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HYPERBOLIC GROUPS HAVE FINITE ASYMPTOTIC DIMENSION

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ABSTRACT. We detail a proof of a result of Gromov, that hyperbolic groups (and metric spaces) have finite asymptotic dimension. This fact has become important in recent work on the Novikov conjecture.

1. Introduction

Let X be a metric space, with basepoint x_0. We use the notation $|x|$ to denote $d(x, x_0)$. If $x, y \in X$, then the *Gromov product* $(x|y)$ is the positive real number $\frac{1}{2}(|x| + |y| - d(x, y))$. By definition [3], X is *hyperbolic* if there is $\delta > 0$ such that

$$(x|z) \geq \min\{(x|y), (y|z)\} - \delta$$

for all $x, y, z \in X$.

Let $\mathcal{U}$ be a family of subsets of X. We say that $\mathcal{U}$ is *d-disconnected* if the minimum distance between any two distinct sets of the family $\mathcal{U}$ is at least d. We say that $\mathcal{U}$ is *r-bounded* if each set in the family has diameter $\leq r$. One says that X has *finite asymptotic dimension* if there is a number N such that for each $d > 0$ there is an $r > 0$ such that X can be covered by at most $N + 1$ d-disconnected, r-bounded families. The least such N is the *asymptotic dimension* of X.

This definition is due to Gromov [4, page 29] and was crucial to the work of Yu [5] on the Novikov conjecture. On page 31 of [4], Gromov remarks that word hyperbolic groups have finite asymptotic dimension. Below we present a short proof of (a slight generalization of) this result. Our proof is related to those of the finite-dimensionality of the Gromov boundary of a hyperbolic group given in [2] and [1].

2. The proof

Let X be a geodesic metric space. Say that X has *bounded growth* if for each $s > 0$ there is a number N_s such that each ball of radius $S + s$ in X can be covered by at most N_s balls of radius S.

Since X is geodesic, one may take for N_s the supremum (if finite) of the cardinalities of s-separated subsets in balls of radius $2s$. This observation shows that a

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space of bounded geometry has bounded growth. In particular, the Cayley graph of a finitely generated group has bounded growth.

Theorem 2.1. *Let X be a hyperbolic geodesic metric space with bounded growth. Then X has finite asymptotic dimension.*

Proof. Fix a basepoint $x_0 \in X$, and let $d > 0$ be given. Suppose that X is δ-hyperbolic. Let A_k denote the annulus $\{x \in X : kd \leq |x| \leq (k+1)d\}$ in X. It will suffice to show that there is a number N, independent of d, such that each annulus A_k can be covered by a family of sets $\{U_i\}$, each having diameter no more than $4d + 4\delta$, and such that no more than N of the sets U_i have nonempty intersection, in A_k, with any set of diameter d.

Let $\{x_i\}$ be a maximal d-separated subset of the sphere $\{x : |x| = kd\}$ of radius kd and define $U_i = \{x \in A_k : (x|x_i) \geq (k - \frac{1}{2})d - \delta\}$. If $x \in A_k$ let x' denote the point where a geodesic from x_0 to x intersects the sphere of radius kd. Then $|x'| = (x|x') = kd$. By maximality there is some i for which $d(x', x_i) \leq d$ and therefore $(x'|x_i) \geq (k - \frac{1}{2})d$. By hyperbolicity $(x|x_i) \geq \min\{(x|x'), (x'|x_i)\} - \delta \geq (k - \frac{1}{2})d - \delta$ and so $x \in U_i$. Thus the U_i cover A_k as asserted.

Suppose $x \in U_i$. Then $d(x, x_i) = |x| + |x_i| - 2(x|x_i) \leq 2d + 2\delta$. Thus the U_i have uniformly bounded diameter.

Suppose that U_i meets the ball of radius d around some $x \in A_k$; let y be a point in the intersection. Let x'' be the point where a geodesic ray from x_0 to x intersects the sphere of radius $(k - \frac{1}{2})d$, so that $|x''| = (x|x'') = (k - \frac{1}{2})d$. We also have $(x|y) \geq (k - \frac{1}{2})d$, and $(x_i|y) \geq (k - \frac{1}{2})d - \delta$, so $(x_i|x'') \geq (k - \frac{1}{2})d - 3\delta$. It follows that $d(x_i, x'') = |x_i| + |x''| - 2(x_i|x'') \leq \frac{1}{2}d + 6\delta$. The maximum number of U_i that meet the ball of radius d around x is therefore bounded by the maximum cardinality of a d-separated subset in a ball of radius $\frac{1}{2}d + 6\delta$. But this cardinality is bounded by the number $N_{6\delta}$ arising from the definition of bounded growth. The proof is complete. □

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