcal{O}$, there is a nonzero prime ideal containing $\mathfrak{f}$ and so $\mathcal{O}$ is not a regular ring.

2. K nonreal

When the field K is nonreal, there is no need to restrict the consideration to global fields. Theorem 2.3 below gives a convenient necessary and sufficient condition for the kernel of the natural ring homomorphism $W(A) \to W(K)$ to be a nilideal. This applies to nonreal global fields. We begin the discussion with the case of local domains.

Theorem 2.1. *Let A be a local domain. If A has finite level, then the kernel of the natural ring homomorphism $W(A) \to W(K)$ is a nilideal.*

Proof. If A has finite level, then $W(A)_t = W(A)$ (see [1, A.4, p. 178]). On the other hand $\ker(W(A) \to W(K)) \subseteq W(A)_0$, since any element in the kernel becomes metabolic over K, hence of even dimension. Thus

$$\ker(W(A) \to W(K)) \subseteq W(A)_0 = W(A)_0 \cap W(A)_t = \operatorname{Nil} W(A),$$

the latter by [1, Theorem 8.9, p. 158]. □

Remark 2.2. Craven, Rosenberg and Ware proved that if A is a regular noetherian local domain, then $\ker(W(A) \to W(K))$ is a nilideal ([3, Thm. 2.4]). Theorem 2.1

shows that if the local domain A has finite level, then the conclusion holds even if it is not regular or noetherian.

For a ring A and a prime ideal $\mathfrak{p}$ of A we write $A(\mathfrak{p})$ for the field of fractions of $A/\mathfrak{p}$. In particular, if A is a domain, $A(0) = K$. We shall use below the following *Dress' local-global principle* (see [3, Cor. 2.7]):

For a commutative ring A and any bilinear A-space E, the class $\langle E\rangle \in W(A)$ is nilpotent if and only if $\langle E\rangle_{A_\mathfrak{p}} \in W(A_\mathfrak{p})$ is nilpotent for all maximal ideals $\mathfrak{p}$ in A.

Theorem 2.3. *Let A be an integral domain. Assume that the field of fractions K is not formally real. Then the following are equivalent:*
(a) $\ker(W(A) \to W(K))$ *is a nilideal.*
(b) *For each prime ideal $\mathfrak{p}$ in A the field $A(\mathfrak{p})$ is not formally real.*

Proof. (a) $\Rightarrow$ (b) The proof in [3, Prop. 3.2] applies with a slight complication. Here are the details. Suppose there is a prime ideal $\mathfrak{p}$ of A such that $A(\mathfrak{p})$ is formally real. Then $r \cdot \langle 1\rangle \neq 0$ in $W(A(\mathfrak{p}))$ for all positive integers r. On the other hand $2^m\langle 1\rangle = 0$ in $W(K)$ for some $m \geq 0$ since K is a nonreal field. Hence $2^m\langle 1\rangle \in \ker(W(A) \to W(K))$. It remains to show that $2^m\langle 1\rangle$ is not a nilpotent element in $W(A)$. If it were, then $2^n\langle 1\rangle = 0$ in $W(A)$ for some n. Then the composition of natural homomorphisms

$$W(A) \to W(A/\mathfrak{p}) \to W(A(\mathfrak{p}))$$

sends $2^n\langle 1\rangle$ to zero in $W(A(\mathfrak{p}))$, a contradiction.

(b) $\Rightarrow$ (a) By [2, Theorem 1], (b) implies that the level of the ring A is finite. Hence the level of the localization $A_\mathfrak{p}$ is also finite for each nonzero prime ideal in A, and so $\ker(W(A_\mathfrak{p}) \to W(K))$ is a nilideal by Theorem 2.1. Now let $\langle E\rangle \in \ker(W(A) \to W(K))$. Then

$$\langle E \otimes_A A_\mathfrak{p}\rangle \in \ker(W(A_\mathfrak{p}) \to W(K)) \subseteq \operatorname{Nil} W(A_\mathfrak{p}).$$

Thus $\langle E\rangle_{A_\mathfrak{p}} = \langle E \otimes_A A_\mathfrak{p}\rangle$ is nilpotent for all nonzero prime ideals $\mathfrak{p}$ in A. By Dress' local-global principle, the class $\langle E\rangle$ is a nilpotent element of the ring $W(A)$. $\square$

If $A = \mathcal{O}$ is any order in a nonreal global field K, then (b) is satisfied, and so $\ker(W(\mathcal{O}) \to W(K))$ is a nilideal. Thus the following theorem, which is the first part of our main result, is a corollary to Theorem 2.3.

Theorem 2.4. *Let $\mathcal{O}$ be an order in a nonreal global field K. Then the kernel of the natural ring homomorphism*

$$\varphi : W(\mathcal{O}) \to W(K)$$

is a nilideal.

3. K formally real

In this section K is a formally real global field, that is, a formally real number field. As in the previous section we will use the local-global principle for nilpotency of elements of the Witt ring. Hence we first establish the following result for the local case.

Theorem 3.1. *Let $\mathfrak{p}$ be a nonzero prime ideal in an order $\mathcal{O}$ of a formally real number field K. Then the kernel of the ring homomorphism $\varphi : W(\mathcal{O}_\mathfrak{p}) \to W(K)$ is a nilideal.*

The proof will use some results on signatures of Witt rings and orderings of rings. We collect the definitions we need following [6]. For an arbitrary commutative ring A we write $\tilde{X}_A$ for the set of all signatures of the ring $W(A)$, i.e., the set of all ring homomorphisms $\sigma : W(A) \to \mathbb{Z}$. Recall that $P \subset A$ is said to be an ordering in A if

(1) $P + P \subset P, \quad P \cdot P \subset P$,
(2) $P \cup -P = A$,
(3) $P \cap -P =: \operatorname{supp} P$, called the support of P, is a prime ideal in A.

The set of all orderings of A is denoted by X_A. Since K is assumed formally real, there is an ordering $P \in X_K$. One can associate a signature $\sigma_P : W(K) \to \mathbb{Z}$ with the ordering P by setting

$$\sigma_P\langle a\rangle = 1 \quad \text{for all} \quad a \in P,\ a \neq 0. \tag{3.1}$$

More generally, if A is a local ring and $P \in X_A$, then there is a signature $\sigma_P : W(A) \to \mathbb{Z}$ satisfying (3.1). This follows from the fact that $W(A)$ is generated by the rank one elements $\langle a\rangle$ with $a \in A^*$. We need a stronger converse statement associating orderings of K to signatures of $\mathcal{O}_\mathfrak{p}$.

Lemma 3.2. *For every signature $\sigma \in \tilde{X}_{\mathcal{O}_\mathfrak{p}}$ there exists an ordering P of the field K such that*

$$\sigma\langle E\rangle = \sigma_P\langle E\rangle_K$$

for all $\langle E\rangle \in W(\mathcal{O}_\mathfrak{p})$.

Proof. Let $\sigma \in \tilde{X}_{\mathcal{O}_\mathfrak{p}}$. Then by [6, Prop. 1.5, p. 145] there exists an ordering $Q \in X_{\mathcal{O}_\mathfrak{p}}$ such that $\sigma\langle a\rangle = \sigma_Q\langle a\rangle$ for $a \in \mathcal{O}_\mathfrak{p}^*$. We must show that the ordering Q on $\mathcal{O}_\mathfrak{p}$ can be extended to an ordering P on K. To this end we show that $\operatorname{supp} Q = (0)$. If $\operatorname{supp} Q \neq (0)$, then $\operatorname{supp} Q = \mathfrak{p}\mathcal{O}_\mathfrak{p}$ since $\mathfrak{p}\mathcal{O}_\mathfrak{p}$ is the only nonzero prime ideal in $\mathcal{O}_\mathfrak{p}$. By a general rule (see [6, Prop. 1.1, p. 96]), there exists an ordering on $\mathcal{O}_\mathfrak{p}/\mathfrak{p}\mathcal{O}_\mathfrak{p}$ whose inverse image under the canonical homomorphism $\mathcal{O}_\mathfrak{p} \to \mathcal{O}_\mathfrak{p}/\mathfrak{p}\mathcal{O}_\mathfrak{p}$ is the ordering Q. However, this is absurd, since the residue class field $\mathcal{O}_\mathfrak{p}/\mathfrak{p}\mathcal{O}_\mathfrak{p} \cong \mathcal{O}/\mathfrak{p}$ is a subfield of the finite field $R/\mathfrak{q}$, where $\mathfrak{q} \cap \mathcal{O} = \mathfrak{p}$, and finite fields have no orderings. This proves that $\operatorname{supp} Q = (0)$. Thus, by [6, Prop. 1.1, p. 96], the ordering Q of $\mathcal{O}_\mathfrak{p}$ extends uniquely to an ordering P of the field K. The associated signature σ_P satisfies

$$\sigma\langle a\rangle = \sigma_Q\langle a\rangle = \sigma_P\langle a\rangle_K \quad \text{for all} \quad a \in \mathcal{O}_\mathfrak{p}^*.$$

Since $W(\mathcal{O}_\mathfrak{p})$ is additively generated by the rank-one classes $\langle a\rangle$, $a \in \mathcal{O}_\mathfrak{p}^*$, we have $\sigma\langle E\rangle = \sigma_P\langle E\rangle_K$ for all $\langle E\rangle \in W(\mathcal{O}_\mathfrak{p})$. □

Proof of Theorem 3.1. Let $\langle E\rangle \in W(\mathcal{O}_\mathfrak{p})$ and $\langle E\rangle_K = 0$. Since

$$\operatorname{Nil} W(\mathcal{O}_\mathfrak{p}) = \bigcap_{\sigma \in \tilde{X}_{\mathcal{O}_\mathfrak{p}}} \ker \sigma$$

(see [1, (7.11), p. 151 and (7.16), p. 152]), it is enough to show that $\sigma\langle E\rangle = 0$ for every signature $\sigma \in \tilde{X}_{\mathcal{O}_\mathfrak{p}}$. By Lemma 3.2, given such a σ, there exists $P \in X_K$ such that for all $\langle E\rangle \in W(\mathcal{O}_\mathfrak{p})$ we have $\sigma\langle E\rangle = \sigma_P\langle E\rangle_K$. Since $\langle E\rangle_K = 0$ we get $\sigma\langle E\rangle = 0$. This finishes the proof. □

Theorem 3.1 and Dress' local-global principle imply the following theorem which is the second part of our main result.

Theorem 3.3. *Let $\mathcal{O}$ be an order in a formally real algebraic number field K. Then the kernel of the natural ring homomorphism*

$$\varphi : W(\mathcal{O}) \to W(K)$$

is a nilideal.

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