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FENCHEL DUALITY, FITZPATRICK FUNCTIONS AND THE KIRSZBRAUN–VALENTINE EXTENSION THEOREM

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ABSTRACT. We present a new proof of the classical Kirszbraun–Valentine extension theorem. Our proof is based on the Fenchel duality theorem from convex analysis and an analog for nonexpansive mappings of the Fitzpatrick function from monotone operator theory.

Let H be a real Hilbert space, let D be a nonempty subset of H and let $T\colon\ D \mapsto H$ be nonexpansive. According to the Kirszbraun–Valentine extension theorem ([5], [12]), there is a nonexpansive extension of T to the whole of H. Proofs of this classical result, as well as related information, can be found in, for instance, [6], [10], [3], [13] and [1]. The purpose of this note is to present a new proof of the Kirszbraun–Valentine extension theorem, which we state as Theorem 5 below. Our proof is based on the Fenchel duality theorem from convex analysis (see Theorem 1 below) and an analog for nonexpansive maps of the Fitzpatrick function from monotone multifunction theory (see [4] or [11] — in the context of monotone multifunctions, related functions appear in [2] and [8]). We start off by stating Rockafellar's version of the Fenchel duality theorem (see [9, Theorem 1, pp. 82–83]). We recall that if $f\colon\ E \mapsto (-\infty,\infty]$ is convex, then the *Fenchel conjugate*, f^*, of f is the function on the dual, E^*, of E defined by $f^*(x^*) = \sup_E [x^* - f]$.

Theorem 1. *Let E be a nonzero normed space, let $f, g\colon\ E \mapsto (-\infty,\infty]$ be proper and convex, let g be finite and continuous at a point where f is finite, and let $f + g \geq 0$ on E. Then there exists $x^* \in E^*$ such that $f^*(x^*) + g^*(-x^*) \leq 0$.*

Definition 2. Let D be a nonempty subset of a Hilbert space H and let $T\colon\ D \mapsto H$. The map T is said to be *nonexpansive* if $x, y \in D \Longrightarrow \|T(x) - T(y)\| \leq \|x - y\|$. We say that T is *maximally nonexpansive* if T is nonexpansive, and there is no nonexpansive extension of T to a proper superset of D.

Lemma 3. *Let $T\colon\ D \mapsto H$ be maximally nonexpansive, and define $\chi\colon\ H \times H \mapsto (-\infty,\infty]$ by*

$$\chi(p,q) := \sup_{d\in D} \big[\|q - T(d)\|^2 - \|p - d\|^2\big].$$

Let $G(T)$ be the graph of T, $\big\{\big(p, T(p)\big)\colon\ p \in D\big\}$. Then:

(a) *$\chi \geq 0$ on $H \times H$.*
(b) *$\chi(p,q) = 0 \iff (p,q) \in G(T)$.*

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Proof. (a) If $(p,q) \in (H \setminus D) \times H$, then we can extend T to a function $\widetilde{T}$ on $D \cup \{p\}$ by defining $\widetilde{T}(p) = q$. Since T is maximally nonexpansive, $\widetilde{T}$ is not nonexpansive on $D \cup \{p\}$, from which it easily follows that there exists $d \in D$ such that $\|\widetilde{T}(p) - \widetilde{T}(d)\| > \|p - d\|$, that is to say, $\|q - T(d)\| > \|p - d\|$. Consequently, $\chi(p,q) > 0$. If, on the other hand, $(p,q) \in D \times H$, then $\chi(p,q) \ge \|q - T(p)\|^2 - \|p - p\|^2 = \|q - T(p)\|^2 \ge 0$. This completes the proof of (a).

(b) If $(p,q) \in G(T)$, then, since T is nonexpansive on D, for all $d \in D$,

$$\|q - T(d)\|^2 - \|p - d\|^2 = \|T(p) - T(d)\|^2 - \|p - d\|^2 \le 0,$$

while $\|q - T(p)\|^2 - \|p - p\|^2 = 0 - 0 = 0$, which shows that $\chi(p,q) = 0$. Conversely, if $\chi(p,q) = 0$, then the proof of (a) shows first that $(p,q) \notin (H \setminus D) \times H$, from which $(p,q) \in D \times H$, and then that $\|q - T(p)\|^2 = 0$, from which $(p,q) \in G(T)$, completing the proof of (b). ☐

Lemma 4. *Let $T\colon\ D \mapsto H$ be maximally nonexpansive and define $\varphi\colon\ H \times H \mapsto (-\infty, \infty]$ by*

$$\varphi(x,x^*) := \tfrac{1}{4}\chi(x + x^*, x - x^*) + \langle x, x^*\rangle.$$

Define the norm of $H \times H$ by $\|(x,x^*)\| := \sqrt{\|x\|^2 + \|x^*\|^2}$. Then:

(a) *For all $(x,x^*) \in H \times H$,*

$$\varphi(x,x^*) = \tfrac{1}{4}\sup\nolimits_{d \in D}\big[2\langle x, d - T(d)\rangle + 2\langle x^*, d + T(d)\rangle + \|T(d)\|^2 - \|d\|^2\big].$$

(b) *For all $d \in D$, $\varphi(\tfrac{1}{2}(d + T(d)), \tfrac{1}{2}(d - T(d))) = \tfrac{1}{4}\|d\|^2 - \tfrac{1}{4}\|T(d)\|^2$.*

(c) *φ is proper, convex and lower semicontinuous.*

(d) *For all $(z^*, z) \in H \times H$, $\varphi^*(z^*, z) \ge \varphi(z, z^*)$.*

(e) *$D \ni 0$.*

(f) *$D = H$.*

Proof. (a) This follows from the observation that, for all $(x,x^*) \in H \times H$ and $d \in D$,

$$\begin{aligned}&\|x - x^* - T(d)\|^2 - \|x + x^* - d\|^2\\ &\quad= \|x - x^*\|^2 - 2\langle x - x^*, T(d)\rangle + \|T(d)\|^2 - \|x + x^*\|^2 + 2\langle x + x^*, d\rangle - \|d\|^2\\ &\quad= -4\langle x, x^*\rangle + 2\langle x, d - T(d)\rangle + 2\langle x^*, d + T(d)\rangle + \|T(d)\|^2 - \|d\|^2.\end{aligned}$$

(b) This follows since, for all $d \in D$, Lemma 3(b) gives

$$\begin{aligned}\varphi(\tfrac{1}{2}(d + T(d)), \tfrac{1}{2}(d - T(d))) &= \tfrac{1}{4}\chi(d, T(d)) + \tfrac{1}{4}\langle d + T(d), d - T(d)\rangle\\ &= \tfrac{1}{4}\langle d + T(d), d - T(d)\rangle = \tfrac{1}{4}\|d\|^2 - \tfrac{1}{4}\|T(d)\|^2.\end{aligned}$$

(c) This is immediate from (a) and (b).

(d) We have from (b) that

$$\begin{aligned}&\varphi^*(z^*, z)\\ &\quad:= \sup_{(x,x^*) \in H \times H}\big[\langle x, z^*\rangle + \langle x^*, z\rangle - \varphi(x,x^*)\big]\\ &\quad\ge \sup_{d \in D}\big[\big\langle \tfrac{1}{2}(d + T(d)), z^*\big\rangle + \big\langle \tfrac{1}{2}(d - T(d)), z\big\rangle - \varphi(\tfrac{1}{2}(d + T(d)), \tfrac{1}{2}(d - T(d)))\big]\\ &\quad= \sup_{d \in D}\big[\big\langle \tfrac{1}{2}(d + T(d)), z^*\big\rangle + \big\langle \tfrac{1}{2}(d - T(d)), z\big\rangle - \tfrac{1}{4}\|d\|^2 + \tfrac{1}{4}\|T(d)\|^2\big]\\ &\quad= \tfrac{1}{4}\sup\nolimits_{d \in D}\big[2\langle d + T(d), z^*\rangle + 2\langle d - T(d), z\rangle - \|d\|^2 + \|T(d)\|^2\big],\end{aligned}$$

and the result now follows from (a).

(e) For all $(x, x^*) \in H \times H$,

$$\varphi(x, x^*) + \tfrac{1}{2}\|(x, x^*)\|^2 = \tfrac{1}{4}\chi(x + x^*, x - x^*) + \langle x, x^*\rangle + \tfrac{1}{2}\|(x, x^*)\|^2.$$

Thus, from Lemma 3(a),

$$\tag{4.1} \begin{cases} \varphi(x, x^*) + \tfrac{1}{2}\|(x, x^*)\|^2 \geq \langle x, x^*\rangle + \tfrac{1}{2}\|(x, x^*)\|^2 \\ \qquad\qquad\qquad\qquad\qquad \geq \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2 - \|x\|\|x^*\| \geq 0. \end{cases}$$

Part (c) and Theorem 1 now give us an element (z^*, z) of $(H \times H)^* = H \times H$ such that $\varphi^*(z^*, z) + \tfrac{1}{2}\|(-z^*, -z)\|^2 \leq 0$. Part (d) and (4.1) imply, in turn, that $\varphi(z, z^*) + \tfrac{1}{2}\|(z, z^*)\|^2 = 0$, that is to say,

$$\tfrac{1}{4}\chi(z + z^*, z - z^*) + \langle z, z^*\rangle + \tfrac{1}{2}\|(z, z^*)\|^2 = 0,$$

which can be rewritten as

$$\tfrac{1}{4}\chi(z + z^*, z - z^*) + \tfrac{1}{2}\|z + z^*\|^2 = 0.$$

It is clear from Lemma 3(a) that $\|z + z^*\|^2 = 0$, that is to say $z + z^* = 0$, and then $\chi(0, z - z^*) = 0$. Lemma 3(b) now gives $(0, z - z^*) \in G(T)$, from which (e) follows.

(f) This is immediate from (e) and a simple translation argument. □

Theorem 5. *Let $T\colon\ D \mapsto H$ be nonexpansive. Then there is a nonexpansive extension of T to H.*

Proof. Zorn's lemma easily gives that there is a maximal nonexpansive extension of T. The result now follows from Lemma 4(f). □

Remark 6. We note the fact that we have actually proved in Lemma 4(e) that

$$\min_{(x,x^*)\in H\times H} \left[\phi(x, x^*) + \tfrac{1}{2}\|(x, x^*)\|^2\right] = 0.$$

Remark 7. The proof of Theorem 1 uses the Eidelheit separation theorem in $E \times \mathbb{R}$. It would be interesting to find a more direct proof for the special case we need for Lemma 4(e). This can be stated without conjugate functions in the following way: *if $f\colon\ H \mapsto (-\infty, \infty]$ is convex and, for all $x \in H$, $f(x) + \tfrac{1}{2}\|x\|^2 \geq 0$, then there exists $y \in H$ such that, for all $x \in H$, $f(x) \geq \langle x, y\rangle + \tfrac{1}{2}\|y\|^2$.*

Remark 8. We note that, in contrast with other proofs, the proof of Theorem 5 given here does not use any compactness arguments. We also remark that it was observed in [7] that Theorem 5 is equivalent to Minty's theorem on maximal monotone multifunctions.

Remark 9. We show in this remark how Theorem 5 can be bootstrapped to give a similar result for maps from one Hilbert space into another. Let H_1 and H_2 be Hilbert spaces, $\emptyset \neq D_1 \subset H_1$ and $U\colon\ D_1 \mapsto H_2$ be nonexpansive. Define $T\colon\ D_1 \times H_2 \mapsto H_1 \times H_2$ by $T(x_1, x_2) := \big(0, U(x_1)\big)$. Then T is nonexpansive. Theorem 5 gives a nonexpansive $\widetilde{T}\colon\ H_1 \times H_2 \mapsto H_1 \times H_2$ such that $\widetilde{T}|_{D_1\times H_2} = T$. If pr_2 is the projection from $H_1 \times H_2$ onto H_2, then the map $\widetilde{U}\colon\ H_1 \mapsto H_2$ defined by $\widetilde{U}(x_1) := \mathrm{pr}_2\widetilde{T}(x_1, 0)$ is a nonexpansive extension of U.

Remark 10. If T is maximally nonexpansive, then the operator S defined by

$$G(S) := \big\{\big((p-q)/2, (p+q)/2\big)\colon\ (p,q) \in G(T)\big\}$$

is maximal monotone. The Fitzpatrick function of S, φ_S, is then given by the formula $\varphi_S(x,x^*) := \sup_{(s,s^*)\in G(S)}[\langle x, s^*\rangle + \langle s, x^*\rangle - \langle s, s^*\rangle]$. This turns out to be identical with $\varphi(x,x^*)$ as defined in Lemma 4.

References

[1] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis*, Amer. Math. Soc. Coll. Pubs. **48**, Providence, RI, 2000. MR1727673 (2001b:46001)

[2] H. Brezis and A. Haraux, *Image d'une somme d'opérateurs monotones at applications*, Israel J. Math. **23** (1976), 165–186. MR0399965 (53:3803)

[3] H. Federer, *Geometric measure theory*, Springer–Verlag, New York, 1969. MR0257325 (41:1976)

[4] S. P. Fitzpatrick, *Representing monotone operators by convex functions*, Workshop/ Miniconference on Functional Analysis and Optimization, Proc. Centre Math. Anal. Austral. Nat. Univ., vol. 20, Austral. Nat. Univ., Canberra, 1988, pp. 59–65. MR1009594 (90i:47054)

[5] M. D. Kirszbraun, *Über die zusammenziehende und Lipschitzsche Transformationen*, Fund. Math. **22** (1934), 77–108.

[6] E. J. Mickle, *On the extension of a transformation*, Bull. Amer. Math. Soc. **55** (1949), 160–164. MR0029974 (10:691b)

[7] S. Reich, *Extension problems for accretive sets in Banach spaces*, J. Functional Analysis **26** (1977), 378–395. MR0477893 (57:17393)

[8] S. Reich, *The range of sums of accretive and monotone operators*, J. Math. Anal. Appl. **68** (1979), 310–317. MR0531440 (80g:47060)

[9] R. T. Rockafellar, *Extension of Fenchel's duality theorem for convex functions*, Duke Math. J. **33** (1966), 81–89. MR0187062 (32:4517)

[10] I. J. Schoenberg, *On a theorem of Kirzbraun and Valentine*, Amer. Math. Monthly **60** (1953), 620–622. MR0058232 (15,341c)

[11] S. Simons and C. Zălinescu, *Fenchel duality, Fitzpatrick functions and maximal monotonicity*, J. Nonlinear Convex Anal., in press.

[12] F. A. Valentine, *A Lipschitz condition preserving extension for a vector function*, Amer. J. Math. **67** (1945), 83–93. MR0011702 (6:203e)

[13] J. H. Wells and L. R. Williams,, *Embeddings and extensions in analysis*, Springer–Verlag, New York–Heidelberg, 1975. MR0461107 (57:1092)

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