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A CAUCHY-SCHWARZ TYPE INEQUALITY FOR BILINEAR INTEGRALS ON POSITIVE MEASURES

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ABSTRACT. If $W\colon \mathbb{R}^n \to [0,\infty]$ is Borel measurable, define for σ-finite positive Borel measures μ,ν on $\mathbb{R}^n$ the bilinear integral expression

$$I(W;\mu,\nu) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} W(x-y)\,d\mu(x)\,d\nu(y)\ .$$

We give conditions on W such that there is a constant $C \ge 0$, independent of μ and ν, with

$$I(W;\mu,\nu) \le C\sqrt{I(W;\mu,\mu)I(W;\nu,\nu)}\ .$$

Our results apply to a much larger class of functions W than known before.

1. Introduction and results

Given a Borel function $W\colon \mathbb{R}^n \to [0,\infty]$, for σ-finite positive measures μ,ν on $\mathbb{R}^n$ define the integral

$$I(W;\mu,\nu) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} W(x-y)\,d\mu(x)\,d\nu(y)\ .$$

Denote for $C \ge 0$ by $\mathcal{W}(n,C)$ the class of Borel functions $W\colon \mathbb{R}^n \to [0,\infty]$ such that for all σ-finite positive measures μ,ν on $\mathbb{R}^n$

$$I(W;\mu,\nu) \le C\sqrt{I(W;\mu,\mu)I(W;\nu,\nu)} \tag{1.1}$$

holds. Moreover, denote

$$\mathcal{W}(n) := \bigcup_{C\ge 0} \mathcal{W}(n,C)\ .$$

If W is an even function and the symmetric bilinear form $I(W;\cdot,\cdot)$ is positive semidefinite, then $W \in \mathcal{W}(n,1)$ (Cauchy-Schwarz inequality). Hence we may regard (1.1) as a generalized form of the Cauchy-Schwarz inequality.

An even function W such that $I(W;\cdot,\cdot)$ is positive semidefinite is called *positive definite*. Roughly speaking, positive definiteness of a function corresponds to nonnegativity of its Fourier transform [6, 5]. The only result regarding (1.1) the author is aware of that goes beyond positive definite functions is given by Mattner [4, Sect. 5.1]: If $\|\cdot\|$ is any norm on $\mathbb{R}^n$, $h\colon [0,\infty) \to [0,\infty]$ is decreasing, and W is given by $W(x) := h(\|x\|)$, then $W \in \mathcal{W}(n)$. Theorem 1.5 below recovers this

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statement and extends it by allowing h to be nonmonotone. Theorem 1.2, the main result of the present paper, yields a criterion for membership in $\mathcal{W}(n)$ for functions W that cannot be written as $h \circ p$ with a seminorm p on $\mathbb{R}^n$.

The study of property (1.1) is motivated by the partial differential equation

$$-\Delta u + Vu = (W * u^2)u, \qquad u \in H^1(\mathbb{R}^n) \ . \tag{1.2}$$

Here $*$ denotes convolution, V in $L^\infty(\mathbb{R}^n)$ is periodic, and 0 lies in a gap of the spectrum of $(-\Delta + V)$; cf. [1]. One is interested in the existence of nontrivial solutions to (1.2). For the special case $n = 3$ and $W(x) = 1/\|x\|_2$ the problem was settled in [2] by using the fact that this particular function W is positive definite. In [1] it is shown that $W \in \mathcal{W}(n)$ (together with appropriate growth conditions) is sufficient to obtain a nontrivial solution of (1.2).

1.1. **Main results.** The statement of our theorems requires us to introduce some notation and definitions. For a topological space X denote by $\mathcal{P}(X)$ the set of Borel functions $f\colon X \to [0,\infty]$. For n in $\mathbb{N}$ denote by $\mathcal{C}(n)$ the class of subsets of $\mathbb{R}^n$ that are closed, convex, and symmetric (i.e. $-A = A$). The dimension $\dim A$ of a convex subset A of $\mathbb{R}^n$ is the dimension of the affine hull of A.

Definition 1.1. For $X, A \subseteq \mathbb{R}^n$, $X \neq \varnothing$, put

$$\kappa(X,A) := \inf\{\, m \in \mathbb{N} \mid X \text{ can be covered by } m \text{ translates of } A \,\}$$

and

$$\alpha(X) := \inf\{\, m \in \mathbb{N} \mid \exists A \in \mathcal{C}(n)\colon\ \dim A = n,\ A \subseteq X \text{ and } \kappa(X,A) = m \,\} \ .$$

For $X = \varnothing$ set $\kappa(\varnothing, A) := 0$ and $\alpha(\varnothing) := 0$.

Given a set X, a map $W\colon X \to \mathbb{R}$ and t in $\mathbb{R}$ denote

$$[W]_t := \{\, x \in X \mid W(x) \geq t \,\} \ .$$

Furthermore, define the class $\mathcal{A}(n)$ by

$$\mathcal{A}(n) := \left\{ W \in \mathcal{P}(\mathbb{R}^n) \;\middle|\; \limsup_{t\to 0} \alpha([W]_t) + \limsup_{t\to\infty} \alpha([W]_t) < \infty \right\}.$$

Our main result then reads:

Theorem 1.2. *For every n in $\mathbb{N}$ the inclusion* $\operatorname{conv}(\mathcal{A}(n)) \subseteq \mathcal{W}(n)$ *holds.*

Remark 1.3. It will be shown in the proof of Theorem 1.2 that $\mathcal{W}(n)$ is a convex cone. Obviously, $\mathcal{A}(n)$ is a cone. The Example 1.6 given below demonstrates that $\mathcal{A}(n)$ is not convex.

The present author does not know whether a function in $\mathcal{W}(n)$ that is sufficiently regular, say continuous, must necessarily belong to $\operatorname{conv}(\mathcal{A}(n))$.

A simpler criterion for membership in $\mathcal{W}(n)$ can be formulated in the case of the composition of a map with a seminorm. To state it we introduce further concepts and notation.

Definition 1.4. For a subset Y of $[0,\infty)$ put $\lambda(Y) := \sup\{\, t > 0 \mid [0,t] \subseteq Y \,\}$ and

$$\beta(Y) := \begin{cases} 0, & Y = \varnothing, \\ \infty, & \lambda(Y) = -\infty \text{ and } Y \neq \varnothing, \\ \sup(Y)/\lambda(Y), & \text{otherwise.} \end{cases}$$

Here we set $\infty/a := \infty$ if $a > 0$ and $\infty/\infty := 1$.

We introduce

$$\mathcal{B} := \left\{ h \in \mathcal{P}([0,\infty)) \;\middle|\; \limsup_{t\to 0} \beta([h]_t) + \limsup_{t\to\infty} \beta([h]_t) < \infty \right\}.$$

Our second result then reads:

Theorem 1.5. *Suppose that $h \in \mathcal{P}([0,\infty))$ and that p is a seminorm on $\mathbb{R}^n$. If $h \in \mathcal{B}$, then $h \circ p \in \mathcal{A}(n)$. If $h \circ p \in \mathcal{A}(n)$ and $\operatorname{codim}(\ker p) \geq 2$, then $h \in \mathcal{B}$.*

We provide some examples to illustrate the concepts introduced so far:

Example 1.6. Denote by h the characteristic function of $[0,1]$, taken as a map from $[0,\infty)$ into $[0,\infty]$. Then $h \in \mathcal{B}$. For $i = 1,2$ define W_i as a map in $\mathcal{P}(\mathbb{R}^2)$ by $W_i(x_1,x_2) := h(|x_i|)$. Theorem 1.5 implies that $W_i \in \mathcal{A}(2)$ for $i = 1,2$, but clearly $W := W_1 + W_2 \notin \mathcal{A}(2)$. Since $\mathcal{A}(2)$ is a cone this implies that $\mathcal{A}(2)$ is not convex. Nevertheless, $W \in \mathcal{W}(2)$ by Theorem 1.2 and since $\mathcal{W}(2)$ is a convex cone.

Example 1.7. We construct a function W in $\mathcal{A}(n)$ that is not even, and hence is neither positive definite nor of the form $h \circ p$ with h in $\mathcal{P}([0,\infty))$ and p a seminorm on $\mathbb{R}^n$. Pick x_0 in $\mathbb{R}^n \setminus \{0\}$ and set

$$\begin{aligned} W_0(x) &:= \frac{1}{\|x\|_2}, \\ W(x) &:= W_0(x) + W_0(x - x_0)\,. \end{aligned}$$

Denoting by $D(r,x)$ the closed ball of radius $r > 0$ with center x, it follows easily that

$$D(1/t,0) \subseteq [W]_t \subseteq D(2/t,0) \cup D(2/t,x_0)$$

for all $t > 0$. This implies that $W \in \mathcal{A}(n)$.

Example 1.8. We show that the assumption on $\operatorname{codim}(\ker p)$ used in Theorem 1.5 is not purely technical. If p is a seminorm on $\mathbb{R}^n$ with $\operatorname{codim}(\ker p) = 0$, then trivially $h \circ p \in \mathcal{A}(n)$ for arbitrary h in $\mathcal{P}([0,\infty))$. Given the seminorm $p(x) := |x|$ in $\mathbb{R}$ with $\operatorname{codim}(\ker p) = 1$, we construct h in $\mathcal{P}([0,\infty))$ such that $W := h \circ p \in \mathcal{A}(1)$ but $h \notin \mathcal{B}$. Put

$$h(s) := \begin{cases} \infty, & s = 0, \\ \exp(-(k-1)^2), & s = \exp(k^2) \text{ for some } k \text{ in } \mathbb{N}, \\ 1/s, & \text{otherwise.} \end{cases}$$

For $t > 1$ we obtain $[h]_t = [0,1/t]$, and for $0 < t \leq 1$ we obtain

$$[h]_t = [0,1/t] \cup \left\{ \exp\left(\left(1 + \left[\sqrt{-\log t} \right] \right)^2 \right) \right\}. \tag{1.3}$$

Recall that $[a]$ denotes the largest integer less than or equal to a if $a \in \mathbb{R}$. From (1.3) it is clear that $\alpha([W]_t) \leq 3$ for all $t \geq 0$, so $W \in \mathcal{A}(1)$. On the other hand, for $t_k := \exp(-k^2)$ we find

$$\beta([h]_{t_k}) = \exp((1+k)^2)\exp(-k^2) = \exp(1+2k)$$

and therefore $\limsup_{t\to 0} \beta([h]_t) = \infty$. Hence $h \notin \mathcal{B}$.

1.2. **General notation.** In $\mathbb{R}^n$ denote by $\|\cdot\|_p$ for p in $[1,\infty]$ the standard $l^p(n)$-norm. In the case of $p=2$ we write $x \cdot y$ for the standard Euclidean scalar product of elements x, y in $\mathbb{R}^n$. If V is a subspace of $\mathbb{R}^n$, denote by $V^\perp$ the orthogonal subspace with respect to the standard scalar product.

The power set of a set X will be written 2^X. The cardinality of X is denoted by $|X|$. Some operators used are: $\operatorname{conv} A$ for the convex hull of A, $\operatorname{cl} A$, $\operatorname{int} A$, and ∂A for closure, interior, and boundary of a subset A of a topological space.

A parallelotope is a rectangular parallelepiped.

2. Some convex geometry

The next lemma allows us to deal with unbounded sets in $\mathcal{C}(n)$ in a convenient manner.

Lemma 2.1. *If $A \in \mathcal{C}(n)$, then there is a unique subspace V of $\mathbb{R}^n$ such that $B := A \cap V^\perp \in \mathcal{C}(n)$ is compact and $A = B + V$.*

Proof. First we remark: If a set A in $\mathcal{C}(n)$ includes a ray (a set $\{x + ty \mid t \geq 0\}$ for some x, y in $\mathbb{R}^n$), then it includes the 1-dimensional subspace parallel to that ray. If A includes a translate of a subspace V of $\mathbb{R}^n$, then A includes V.

Now fix A in $\mathcal{C}(n)$. From [3, Lemma 2.5.4] we obtain a unique subspace V of $\mathbb{R}^n$ of maximal dimension such that a translate of V and thus V is included in A. Moreover, by that lemma it also holds that $B := A \cap V^\perp \in \mathcal{C}(n)$ does not include a line (the translate of a 1-dimensional subspace) and $A = B + V$. If B was not bounded, then it included a ray by [3, Lemma 2.5.1]. Since B is symmetric it therefore included a line also. Contradiction. Since A is closed B must therefore be compact.

If, on the other hand, for some subspace V of $\mathbb{R}^n$, $B = A \cap V^\perp$ is compact and $A = B + V$, then V is included in A. If A includes a translate of another subspace W, and thus includes W, then $W \subseteq V$. Hence V has maximal dimension among the subspaces included in A, and it is unique, again by Lemma 2.5.4 *loc. cit.* $\square$

Definition 2.2. We call the pair (B, V) given for A in $\mathcal{C}(n)$ by Lemma 2.1 the *splitting of A*.

Definition 2.3. Denote for $X \subseteq \mathbb{R}^n$ by

$$\operatorname{ccs} X := \operatorname{cl}(\operatorname{conv} \tfrac{1}{2}(X - X)) \in \mathcal{C}(n)$$

the closed convex hull of the symmetrization of X.

Remark 2.4. For $A, B \subseteq \mathbb{R}^n$ we have $\operatorname{conv}(A + B) = \operatorname{conv} A + \operatorname{conv} B$. Thus

$$\operatorname{ccs} X = \operatorname{cl} \tfrac{1}{2}(\operatorname{conv} X - \operatorname{conv} X)\ .$$

From this it also follows that $\operatorname{ccs}(X + Y) = \operatorname{ccs} X + \operatorname{ccs} Y$ if one of X and Y is relative compact. Moreover, $\operatorname{ccs} A = A$ if $A \in \mathcal{C}(n)$.

Definition 2.5. If $X \subseteq \mathbb{R}^n$ and (A, V) is the splitting of $\operatorname{ccs} X$, put $\gamma(X) := \dim V$.

Lemma 2.6. *The map $\gamma\colon 2^{\mathbb{R}^n} \to \{0, 1, 2, \ldots, n\}$ is monotone increasing with respect to the partial order induced on $2^{\mathbb{R}^n}$ by inclusion. If $X \subseteq Y \subseteq \mathbb{R}^n$ and $\gamma(X) = \gamma(Y)$, then from $A \in \mathcal{C}(n)$ with $\dim A = n$ and $\kappa(X, A) < \infty$ it follows that $\kappa(Y, A) < \infty$.*

Proof. Monotonicity of γ is obvious. Fix $X \subseteq Y$ with $\gamma(X) = \gamma(Y)$, and suppose we are given A in $\mathcal{C}(n)$ with $\dim A = n$ and $\kappa(X, A) < \infty$. Let (B, V) be the splitting of A and let $\mathcal{I} \subseteq \mathbb{R}^n$ be finite with $X \subseteq \mathcal{I} + A = \mathcal{I} + B + V$. Since $\mathcal{I} + B$ is compact, in view of Remark 2.4 we obtain

$$\operatorname{ccs} X \subseteq \operatorname{ccs}(\mathcal{I} + B + V) = \operatorname{ccs}(\mathcal{I} + B) + V \ . \tag{2.1}$$

Since $\operatorname{ccs} X \subseteq \operatorname{ccs} Y$ and $\gamma(X) = \gamma(Y)$ there is a subspace W of $\mathbb{R}^n$ with $\dim W = \gamma(X)$ and there are splittings (B_1, W) and (B_2, W) of $\operatorname{ccs} X$ and $\operatorname{ccs} Y$, respectively, with $B_1 \subseteq B_2$. Put $A_1 := A \cap W^\perp$. Now (2.1) implies $W \subseteq V$, and hence $A_1 + W = A$. Therefore $\dim A = n$ yields $\dim A_1 = \dim W^\perp = n - \gamma(X)$, and relint A_1 (the interior of A_1 relative to the smallest subspace including A_1) is open in $W^\perp$. Since $B_2 \subseteq W^\perp$ is compact there is a finite set $\mathcal{J} \subseteq W^\perp$ with $B_2 \subseteq \mathcal{J} + A_1$. It follows that

$$Y \subseteq \operatorname{ccs} Y = B_2 + W \subseteq \mathcal{J} + A_1 + W = \mathcal{J} + A$$

and thus $\kappa(Y, A) < \infty$. □

Lemma 2.7. *For all n in $\mathbb{N}$ there is a constant $C_1(n) \geq 0$ such that for all A in $\mathcal{C}(n)$ with $\dim A = n$ the following hold:*

a) $\kappa(A, \frac{1}{2}A) \leq C_1(n)$,
b) *there is a discrete subgroup G of the additive group of $\mathbb{R}^n$ such that $\mathbb{R}^n = G + A$ and $\sup_{x \in \mathbb{R}^n} |(x + 3A) \cap G| \leq C_1(n)$.*

Proof. From [7, Lemma 2.4] we obtain for all m in $\mathbb{N}$ a constant $C_2(m)$, monotone increasing in m, such that for every m-dimensional compact B in $\mathcal{C}(m)$ there is a parallelotope $P \subseteq \mathbb{R}^m$, centered at the origin, with

$$P \subseteq B \subseteq C_2(m)P \ . \tag{2.2}$$

Now set

$$C_1(n) := [3C_2(n) + 1]^n,$$

where $[a]$ denotes the largest integer below or equal to a if $a \in \mathbb{R}$.

Fix A in $\mathcal{C}(n)$ and let (B, V) be the splitting of A. Since $\dim A = n$ we have $\dim B + \dim V = n$. We may assume $\dim B = m$ and $V = \{0\} \times \mathbb{R}^{n-m}$ as a subspace of $\mathbb{R}^n$. We identify $\mathbb{R}^m$ with $\mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^n$ so that $B \subseteq \mathbb{R}^m$, and we choose a parallelotope $P \subseteq \mathbb{R}^m$ for B as in (2.2). Then from $2C_2(m) \leq 3C_2(n)$ and the definition of $C_1(n)$ we obtain

$$\kappa(A, \tfrac{1}{2}A) = \kappa(B, \tfrac{1}{2}B) \leq \kappa(C_2(m)P, \tfrac{1}{2}P) \leq \kappa(3C_2(n)P, P) \leq C_1(n) \ .$$

For the second assertion we use for P from above the representation

$$P = [-r_1, r_1] \times [-r_2, r_2] \times \cdots \times [-r_m, r_m]$$

with some $r_1, r_2, \ldots, r_m > 0$ and put $G_0 := 2r_1\mathbb{Z} \times 2r_2\mathbb{Z} \times \cdots \times 2r_m\mathbb{Z} \subseteq \mathbb{R}^m$. Then G_0 is an additive subgroup of $\mathbb{R}^m$ with $G_0 + B \supseteq G_0 + P = \mathbb{R}^m$. Now set $G := G_0 \times \{0\} \subseteq \mathbb{R}^n$. Then $G + A = G + B + V = \mathbb{R}^n$. On the other hand we have for every x in $\mathbb{R}^n$

$$(x + 3A) \cap G = (x + 3B) \cap G \subseteq (x + 3C_2(m)P) \cap G \subseteq (x + 3C_2(n)P) \cap G$$

and hence

$$|(x + 3A) \cap G| \leq |(x + 3C_2(n)P) \cap G| \leq C_1(n) \ .$$

This completes the proof. □

Lemma 2.8. *Suppose that p is a seminorm on $\mathbb{R}^n$ and that $Y \subseteq [0,\infty)$. Put $X := p^{-1}(Y)$. Then $\alpha(X) \le C_3(n)\beta(Y)^n$ for some constant $C_3(n)$. If $\operatorname{codim}(\ker p) \ge 2$, then $\alpha(X) \ge \beta(Y)/2$.*

Proof. For $r > 0$ put $A(r) := \{\, x \in \mathbb{R}^n \mid p(x) \le r \,\} \in \mathcal{C}(n)$. Let $(B(1), V)$ be the splitting of $A(1)$ and put $B(r) := rB(1)$ for $r > 0$. Then $(B(r), V)$ is the splitting of $A(r)$. Moreover, $V = \ker p$. Set $m := \operatorname{codim} V$, so $\dim B(1) = m$.

Define $f, g\colon [0,\infty) \to \mathbb{N}$ by setting $f(0) := g(0) := 1$ and, for $t > 0$, $f(t) := \kappa(\partial A(t), A(1)) = \kappa(\partial B(t), B(1))$ and $g(t) := \kappa(A(t), A(1)) = \kappa(B(t), B(1))$. Then f and g are monotone increasing, $f \le g$, and

$$\kappa(\partial A(r), A(s)) = f(r/s),$$
$$\kappa(A(r), A(s)) = g(r/s)$$

for $r, s > 0$. As in the beginning of the proof of Lemma 2.7 we obtain

$$g(t) = \kappa(B(t), B(1)) \le \kappa(tC_2(m)P, P) = [tC_2(m) + 1]^m \ . \tag{2.3}$$

Here $P \subseteq B(1)$ is a parallelotope chosen as for (2.2). If $m \ge 2$, then

$$f(t) = \kappa(\partial B(t), B(1)) \ge t \ . \tag{2.4}$$

This can be seen as follows: Consider $B(1)$ as a subset of $\mathbb{R}^m$. Fix x_0 in $\partial B(1)$ such that $2\|x_0\|_2 = \operatorname{diam} B(1)$. Let Q be the orthogonal projection in $\mathbb{R}^m$ onto $\operatorname{span}\{x_0\}$ and $L := \ker Q$. Then $\dim L \ge 1$. It follows that for every x in $[-tx_0, tx_0]$ (the segment joining $-tx_0$ and tx_0) the set $(x + L) \cap \partial B(t)$ is not empty. Moreover, from $B(1) \in \mathcal{C}(n)$ it follows that $x_0 + L$ is a supporting hyperplane for $B(1)$. If $x_1, x_2, \dots, x_k \in \mathbb{R}^m$ are such that

$$\partial B(t) \subseteq \bigcup_{l=1}^{k} (x_l + B(1)) \ ,$$

then from the above it is clear that

$$[-tx_0, tx_0] \subseteq \bigcup_{l=1}^{k} (Qx_l + B(1))$$

and therefore $k \ge [t + 1] \ge t$. This yields (2.4).

Let us consider the case $0 < \lambda(Y) \le \sup Y < \infty$. There is

$$\varepsilon \in [0, \lambda(Y)/2]$$

such that $[0, \lambda(Y) - \varepsilon] \subseteq Y$. It follows that

$$A(\lambda(Y) - \varepsilon) \subseteq X \subseteq A(\sup Y) \ .$$

Thus, using (2.3), we obtain

$$\alpha(X) \le \kappa(A(\sup Y), A(\lambda(Y) - \varepsilon)) = g\left(\frac{\sup Y}{\lambda(Y) - \varepsilon}\right) \le g(2\beta(Y)) \le C_3(n)\beta(Y)^n$$

for some constant $C_3(n) \ge 1$.

There is ε in $[0, \sup Y/2]$ such that $\sup Y - \varepsilon \in Y$ and therefore

$$\partial A(\sup Y - \varepsilon) \subseteq X \ . \tag{2.5}$$

Every A in $\mathcal{C}(n)$ with $A \subseteq X$ is path connected and satisfies $0 \in A$. Since p is continuous, $p(A)$ is included in the path component of Y containing 0. Therefore $p(A) \subseteq [0, \lambda(Y)]$ and $A \subseteq A(\lambda(Y))$. This shows that

$$\kappa(X, A) \geq \kappa(X, A(\lambda(Y)))$$

for all A in $\mathcal{C}(n)$. Hence we find for $m \geq 2$, applying (2.4) and (2.5):

$$\alpha(X) \geq \kappa(\partial A(\sup Y - \varepsilon), A(\lambda(Y))) = f\left(\frac{\sup Y - \varepsilon}{\lambda(Y)}\right) \geq f(\beta(Y)/2) \geq \beta(Y)/2 \,.$$

The case $\lambda(Y) > 0$, $\sup(Y) = \infty$ is handled similarly, and in all other cases the assertion is trivial. □

3. Proof of the theorems

Let us first prove Theorem 1.5. Suppose that we are given $h \in \mathcal{P}([0,\infty))$ and a seminorm p on $\mathbb{R}^n$. Set $W := h \circ p$. Then $[W]_t = p^{-1}([h]_t)$ for every $t > 0$. Now Lemma 2.8 yields $\alpha([W]_t) \leq C\beta([h]_t)^n$ with some positive constant C. Moreover, if $\operatorname{codim}(\ker p) \geq 2$ Lemma 2.8 implies that $\beta([h]_t) \leq 2\alpha([W]_t)$. From these facts the theorem follows.

The proof of Theorem 1.2, taken up next, is divided into the following steps:

(i) $\mathcal{W}(n, C)$ is closed under increasing pointwise limits for every $C \geq 0$.
(ii) $\mathcal{W}(n, C)$ is a convex cone for every $C \geq 0$.

Now suppose that $W \in \mathcal{P}(\mathbb{R}^n)$.

(iii) If A in $\mathcal{C}(n)$ has dimension n, if $\kappa(\operatorname{supp} W, A) < \infty$, if there is $a > 0$ such that $W \geq a$ on $2A$, and if W is bounded with $b := \sup W(\mathbb{R}^n)$, then $W \in \mathcal{W}(n, C)$ for $C := C_1(n)^3 \kappa(\operatorname{supp} W, A) b/a$, where $C_1(n)$ is the constant given in Lemma 2.7.
(iv) If $\sup_{t \geq 0} \alpha([W]_t) < \infty$, then $W \in \mathcal{W}(n, C)$ for some $C \geq 0$.
(v) If $\limsup_{t\to 0} \alpha([W]_t) + \limsup_{t\to\infty} \alpha([W]_t) < \infty$, then $W \in \mathcal{W}(n, C)$ for some $C \geq 0$.

Theorem 1.2 is then a consequence of (ii) and (v).

Statements (i) and (ii) were proven in [4, Sect. 5.1]. For completeness we repeat the argument here. Suppose that $C \geq 0$. Fix two σ-finite positive Borel measures μ, ν on $\mathbb{R}^n$. If W is the pointwise limit of an increasing sequence of functions in $\mathcal{W}(n, C)$, then (1.1) follows from Lebesgue's Monotone Convergence Theorem. This proves (i) since μ, ν were chosen arbitrarily.

Consider the implication

$$\left(u \leq C\sqrt{vw} \quad \text{and} \quad x \leq C\sqrt{yz}\right) \qquad \Rightarrow \qquad (u+x)^2 \leq C^2(v+y)(w+z) \tag{3.1}$$

for u, v, w, x, y, z in $[0, \infty)$, which is a consequence of $2\sqrt{vwyz} \leq vz + yw$. If $W_1, W_2 \in \mathcal{W}(n, C)$, then (3.1) implies that $W_1 + W_2 \in \mathcal{W}(n, C)$. Since $\mathcal{W}(n, C)$ is a cone, $\mathcal{W}(n, C)$ is convex.

To show (iii) choose a discrete additive subgroup G of $\mathbb{R}^n$ for A as in Lemma 2.7b). Let $\mathcal{I}$ be a finite subset of $\mathbb{R}^n$ with $\operatorname{supp} W \subseteq \mathcal{I} + A$ and $|\mathcal{I}| = \kappa(\operatorname{supp} W, A)$. Put $\mathcal{J} := (\mathcal{I} + 3A) \cap G$. From the choice of G it follows that

$$|\mathcal{J}| \leq C_1(n)|\mathcal{I}| \,. \tag{3.2}$$

Define $\overline{W}\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $\overline{W}(x,y) := W(x-y)$. Then $\overline{W}$ is a Borel function. We claim that

$$\operatorname{supp} \overline{W} \subseteq \bigcup_{\substack{u,v\in G\\ u-v\in\mathcal{J}}} (u+A)\times(v+A) \, .$$

To see this, suppose that $(x,y) \in \operatorname{supp}\overline{W}$, or equivalently $x-y \in \operatorname{supp} W$. There is w in $\mathcal{I}$ such that $x-y \in w+A$, and there are u,v in G such that $x \in u+A$ and $y \in v+A$. It follows that $u-v \in x-y+2A \subseteq w+3A \subseteq \mathcal{I}+3A$. Also $u-v\in G$ because G is a subgroup. This proves the claim.

Now the Cauchy-Schwarz inequality for sums yields

$$\begin{aligned}(3.3)\quad I(W;\mu,\nu) &= \int_{\mathbb{R}^n\times\mathbb{R}^n} \overline{W}\, d(\mu\times\nu) \le b\int_{\operatorname{supp}\overline{W}} d(\mu\times\nu)\\ &\le b\sum_{\substack{u,v\in G\\ u-v\in\mathcal{J}}} \mu(u+A)\nu(v+A) \le b\Bigg(\sum_{\substack{u,v\in G\\ u-v\in\mathcal{J}}}\mu(u+A)^2 \sum_{\substack{u,v\in G\\ u-v\in\mathcal{J}}}\nu(v+A)^2\Bigg)^{\frac12} .\end{aligned}$$

We need to estimate the sums in the last term. For every x in $\mathbb{R}^n$, from $A\in\mathcal{C}(n)$ it follows that the statement ($u\in G$ and $x\in u+A$) is equivalent to the statement $u\in(x+A)\cap G$. By the choice of G this leads to

$$|\{\, u\in G \mid x\in u+A \,\}| = |(x+A)\cap G| \le |(x+3A)\cap G| \le C_1(n)$$

and thus for all x,y in $\mathbb{R}^n$

$$(3.4)\qquad |\{\, u\in G \mid (x,y)\in(u+A)\times(u+A) \,\}| \le C_1(n)^2 \, .$$

We also have

$$(3.5)\qquad \bigcup_{u\in G}(u+A)\times(u+A) \subseteq \{\,(x,y)\in\mathbb{R}^n\times\mathbb{R}^n \mid x-y\in 2A\,\} =: D$$

and $\overline{W}\ge a$ on D. Using (3.2), (3.4) and (3.5) we calculate

$$\begin{aligned}\sum_{\substack{u,v\in G\\ u-v\in\mathcal{J}}}\mu(u+A)^2 &= |\mathcal{J}|\sum_{u\in G}\mu(u+A)^2 = |\mathcal{J}|\sum_{u\in G}\int_{(u+A)\times(u+A)} d(\mu\times\mu)\\ &\le C_1(n)^2|\mathcal{J}|\int_D d(\mu\times\mu) \le \frac{C_1(n)^3|\mathcal{I}|}{a}\int_{\mathbb{R}^n\times\mathbb{R}^n}\overline{W}\, d(\mu\times\mu)\\ &= \frac{C_1(n)^3|\mathcal{I}|}{a} I(W;\mu,\mu) \, ,\end{aligned}$$

a similar estimate holding for the sum over $\nu(v+A)^2$. This proves (iii) in view of (3.3).

To show (iv) suppose that $M := \sup_{t\ge0}\alpha([W]_t) < \infty$. For m in $\mathbb{N}$ and $1\le i\le m2^m$ define $W_{m,i}$ and W_m in $\mathcal{P}(\mathbb{R}^n)$ by setting

$$\begin{aligned} W_{m,i} &:= \frac{1}{2^m}\chi_{[W]_{i/2^m}} ,\\ W_m &:= \sum_{i=1}^{m2^m} W_{m,i} \, .\end{aligned}$$

Here χ_A denotes for $A \subseteq \mathbb{R}^n$ the characteristic function of A. The sequence (W_m) is increasing and converges pointwise to W. Fix m and i. There is A in $\mathcal{C}(n)$ such that $\dim A = n$, $A \subseteq [W]_{i/2^m}$, and $\kappa([W]_{i/2^m}, A) \leq M$. Since A is closed $\operatorname{supp} W_{m,i} = \operatorname{cl}[W]_{i/2^m}$ can be covered by the same number of translates of A as $[W]_{i/2^m}$, i.e. $\kappa(\operatorname{supp} W_{m,i}, A) = \kappa([W]_{i/2^m}, A)$. Using Lemma 2.7 we thus obtain

$$\kappa(\operatorname{supp} W_{m,i}\,, \tfrac{1}{2}A) \leq C_1(n)\kappa(\operatorname{supp} W_{m,i}\,, A) \leq C_1(n)M\ .$$

Moreover, $W_{m,i} = 1/2^m$ on A and $W_{m,i} \leq 1/2^m$ on $\mathbb{R}^n$. By (iii) $W_{m,i} \in \mathcal{W}(n, C)$ for $C = C_1(n)^4 M$, independently of m and i. By (ii) $W_m \in \mathcal{W}(n, C)$ for every m, and thus (i) yields the desired result.

The remaining case (v) is handled as follows: We can assume that $W \neq 0$, otherwise there is nothing to do. By our assumptions there are $M > 0$ and $0 < t_1 < t_0$ such that $\alpha([W]_t) \leq M$ for t in $(0, t_1] \cup [t_0, \infty)$ and $[W]_t \neq \varnothing$ for t in $(0, t_1]$. Consider $\gamma([W]_t)$ as a function of t sending $(0, \infty)$ into $\{\,0, 1, 2, \ldots, n\,\}$ (γ is given in Definition 2.5). We can choose $0 < t_3 < t_2 \leq t_1$ with $\gamma([W]_{t_2}) = \gamma([W]_{t_3})$. For x in $\mathbb{R}^n$ put $W_1(x) := \min\{t_3, W(x)\}$ and $W_2(x) := \min\{t_0 - t_3, W(x) - W_1(x)\}$. Also put $W_3 := W - W_1 - W_2$. Then $W_1 \leq t_3$, $W_2 \leq t_0 - t_3$, and $W_i \geq 0$ for $i = 1, 2, 3$. Moreover, we have

$$[W_1]_t = \begin{cases} [W]_t, & 0 \leq t \leq t_3, \\ \varnothing, & t_3 < t, \end{cases}$$

$$[W_2]_t = \begin{cases} [W]_{t+t_3}, & 0 \leq t \leq t_0 - t_3, \\ \varnothing, & t_0 - t_3 < t, \end{cases}$$

$$[W_3]_t = [W]_{t+t_0}\ .$$

From (iv) it follows that $W_1, W_3 \in \mathcal{W}(n, C)$ for some $C \geq 0$. Since $[W]_{t_2} \neq \varnothing$ and $\alpha([W]_{t_2}) < \infty$ there is A in $\mathcal{C}(n)$ with $\dim A = n$, $A \subseteq [W]_{t_2}$ and $\kappa([W]_{t_2}, A) < \infty$. By Lemma 2.6 $\kappa([W]_{t_3}, A) < \infty$ also, and by Lemma 2.7a) $\kappa([W]_{t_3}, \frac{1}{2}A) < \infty$. Hence the closedness of A and $\operatorname{supp} W_2 \subseteq \operatorname{cl}[W]_{t_3}$ imply that $\kappa(\operatorname{supp} W_2, \frac{1}{2}A) < \infty$. We also have $W_2 \geq t_2 - t_3$ on A and $W_2 \leq t_0 - t_3$ on $\mathbb{R}^n$. Now (iii) implies that $W_2 \in \mathcal{W}(n, C)$ for some C, and by (ii) the same holds for $W = W_1 + W_2 + W_3$. This finishes the proof of (v).

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