ASYMPTOTIC EXPANSION OF SOLUTIONS TO NONLINEAR ELLIPTIC EIGENVALUE PROBLEMS

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Abstract. We consider the nonlinear eigenvalue problem

\[-\Delta u + g(u) = \lambda \sin u \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,\]

where \(\Omega \subset \mathbb{R}^N (N \geq 2)\) is an appropriately smooth bounded domain and \(\lambda > 0\) is a parameter. It is known that if \(\lambda \gg 1\), then the corresponding solution \(u_\lambda\) is almost flat and almost equal to \(\pi\) inside \(\Omega\). We establish an asymptotic expansion of \(u_\lambda(x) (x \in \Omega)\) when \(\lambda \gg 1\), which is explicitly represented by \(g\).

1. Introduction

We consider the nonlinear eigenvalue problem

\[
\begin{align*}
-\Delta u + g(u) &= \lambda \sin u \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \(\Omega \subset \mathbb{R}^N (N \geq 2)\) is an appropriately smooth bounded domain and \(\lambda > 0\) is a parameter. We assume the following conditions (A.1)–(A.3):

(A.1) \(g \in C^{m,\gamma}(\mathbb{R}) (m \geq 1, 0 < \gamma < 1)\) and \(g(u) > 0\) for \(u > 0\).

(A.2) \(g(0) = g'(0) = 0\).

(A.3) \(g(u)/u\) is strictly increasing for \(0 < u < \pi\).

The typical example of \(g(u)\) is \(g(u) = |u|^{m-1}u\ (m > 1)\).

The equation (1.1)–(1.3) is regarded as the equation of a simple pendulum with a nonlinear self-interaction term \(g(u)\), and the following (P.1) and (P.2) are well known and easy to show (cf. [1], [2], [3], [4], [5]).

(P.1) For a given \(\lambda \in \mathbb{R}\), (1.1)–(1.3) has a unique solution \(u_\lambda \in C^3(\bar{\Omega})\) if and only if \(\lambda > \lambda_1\), where \(\lambda_1 > 0\) is the first eigenvalue of \(-\Delta\) with Dirichlet zero boundary condition.

(P.2) \(\|u_\lambda\|_\infty < \pi\) and \(u_\lambda \to \pi\) locally uniformly in \(\Omega\) as \(\lambda \to \infty\).

Therefore, we see from (P.2) that \(u_\lambda\) is almost flat inside \(\Omega\), and one common interest to study (1.1)–(1.3) is to investigate precisely the asymptotic behavior of \(u_\lambda\) inside \(\Omega\). In other words, we are interested in “how flat \(u_\lambda\) is inside \(\Omega\)”.

In the case when \(g \equiv 0\), there are many works concerning the asymptotic behavior of the solutions of (1.1)–(1.3) inside \(\Omega\) as \(\lambda \to \infty\), since the properties (P.1) and

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(P.2) are valid when \( g \equiv 0 \) (cf. [2], [3] and the references therein). To take a simple example, let us consider the case \( N = 1, \Omega = (-R, R) \) and \( g \equiv 0 \). In this case, we denote by \( u_{0, \lambda} \) the unique solution associated with given \( \lambda > \lambda_1 \). Then it is known (cf. [6]) that as \( \lambda \to \infty \),

\[
\|u_{0, \lambda}\|_\infty = \pi - 8(1 + o(1))e^{-\sqrt{\lambda}(1 + o(1))R}.
\]

We remark that the second term in the right-hand side of (1.4) decays exponentially as \( \lambda \to \infty \).

However, as far as the author knows, there are no works concerning the precise asymptotic analysis of the interior behavior of \( u_\lambda \) as \( \lambda \to \infty \) when \( g \neq 0 \). So the natural and fundamental questions we have to ask here are as follows:

- (Q.1) Does the second term of \( \|u_\lambda\|_\infty \) decay exponentially, too?
- (Q.2) If the answer to question (Q.1) is in the negative, then what is the second term of \( \|u_\lambda\|_\infty \) as \( \lambda \to \infty \)?
- (Q.3) How flat is \( u_\lambda \) inside \( \Omega \) when \( \lambda \gg 1 \)?

To answer these questions, we establish an asymptotic expansion of \( u_\lambda(x) \) as \( \lambda \to \infty \), which is explicitly represented by \( g \) and show that the second term and the remainder terms decay algebraically as \( \lambda \to \infty \).

Now we state our results. Let \( G(u) := \int_0^u g(s)ds \).

**Theorem 1.** Let \( x \in \Omega \) be fixed. Then the following asymptotic formula holds as \( \lambda \to \infty \):

\[
u_\lambda(x) = \pi - \sum_{k=1}^{m+1} \frac{b_k}{\lambda^k} + o\left(\frac{1}{\lambda^{m+1}}\right),
\]

where \( b_k \ (k = 1, 2, \ldots, m + 1) \) are constants determined by \( \{g^{(j)}(\pi)\}_{j=0}^{k-1} \).

For example,

\[
\begin{align*}
b_1 &= g(\pi), \\
b_2 &= -g(\pi)g'(\pi), \\
b_3 &= \frac{1}{6}g(\pi)^3 + g(\pi)^3g'(\pi) + \frac{1}{2}g(\pi)^2g''(\pi).
\end{align*}
\]

We explain the idea of the proof of Theorem 1 briefly. We first consider (1.1)–(1.3) for the case \( \Omega = B_R := \{x \in \mathbb{R}^N : |x| < R\} \). In this case, (1.1)–(1.3) are equivalent to the ordinary differential equation

\[
\begin{align*}
u''(r) + \frac{N - 1}{r}u'(r) - g(u(r)) + \lambda \sin u(r) &= 0, \quad 0 < r < R, \\
u(r) &> 0, \quad 0 \leq r < R, \\
u'(0) &= u(R) = 0.
\end{align*}
\]

Then we prove (1.5) for \( u_\lambda(0) (= \|u_\lambda\|_\infty) \). Then by using the fact that \( \|u_\lambda\|_\infty \) does not depend on the radius \( R \) of the ball \( B_R \), we prove Theorem 1. Therefore, to prove (1.5) for \( \Omega = B_R \) and \( x = 0 \) is crucial. The difficulty we encounter for the case \( \Omega = B_R \ (N \geq 2) \) is as follows. If we directly follow the argument in [8], in which (1.5) was obtained for the case \( N = 1, \Omega = (-R, R) \) and \( x = 0 \), then we find that the second term \( (N-1)u''_\lambda(r)/r \) in (1.7), which does not appear in the case \( N = 1 \), is quite difficult to treat. We concentrate our attention on treating this characteristic term appropriately and developing new devices to deal with the
radial solution of (1.7)--(1.9) for the case $N \geq 2$. Then we obtain the answer to (Q.1)--(Q.3).

2. Proof of Theorem 1 for $\Omega = B_R$ and $x = 0$

In this section, we consider (1.7)--(1.9) and establish (1.5) for $x = 0$. Note that $u_\lambda(0) = \|u_\lambda\|_\infty$. We begin with the fundamental equalities which play important roles in this section. Multiply (1.7) by $u_\lambda'$. Then for $r \in [0, R]$,

$$
\left\{ u_\lambda''(r) + \frac{N - 1}{r} u_\lambda'(r) + \lambda \sin u_\lambda(r) - g(u_\lambda(r)) \right\} u_\lambda'(r) = 0.
$$

Since $u_\lambda(0) = \|u_\lambda\|_\infty$, this implies that for $r \in [0, R]$,

$$
\begin{align*}
(2.1) \quad & \frac{1}{2} u_\lambda'(r)^2 + \int_0^r \frac{N - 1}{s} u_\lambda'(s)^2 ds - \lambda \cos u_\lambda(r) - G(u_\lambda(r)) \equiv \text{constant} \\
& = -\lambda \cos \|u_\lambda\|_\infty - G(\|u_\lambda\|_\infty) \quad \text{(put $r = 0$)} \\
& = \frac{1}{2} u_\lambda'(R)^2 + \int_0^R \frac{N - 1}{s} u_\lambda'(s)^2 ds - \lambda \quad \text{(put $r = R$)}.
\end{align*}
$$

Let $M_\lambda := \inf\{\theta > 0 : \lambda \sin \theta = g(\theta)\}$. It is clear that $M_\lambda < \pi$ and $\lambda \sin \theta > g(\theta)$ for $0 < \theta < M_\lambda$. We know from $[\text{II}]$ that $\|u_\lambda\|_\infty < M_\lambda$. Therefore, for $0 \leq r < R$, we have

$$
(2.2) \quad \lambda \sin u_\lambda(r) > g(u_\lambda(r)).
$$

Further, it is well known that

$$
(2.3) \quad u_\lambda'(r) < 0 \quad (0 < r \leq R).
$$

We begin with the following fundamental lemma.

**Lemma 2.1.** Let $0 < r_0 < R$ be fixed. Then $u_\lambda''(r) < 0$ for $0 \leq r \leq r_0$ and $\lambda \gg 1$.

**Proof.** We know that for $0 \leq r \leq R$,

$$
(2.4) \quad (r^{N - 1} u_\lambda'(r))^2 = r^{N - 1}(g(u_\lambda(r)) - \lambda \sin u_\lambda(r)).
$$

Integrate (2.4) over $[0, r]$ to obtain

$$
(2.5) \quad u_\lambda'(r) = \frac{1}{r^{N - 1}} \int_0^r s^{N - 1}(g(u_\lambda(s)) - \lambda \sin u_\lambda(s))ds.
$$

By this, we obtain

$$
(2.6) \quad u_\lambda''(r) = g(u_\lambda(r)) - \lambda \sin u_\lambda(r) \\
+ (N - 1) r^{-N} \int_0^r s^{N - 1}(\lambda \sin u_\lambda(s) - g(u_\lambda(s)))ds.
$$

Then by integration by parts,

$$
(2.7) \quad \int_0^r s^{N - 1}(\lambda \sin u_\lambda(s) - g(u_\lambda(s)))ds \\
= \frac{r^N}{N} (\lambda \sin u_\lambda(r) - g(u_\lambda(r))) \\
- \int_0^r \frac{s^N}{N} (\lambda \cos u_\lambda(s) - g'(u_\lambda(s)))u_\lambda'(s)ds.
$$
By this, (P.2), (2.2), (2.3) and (2.6), for \( \lambda \gg 1 \), we obtain
\[
\frac{u''_\lambda (r)}{r} = \left( -\frac{1}{N} (\lambda \sin u_\lambda (r) - g(u_\lambda (r))) + \frac{N-1}{N} \int_0^r (\lambda \cos u_\lambda (s) - g'(u_\lambda (s))) u'_\lambda (s) s^N ds \right) < 0.
\]
Thus the proof is complete. \( \square \)

**Lemma 2.2.** Let an arbitrary \( 0 < r_0 < R \) be fixed. Then \( u''_\lambda (r) < 0 \) for \( 0 \leq r \leq r_0 \) and \( \lambda \gg 1 \).

**Proof.** By (2.6), we have
\[
-u''_\lambda (r) = (\lambda \cos u_\lambda (r) - g'(u_\lambda (r))) u'_\lambda (r)
\]
\[
+ (N-1)r^{-N-1} \int_0^r s^{N-1}(\lambda \sin u_\lambda (s) - g(u_\lambda (s)))ds
\]
\[
- (N-1)r^{-1}(\lambda \sin u_\lambda (r) - g(u_\lambda (r))).
\]
By this and (2.7),
\[
(2.8) - u''_\lambda (r) = (\lambda \cos u_\lambda (r) - g'(u_\lambda (r))) u'_\lambda (r)
\]
\[
- (N-1)r^{-N-1} \int_0^r s^{N}(\lambda \cos u_\lambda (s) - g'(u_\lambda (s))) u'_\lambda (s) ds.
\]
Since \( \lambda \gg 1 \), by (P.2) and integration by parts, we obtain
\[
\int_0^r s^{N}(\lambda \cos u_\lambda (s) - g'(u_\lambda (s))) u'_\lambda (s) ds
\]
\[
= -(\lambda + g'(\pi))(1 + o(1)) \int_0^r s^{N}u'_\lambda (s) ds
\]
\[
= -(\lambda + g'(\pi))(1 + o(1)) \left( \frac{1}{N+1} r^{N+1} u'_\lambda (r) - \int_0^r \frac{1}{N+1} s^{N+1} u''_\lambda (s) ds \right).
\]
By this, Lemma 2.1, (2.3) and (2.8), for \( 0 \leq r \leq r_0 \) and \( \lambda \gg 1 \), we obtain
\[
(2.9) - u''_\lambda (r) = \frac{2}{N+1}(\lambda + g'(\pi))(1 + o(1))(-u'_\lambda (r))
\]
\[
+ \frac{N-1}{N+1} r^{-N-1}(\lambda + g'(\pi))(1 + o(1)) \int_0^r s^{N+1}(-u''_\lambda (s)) ds
\]
\[
> 0.
\]
Thus the proof is complete. \( \square \)

We put
\[
(2.10) \quad \xi_\lambda := \lambda \sin \|u_\lambda\|_\infty - g(\|u_\lambda\|_\infty).
\]
By l'Hopital’s rule,
\[
\lim_{r \to 0} \frac{u'_\lambda (r)}{r} = \lim_{r \to 0} u''_\lambda (r) = u''_\lambda (0).
\]
By this and (1.7), we obtain
\[
Nu''_\lambda (0) + \lambda \sin \|u_\lambda\|_\infty - g(\|u_\lambda\|_\infty) = 0.
\]
This along with (2.10) and Lemma 2.1 implies that

\[(2.11)\quad \xi_\lambda = - Nu''_\lambda(0) > 0.\]

Furthermore, we put

\[(2.12)\quad I_\lambda(r) := \lambda (\cos u_\lambda(r) - \cos \|u_\lambda\|_\infty) + G(u_\lambda(r)) - G(\|u_\lambda\|_\infty),\]

\[(2.13)\quad II_\lambda(r) := \int_0^r \frac{N - 1}{s} u'_\lambda(s)^2 ds.\]

Then for \(r \in [0, R]\), by (2.1), we obtain

\[(2.14)\quad \frac{1}{2} u'_\lambda(r)^2 = I_\lambda(r) - II_\lambda(r).\]

**Lemma 2.3.** Let \(0 < r_0 < R\) be fixed. Then for \(0 \leq r \leq r_0\) and \(\lambda \gg 1\)

\[(2.15)\quad I_\lambda(r) = \xi_\lambda(\|u_\lambda\|_\infty - u_\lambda(r)) + \frac{1}{2} (\lambda + g'(\pi))(1 + o(1))(\|u_\lambda\|_\infty - u_\lambda(r))^2.\]

**Proof.** By Taylor expansion and (P.2), for \(0 \leq r \leq r_0\) and \(\lambda \gg 1\), we obtain

\[(2.16)\quad \cos u_\lambda(r) = \cos \|u_\lambda\|_\infty - \sin \|u_\lambda\|_\infty (u_\lambda(r) - \|u_\lambda\|_\infty) + \frac{1}{2} \cos \|u_\lambda\|_\infty (u_\lambda(r) - \|u_\lambda\|_\infty)^2 (1 + o(1)),\]

\[(2.17)\quad G(u_\lambda(r)) = G(\|u_\lambda\|_\infty) + g(\|u_\lambda\|_\infty)(u_\lambda(r) - \|u_\lambda\|_\infty) + \frac{1}{2} (g'(\|u_\lambda\|_\infty) + o(1))(u_\lambda(r) - \|u_\lambda\|_\infty)^2.\]

Then by (P.2), (2.12), (2.16) and (2.17), for \(0 \leq r \leq r_0\) and \(\lambda \gg 1\), we have

\[
I_\lambda = \lambda (\cos u_\lambda(r) - \cos \|u_\lambda\|_\infty) + G(u_\lambda(r)) - G(\|u_\lambda\|_\infty)
\]

\[
= \xi_\lambda(\|u_\lambda\|_\infty - u_\lambda(r)) + \frac{1}{2} (\lambda + g'(\pi))(1 + o(1))(\|u_\lambda\|_\infty - u_\lambda(r))^2.
\]

\[\square\]

**Lemma 2.4.** Let \(0 < r_0 < R\) be fixed. Then for \(0 \leq r \leq r_0\) and \(\lambda \gg 1\),

\[(2.18)\quad II_\lambda(r) \leq \frac{N - 1}{N} \xi_\lambda(\|u_\lambda\|_\infty - u_\lambda(r)) + \frac{N - 1}{2(N + 1)} (\lambda + g'(\pi))(1 + o(1))(\|u_\lambda\|_\infty - u_\lambda(r))^2.\]

**Proof.** Since (2.18) is valid for \(r = 0\), let \(r > 0\). By l'Hopital's rule and (2.11),

\[(2.19)\quad \lim_{r \to 0} \frac{II_\lambda(r)}{\|u_\lambda\|_\infty - u_\lambda(r)} = \lim_{r \to 0} \frac{-(N - 1)u'_\lambda(r)}{r} = \lim_{r \to 0} (1 - N)u''_\lambda(r)
\]

\[= (1 - N)u''_\lambda(0) = \frac{N - 1}{N} \xi_\lambda.\]
Then by this and Cauchy’s mean value theorem, for $0 < r \leq r_0$ and $\lambda \gg 1$, we obtain

\[
(2.20) \hspace{1cm} \frac{II_\lambda(r) - ((N-1)\xi_\lambda/N)(\|u_\lambda\|_\infty - u_\lambda(r))}{(\|u_\lambda\|_\infty - u_\lambda(r))^2} = \frac{(N-1)u_\lambda'(r_1) + (N-1)\xi_\lambda r_1/N}{-2r_1(\|u_\lambda\|_\infty - u_\lambda(r_1))} = \frac{(N-1)u_\lambda''(r_2) + (N-1)\xi_\lambda/N}{-2(\|u_\lambda\|_\infty - u_\lambda(r_2)) + 2ru_\lambda'(r_2)} = \frac{(N-1)u_\lambda''(r_3)}{4u_\lambda'(r_3) + 2ru_\lambda'(r_3)}
\]

where $0 < r_3 < r_2 < r_1 < r$. By Lemma 2.2, we see that $-u_\lambda''(r)$ is increasing for $0 \leq r \leq r_0$. Then by (2.9), for any $0 \leq r \leq r_0$ and $\lambda \gg 1$, we obtain

\[
-\frac{u_\lambda''(r)}{u_\lambda'(r)} \leq \frac{2}{N+2}(\lambda + o'()((1 + o(1))(-u_\lambda'(r)) + \frac{N-1}{N+1}r^{-N-1}(\lambda + o'())((1 + o(1))(-u_\lambda'(r))) \int_0^r s^N ds + \frac{N-1}{(N+1)(N+2)}(\lambda + o'())((1 + o(1))r(-u_\lambda''(r))) \\
\leq \frac{1}{2(N+1)}(\lambda + o'())((1 + o(1))(-4u_\lambda'(r) - 2ru_\lambda'(r)).
\]

Put $r = r_3$ in (2.21). Then by this and (2.20), we obtain (2.18). Thus the proof is complete.

\begin{lemma}
Let $0 < r_0 < R$ be fixed. Then $\xi_\lambda = o\left(\lambda e^{-\sqrt{2\lambda(1+o(1))/(N+1)r_0}}\right)$ as $\lambda \to \infty$.
\end{lemma}

\textbf{Proof.} By (2.14), Lemmas 2.3 and 2.4, for $0 \leq r \leq r_0$ and $\lambda \gg 1$,

\[
(2.22) \hspace{1cm} \frac{1}{2}u_\lambda'(r)^2 = I_\lambda(r) - II_\lambda(r) \geq \frac{1}{N}\xi_\lambda(\|u_\lambda\|_\infty - u_\lambda(r)) + \frac{1}{N+1}(1 + o(1))(g'() + \lambda)(\|u_\lambda\|_\infty - u_\lambda(r))^2 \\
\geq \frac{1}{2}P_\lambda(\|u_\lambda\|_\infty - u_\lambda(r)) + \frac{1}{2}Q_\lambda(\|u_\lambda\|_\infty - u_\lambda(r))^2;
\]
where \( P_\lambda := 2Q_\lambda/N, Q_\lambda := 2\lambda(1+o(1))/(N+1) \). We put \( t = \sqrt{s/(P_\lambda + Q_\lambda s)} \). Then by (2.14) and (2.22), for \( \lambda \gg 1 \), we obtain

\[
r_0 = \int_0^{r_0} 1dt = \int_0^{r_0} \frac{-u'_\lambda(r)}{\sqrt{2(I_\lambda(r) - H_\lambda(r))}} dr
\]

\[
\leq \int_{u_\lambda(r_0)}^{\|u_\lambda\|_\infty} \frac{1}{\sqrt{P_\lambda\|u_\lambda\| - \theta} + Q_\lambda\|u_\lambda\| - \theta}^2 d\theta
\]

\[
= \int_{u_\lambda(r_0)}^{\|u_\lambda\|_\infty - u_\lambda(r_0)} \frac{1}{\sqrt{P_\lambda s + Q_\lambda s^2}} ds = \int_0^{A_\lambda} \frac{2}{1 - Q_\lambda t^2} dt
\]

\[
geq \frac{1}{\sqrt{Q_\lambda}} \log \left( \frac{1/\sqrt{Q_\lambda} + A_\lambda}{1/\sqrt{Q_\lambda} - A_\lambda} \right),
\]

where

\[
A_\lambda := \sqrt{\frac{\|u_\lambda\|_\infty - u_\lambda(r_0)}{P_\lambda + Q_\lambda(\|u_\lambda\|_\infty - u_\lambda(r_0))}}.
\]

This implies that

\[
\sqrt{\frac{(\|u_\lambda\|_\infty - u_\lambda(r_0))}{P_\lambda + Q_\lambda(\|u_\lambda\|_\infty - u_\lambda(r_0))}} \geq \frac{1}{\sqrt{Q_\lambda}} \frac{e^{\sqrt{Q_\lambda r_0}} - 1}{e^{\sqrt{Q_\lambda r_0}} + 1}.
\]

By this and (P.2), for \( \lambda \gg 1 \), we obtain

\[
\frac{2}{N} \xi_\lambda = P_\lambda \leq \frac{4Q_\lambda e^{\sqrt{Q_\lambda r_0}}(\|u_\lambda\|_\infty - u_\lambda(r_0))}{(e^{\sqrt{Q_\lambda r_0}} - 1)^2} = o \left( \lambda e^{-\sqrt{2\lambda(1+o(1))/(N+1)}} \right).
\]

Thus the proof is complete. \( \square \)

**Proof of Theorem 1 for \( \Omega = B_R \) and \( x = 0 \).** We put \( \zeta_1(\lambda) := \pi - \|u_\lambda\|_\infty \). Then \( \zeta_1(\lambda) \to 0 \) as \( \lambda \to \infty \) by (P.2). Then by Lemma 2.5, we obtain

\[
\lambda \sin \|u_\lambda\|_\infty = \lambda \sin(\pi - \zeta_1(\lambda)) = \lambda \sin \zeta_1(\lambda) = \lambda(1 + o(1)) \zeta_1(\lambda) = g(\|u_\lambda\|_\infty) + o(\lambda e^{-\sqrt{2\lambda(1+o(1))/(N+1)}}) = g(\pi)(1 + o(1)).
\]

This implies that

\[
\zeta_1(\lambda) = \frac{g(\pi)}{\lambda} + o \left( \frac{1}{\lambda} \right).
\]

Therefore, we obtain the second term in (1.5). Then (1.5) is proved by completely the same argument as that in [8] Proof of Theorem 2, which is proceeded by the mathematical induction. Thus the proof of Theorem 1 for the case \( \Omega = B_R \) and \( x = 0 \) is complete. \( \square \)

### 3. Proof of Theorem 1

Now we consider (1.1)–(1.3). Let \( x \in \Omega \) be fixed. Let \( \delta_1 := \text{dist}\{x, \partial \Omega\} > 0 \) and \( L_1 := \sup\{|y_1 - y_2| : y_1, y_2 \in \Omega\} \). Since (1.1)–(1.3) is autonomous, by translation of the coordinate system, we may assume that \( x = 0 \). Let \( B_1 = B_{\delta_1/2} \) and \( B_2 := B_{2L_1} \). Furthermore, let \( u_{\lambda,1} \) and \( u_{\lambda,2} \) be the solutions of (1.7)–(1.9) for \( R = \delta_1/2 \) and \( R = 2L_1 \), respectively. It is clear that \( u_\lambda \) is a super-solution of (1.1)–(1.3) for \( B_1 \).
Further, for $0 < \epsilon \ll 1$, we see from (A.2) that $v_{\epsilon}(|x|) = \epsilon \varphi_1(|x|)$ is a sub-solution of (1.1)–(1.3) for $B_1$, where $\varphi_1$ is the first positive eigenfunction of $-\Delta$ in $B_1$ with the Dirichlet zero boundary condition. Since $0 < \epsilon \ll 1$, we see that $\epsilon \varphi_1 < u_\lambda$. Then since $u_{\lambda,1}$ is a unique solution of (1.1)–(1.3) for $\Omega = B_1$, we see from [7, p. 24] that for $x \in B_1$

\begin{equation}
\epsilon \varphi_1(x) \leq u_{\lambda,1}(x) \leq u_\lambda(x).
\end{equation}

By the same argument as above, for $x \in \Omega$, we have

\begin{equation}
u_{\lambda}(x) \leq u_{\lambda,2}(x).
\end{equation}

In particular, by putting $x = 0$, we obtain

\begin{equation}
\|u_{\lambda,1}\|_{\infty} \leq u_\lambda(0) \leq \|u_{\lambda,2}\|_{\infty}.
\end{equation}

Since the formula (1.5) holds for $\|u_{\lambda,1}\|_{\infty}$ and $\|u_{\lambda,2}\|_{\infty}$, by (3.2) and (3.3), we immediately obtain (1.5). Thus the proof is complete.

References


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