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COMPLEMENTED SPACES OF OPERATORS

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ABSTRACT. Classical results of Kalton are used to study the complementation of the space $W(X, Y)$ of weakly compact operators and the space $K(X, Y)$ of compact operators in the space $L(X, Y)$ of bounded linear operators from X to Y.

Throughout this paper X and Y denote Banach spaces. Notation is consistent with that used in Diestel [2]. Let (e_n) be the canonical base of c_0 and let (e_n^*) be the canonical base of l_1. Let η denote the canonical embedding of X in X^{**}.

Numerous authors have studied the complementation of the spaces of weakly compact operators $W(X, Y)$ and compact operators $K(X, Y)$ in the space $L(X, Y)$ of all continuous operators from X to Y. See Bator and Lewis [1], Kalton [9], Emmanuele [4], [5], Emmanuele and John [6], and Feder [7]. The following theorem generalizes the results in [5], [1], and [7].

Theorem 1. (i) *Let X and Y be Banach spaces with the following properties: There exists a Banach space G with an unconditional basis (g_n), biorthogonal coefficients (g_n^*) and operators $R : G \to Y$ and $S : X \to G$ such that $(R(g_i))$ is a seminormalized basic sequence in Y and $\{S^*(g_i^*) : i \in \mathbb{N}\}$ is not relatively weakly compact. Then $W(X, Y)$ is not complemented in $L(X, Y)$.*

(ii) *Let X and Y be Banach spaces with the following properties: There exists a Banach space G with an unconditional basis (g_n), biorthogonal coefficients (g_n^*) and operators $R : G \to Y$ and $S : X \to G$ such that $(R(g_i))$ is a seminormalized basic sequence in Y and $\{S^*(g_i^*) : i \in \mathbb{N}\}$ is not relatively compact. Then $K(X, Y)$ is not complemented in $L(X, Y)$.*

Proof. (i) Let $(g_{i_j}^*)$ be a subsequence of (g_i^*) such that the sequence $(S^*(g_{i_j}^*))$ has no weakly convergent subsequence. Let $E = [g_{i_j}]$ be the closed linear span of $\{g_{i_j} : j \in \mathbb{N}\}$, $y_j = R(g_{i_j})$, for $j \geq 1$, and let (y_j^*) be a sequence of biorthogonal coefficients corresponding to (y_j). Let $p : G \to E$ be a projection. Define $T : X \to E$ by $T = pS$ and $B : E \to Y$ by $B = R_{|E}$.

Since $\sum g_n^*(g)g_n$ converges unconditionally to g for all $g \in G$,

$$B(\sum_j g_{i_j}^*(Tx)g_{i_j}) = \sum_j g_{i_j}^*(Tx)B(g_{i_j}) = \sum_j T^*(g_{i_j}^*)(x)y_j$$

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converges unconditionally for all $x \in X$. Let $D = \{T^*(g_{i_j}^*) : j \in \mathbb{N}\}$. Let X_0 be a separable subspace of X such that $[D]_{|X_0}$ is an isometry and let R_1 denote the restriction map to X_0. Define $\Psi : l_\infty \to L(X, Y)$ by

$$\Psi(b)(x) = \sum_j b_j T^*(g_{i_j}^*)(x) y_j$$

for $b \in l_\infty$ and $x \in X$. Let $J : [y_j] \to l_\infty$ be a linear isometry and let $A : Y \to l_\infty$ be a continuous linear extension of J.

Suppose that $W(X, Y) \overset{c}{\hookrightarrow} L(X, Y)$, and let $P : L(X, Y) \to W(X, Y)$ be a projection. Consider the operators $R_1 A P \Psi : l_\infty \to W(X_0, l_\infty)$ and $R_1 A \Psi : l_\infty \to L(X_0, l_\infty)$. Since $\Psi(e_j) = (S^*(g_{i_j}^*)) \otimes y_j$, $\Psi(e_j)$ is a rank one operator, thus compact and weakly compact. Then

$$R_1 A P \Psi(e_j) = R_1 A \Psi(e_j) \text{ for each } j.$$

An application of Proposition 5 of Kalton [9] produces an infinite subset M of $\mathbb{N}$ such that

$$R_1 A P \Psi(b) = R_1 A \Psi(b), b \in l_\infty(M).$$

Therefore $R_1 A \Psi(\chi_M)$ is weakly compact. But $\Psi(\chi_M)^*(y_j^*) = S^*(g_{i_j}^*), j \in M$, and $\{S^*(g_{i_j}^*) : j \in M\}$ is not relatively weakly compact. Therefore $\Psi(\chi_M)_{|X_0}$ is not weakly compact. However this is a contradiction since $A_{|[y_n]}$ is an isometry and $R_1 A \Psi(\chi_M)$ is weakly compact.

(ii) The proof is essentially the same as the proof of (i), replacing "relatively weakly compact" with "relatively compact". □

Corollary 2 ([5], Theorems 2 and 3; [1], Theorem 4)**.** *If $c_0 \hookrightarrow Y$ and X^* contains a w^*-null sequence (x_n^*) which is not w-null, then $W(X, Y)$ is not complemented in $L(X, Y)$.*

Proof. Let $G = c_0$ and $R : c_0 \to Y$ be an embedding. Assume without loss of generality that (x_n^*) has no weakly convergent subsequence. Define $S : X \to c_0$ by $S(x) = (x_n^*(x))$. Then $S^*(e_n^*) = x_n^*$ for each n. Apply Theorem 1 now. □

Corollary 3 ([5], Corollary 4)**.** *Assume that X contains a complemented copy of c_0 and $c_0 \hookrightarrow Y$. Then $W(X, Y)$ is not complemented in $L(X, Y)$.*

Proof. If $c_0 \overset{c}{\hookrightarrow} X$ and $W(X, Y) \overset{c}{\hookrightarrow} L(X, Y)$, then $W(c_0, Y) \overset{c}{\hookrightarrow} L(c_0, Y)$. An application of Corollary 2 concludes the proof. □

Corollary 4 ([7], Corollary 4)**.** *If $c_0 \hookrightarrow Y$ and X is an infinite-dimensional Banach space, then $K(X, Y)$ is not complemented in $L(X, Y)$.*

Proof. Let (x_n^*) be a w^*-null sequence of norm one elements in X^*. Define $S : X \to c_0$ by $S(x) = (x_n^*(x))$. Clearly $(S^*(e_n^*)) = (x_n^*)$ is not relatively compact. Let $G = c_0$ and $R : c_0 \to Y$ be an embedding. Theorem 1 now gives the conclusion. □

Corollary 5 ([5], Theorem 5)**.** *Assume that $L(X, l_1) \neq K(X, l_1)$ and that Y contains a copy of l_1. Then $W(X, Y)$ is not complemented in $L(X, Y)$.*

Proof. Since l_1 has the Schur property, $W(X, l_1) = K(X, l_1)$. Let $T : X \to l_1$ be an operator that is not weakly compact. Note that $T^*_{|c_0}$ is not weakly compact, otherwise $T^{**} : X^{**} \to l_1$ would be weakly compact. But $T^{**}_{|X} = T^{**}\eta = T$ is not weakly compact. Let $G = l_1$ and $R : l_1 \to Y$ be an embedding. By Theorem 1, $W(X, Y)$ is not complemented in $L(X, Y)$. □

Corollary 6 ([9], Lemma 3)**.** *If X contains a complemented copy of l_1 and Y is infinite-dimensional, then $K(X,Y)$ is not complemented in $L(X,Y)$.*

Proof. Let P be a projection from X to l_1. Take $G = l_1$. Since P restricted to l_1 is the identity, P is not compact. We claim that $P^*_{|c_0}$ is not compact. If it were, its adjoint $P^{**} : X^{**} \to l_1$ would be compact. But then $P^{**}\eta = P$ would be compact, a contradiction.

Therefore $(P^*(e_n))$ is not relatively compact. Let (y_n) be a normalized basic sequence in Y. Define $R : l_1 \to Y$ by $R(b_n) = \sum b_n y_n$. Then $(R(e_n^*)) = (y_n)$ is basic and normalized. Apply Theorem 1 now. □

We conclude by presenting a new and very short proof of the main result in Emmanuele [4].

Corollary 7 ([4], Theorem 2)**.** *If c_0 embeds in $K(X,Y)$, then $K(X,Y)$ is not complemented in $L(X,Y)$.*

Proof. An application of Corollaries 3 and 6 shows that we may assume that c_0 embeds in neither X^* nor Y. Thus, by Theorem 4 of Kalton [9], l_∞ does not embed in $K(X,Y)$. If (T_i) is a copy of the unit vector basis of c_0 in $K(X,Y)$, the Diestel-Faires Theorem ([3]) produces a subsequence (T_{i_j}) so that

$$\sum_j T_{i_j} \qquad \text{(strong operator topology)}$$

produces a noncompact operator. Apply the main result of Feder [7]. □

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