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ORDERED FIELDS SATISFYING PÓLYA'S THEOREM

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Abstract. The purpose of this paper is to characterize ordered fields satisfying Pólya's theorem on positive representations of polynomials. As a main result, it is proved that an ordered field $(F, \leq)$ satisfies Pólya's theorem if and only if $\leq$ is an archimedean ordering and F is a real closed field.

1. Introduction

Early in 1928, G. Pólya [8] established the following interesting result (see also Theorem 56 on page 57 of [3]):

Theorem 1.1 (Pólya's Theorem)**.** *Let $f := f(x_1, ..., x_n)$ be a homogeneous polynomial in n variables over the field $\mathbb{R}$ of real numbers such that $f(a_1, ..., a_n) > 0$ for all nonnegative numbers a_1, ..., $a_n \in \mathbb{R}$ with $\sum_{i=1}^{n} a_i > 0$. Then there exists a natural number r such that all the coefficients of $(x_1 + ... + x_n)^r f$ are strictly positive.*

Based on Pólya's theorem, W. Habicht [2] proposed a constructive method to represent a positive definite form over $\mathbb{R}$ as a quotient of two sums of squares of homogeneous polynomials. Also, some important applications of Pólya's theorem may be found in many areas, e.g. see [5, 7, 9, 12].

It is natural to establish new versions of Pólya's theorem in a wider category of ordered fields. A. Robinson established an appropriate modified version of Pólya's theorem for non-archimedean real closed fields containing $\mathbb{R}$ by giving a necessary and sufficient condition on homogeneous polynomials (see Theorem 8.1 in [11]). The purpose of this paper is to characterize ordered fields to which Pólya's theorem carries over. For this purpose, we need some relevant definitions.

Definition 1.2. Let m be a fixed natural number. An ordered field $(F, \leq)$ is called to satisfy Pólya's theorem on homogeneous polynomials in m variables, if for every homogeneous polynomial $f := f(x_1, ..., x_m) \in F[x_1, ..., x_m]$, there exists a natural number r such that all the coefficients of $(x_1 + ... + x_m)^r f$ are strictly positive, whenever $f(a_1, ..., a_m) > 0$ for all nonnegative elements a_1, ..., $a_m \in F$ with $\sum_{i=1}^{m} a_i > 0$.

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An ordered field $(F, \leq)$ is called to satisfy Pólya's theorem, if $(F, \leq)$ satisfies Pólya's theorem on homogeneous polynomials in n variables for every natural number n.

For generality of discussion, we further give the definition as follows:

Definition 1.3. Let m be a fixed natural number, let $(K, \leq)$ be an ordered field, and F a subfield of K. F is called to satisfy Pólya's theorem on homogeneous polynomials in m variables with respect to $(K, \leq)$, if for every homogeneous polynomial $f := f(x_1, ..., x_m) \in F[x_1, ..., x_m]$, there exists a natural number r such that all the coefficients of $(x_1 + ... + x_m)^r f$ are strictly positive, whenever $f(a_1, ..., a_m) > 0$ for all nonnegative elements a_1, ..., $a_m \in K$ with $\sum_{i=1}^{m} a_i > 0$.

F is called to satisfy Pólya's theorem with respect to $(K, \leq)$, if F satisfies Pólya's theorem on homogeneous polynomials in n variables with respect to $(K, \leq)$ for every natural number n.

Clearly, when $F = K$, Definition 1.2 is immediately derived from Definition 1.3. Generally, Pólya's theorem is not true for an arbitrary ordered field. The following example illustrates that Pólya's theorem is false for non-archimedean ordered fields.

Example 1.4. Let $(F, \leq)$ be a non-archimedean ordered field. Then there exists an element ϵ in F such that ϵ is positive and infinitesimal over the subfield $\mathbb{Q}$ of rational numbers.

Now consider the binary polynomial $f(x, y) = (x - y)^2 + \epsilon xy$ over F. Obviously, $f(x, y)$ is homogeneous, and for all a, $b \in F$, $f(a, b) > 0$, whenever $a \geq 0$, $b \geq 0$, and $a + b > 0$.

However, we may assert that at least one term of $(x + y)^r f(x, y)$ has a negative element in F as its coefficient for every natural number r.

Indeed, we have the following possible cases for r:

When $r = 1$, the term xy^2 in $(x + y)f(x, y)$ has the negative coefficient $\epsilon - 1$.

When $r = 2k$ is even, the coefficient of the term $x^{k+1}y^{k+1}$ in $(x + y)^r f(x, y)$ is $2\binom{2k}{k-1} - (2 - \epsilon)\binom{2k}{k}$, and

$$2\binom{2k}{k-1} - (2-\epsilon)\binom{2k}{k} = \binom{2k}{k}\epsilon - \frac{2(2k)!}{(k-1)!k!}\left(\frac{1}{k} - \frac{1}{k+1}\right) < 0.$$

When $r = 2k+1$ with $k > 1$, the coefficient of the term $x^{k+1}y^{k+2}$ in $(x+y)^r f(x, y)$ is $\binom{2k+1}{k-1} + \binom{2k+1}{k+1} - (2 - \epsilon)\binom{2k+1}{k}$, and

$$\begin{aligned}&\binom{2k+1}{k-1} + \binom{2k+1}{k+1} - (2-\epsilon)\binom{2k+1}{k}\\ &\quad = \binom{2k+1}{k}\epsilon - \frac{(2k+1)!}{(k-1)!(k+1)!}\left(\frac{1}{k} - \frac{1}{k+2}\right) < 0.\end{aligned}$$

This example shows that to possess an archimedean ordering is a necessary condition for an ordered field to satisfy Pólya's theorem. In this paper, we will establish some necessary and sufficient conditions for an ordered field to satisfy Pólya's theorem.

2. Main results

In this section, we give some characterizations of ordered fields satisfying Pólya's theorem. First, we establish the main result as follows.

Theorem 2.1. *Let $(K, \leq)$ be an ordered field, and F a subfield of K. Then the following statements are equivalent:*

(1) *F satisfies Pólya's theorem with respect to $(K, \leq)$.*
(2) *For some natural number m with $m > 1$, F satisfies Pólya's theorem on homogeneous polynomials in m variables with respect to $(K, \leq)$.*
(3) *F satisfies Pólya's theorem on binary homogeneous polynomials with respect to $(K, \leq)$, i.e. the following statement is true:*
If $f(x, y)$ is a binary homogeneous polynomial in $F[x, y]$ such that $f(a, b) > 0$ for all nonnegative elements a, $b \in K$ with $a + b > 0$, then there exists a natural number r such that all the coefficients of $(x + y)^r f$ are strictly positive.
(4) *The algebraic closure of F in K is real closed, and $\leq_F$ is an archimedean ordering of F, where $\leq_F$ is the restriction of the ordering $\leq$ on F.*

Proof. (1) $\Longrightarrow$ (2): Obvious.

(2) $\Longrightarrow$ (3): Let $f(x, y)$ be a binary homogeneous polynomial in $F[x, y]$ such that $f(a, b) > 0$ for all nonnegative elements a, $b \in K$ with $a + b > 0$. Observe that $m \geq 2$. Put $g(x, y, x_3, ..., x_m) = f(x, y) + x_3^d + \cdots + x_m^d$, where d is the total degree of $f(x, y)$. Then g is a homogeneous polynomial in m variables over F. It is easy to see that $g(a_1, ..., a_m) > 0$ for all nonnegative elements a_1, ..., $a_m \in K$ with $\sum_{i=1}^m a_i > 0$. By statement (2), there exists a natural number r such that all the coefficients of $(x + y + x_3 + ... + x_m)^r g$ are strictly positive.

Putting $h := (x + y + x_3 + ... + x_m)^r g - (x + y)^r f$, we have

$$(x + y + x_3 + ... + x_m)^r g = (x + y)^r f + h.$$

Since every term in h contains at least one variable in $\{x_3, ..., x_m\}$, $(x + y)^r f$ and h have no same term. Hence all the coefficients of $(x + y)^r f$ are also strictly positive.

(3) $\Longrightarrow$ (4): Suppose that $\leq_F$ is not an archimedean ordering. Then there exists an element ϵ in F such that ϵ is positive and infinitesimal over the subfield $\mathbb{Q}$ of rational numbers. According to the argument about Example 1.4, the binary homogeneous polynomial $f(x, y) = (x - y)^2 + \epsilon xy$ over F is such that $f(a, b) > 0$ for all nonnegative elements a, $b \in K$ with $a + b > 0$, but at least one term of $(x + y)^r f(x, y)$ has a negative element in F as its coefficient for every natural number r. This contradicts statement (3). Hence, $\leq_F$ is an archimedean ordering. It remains to prove that the algebraic closure of F in K is a real closed field.

Let R be the real closure of $(K, \leq)$, denote by $\leq_R$ the unique ordering of R, and write K_a, R_a for the algebraic closures of F in K, R respectively. Clearly, $K_a \subseteq R_a$. By a familiar fact about real closed fields (see Lemma 3.13 in [10]), R_a is real closed. Denote by $\leq_{R_a}$ the unique ordering of R_a. Then $\leq_{R_a}$ is just the restriction of $\leq_R$ on R_a. Observe that $(R_a, \leq_{R_a})$ is an algebraic ordered extension of the ordered field $(F, \leq_F)$. By Proposition 2.8 in [6], $\leq_{R_a}$ is also archimedean, since $\leq_F$ is an archimedean ordering. This implies that the field $\mathbb{Q}$ of rational numbers is dense in R_a with respect to $\leq_{R_a}$.

Let $\alpha \in R_a$. Denote by $h(x)$ the minimal polynomial of α over F, and assume that all the roots of $h(x)$ in R_a are as follows: $\alpha_1 = \alpha$, α_2, ..., α_s, where $s \geq 1$. By the density of $\mathbb{Q}$ in R_a, there exist δ, $d \in \mathbb{Q}$ such that $\delta >_{R_a} 0$, $(\alpha - d)^2 <_{R_a} \delta$, but $(\alpha_i - d)^2 >_{R_a} \delta$, $i = 2$, ..., s. Obviously, $[\delta - (\alpha - d)^2]^{-1} = \beta^2$ for some element $\beta \in R_a$.

By a familiar result on primitive elements (see Theorem 14 in [4]), there is an element $\gamma \in R_a$ such that $F(\alpha, \beta) = F(\gamma)$, $\alpha = u(\gamma)$, and $\beta = v(\gamma)$, where $u(x)$, $v(x) \in F[x]$. Without loss of generality, we may assume that $\gamma \geq_{R_a} 0$. Put $g(x) := h^2(u(x)) + \Big(v^2(x)[\delta - (u(x) - d)^2] - 1\Big)^2$. Then $g(x) \in F[x]$, and $g(\gamma) = h^2(\alpha) + \Big(\beta^2[\delta - (\alpha - d)^2] - 1\Big)^2 = 0$. Obviously, $g(x)$ may be expressed in the form as follows:

$$g(x) = c_0 x^k + c_1 x^{k-1} + \cdots + c_k,$$

where $c_i \in F$, $i = 0$, 1, ..., k, and $c_0 > 0$.

Put $f(x, y) = c_0 x^k + c_1 x^{k-1} y + \cdots + c_k y^k$. Obviously, $f(x, y)$ is a binary homogeneous polynomial in $F[x, y]$. Suppose that all the coefficients of $(x + y)^t f$ are strictly positive for some natural number t. Then we have $(\gamma+1)^t f(\gamma, 1) >_{R_a} 0$, and $f(\gamma, 1) >_{R_a} 0$. But $f(\gamma, 1) = g(\gamma) = 0$, a contradiction. Hence, at least one term of $(x + y)^n f(x, y)$ has a negative element in F as its coefficient for every natural number n.

By statement (3), there exist two nonnegative elements a, $b \in K$ such that $a + b > 0$, but $f(a, b) \leq 0$. Moreover, we may assert that $b \neq 0$. Indeed, if $b = 0$, $c_0 a^t \leq 0$. Observing that $c_0 > 0$, we have $a = 0$, and $a + b = 0$, a contradiction. This yields $g(ab^{-1}) \leq 0$, i.e. $h^2(u(ab^{-1})) + \Big(v^2(ab^{-1})[\delta - (u(ab^{-1}) - d)^2] - 1\Big)^2 = 0$. Necessarily, $h(u(ab^{-1})) = v^2(ab^{-1})[\delta - (u(ab^{-1}) - d)^2] - 1 = 0$. This implies that $u(ab^{-1})$ is a root of $h(x)$ in K. Hence, $u(ab^{-1}) \in K_a$, and $u(ab^{-1}) = \alpha_j$ for some $j \in \{1, ..., s\}$. It follows that $\delta - (\alpha_j - d)^2 = v^{-2}(ab^{-1}) >_{R_a} 0$. This implies $j = 1$. So we have $\alpha = u(ab^{-1}) \in K_a$, and $K_a = R_a$. Therefore, K_a is real closed.

(4) $\Longrightarrow$ (1): Write K_a for the algebraic closure of F in K, and denote by $\leq_{K_a}$ the restriction of $\leq$ on K_a. By statement (4), K_a is a real closed field, and K_a is just the real closure of $(F, \leq_F)$ with unique ordering $\leq_{K_a}$. By Proposition 2.8 in [6], $\leq_{K_a}$ is an archimedean ordering, since $\leq_F$ is archimedean. According to Theorem 2.7 in [6], there is an unique preserving-order injection τ of K_a into the field $\mathbb{R}$ of real numbers. By identifying α with $\tau(\alpha)$ for all $\alpha \in K_a$, we may assume $F \subseteq K_a \subseteq \mathbb{R}$.

Now, let $f := f(x_1, ..., x_n)$ be any homogeneous polynomial in $F[x_1, ..., x_n]$ such that $f(a_1, ..., a_n) > 0$ for all nonnegative elements a_1, ..., $a_n \in K$ with $\sum_{i=1}^n a_i > 0$, where n is an arbitrary natural number. By the preceding argument, f may be considered as a real polynomial function on $\mathbb{R}$. In this case, we have the claim as follows:

Claim. For all nonnegative numbers a_1, ..., $a_n \in \mathbb{R}$ with $\sum_{i=1}^n a_i > 0$, $f(a_1, ..., a_n) > 0$.

Indeed, if not, then there exist some nonnegative numbers a_1, ..., $a_n \in \mathbb{R}$ such that $\sum_{i=1}^n a_i > 0$, but $f(a_1, ..., a_n) \leq 0$. By the well-known Lang's Homomorphism Theorem (see Theorem 4.1.2 in [1] or Theorem 6.2 in [6]), there is an F-homomorphism π of $F[\sqrt{a_1}, ..., \sqrt{a_n}, (\sum_{i=1}^n a_i)^{-1}, \sqrt{-f(a_1, ..., a_n)}]$ into K_a such that $\pi(a_i) \geq_{K_a} 0$ for $i = 1$, ..., n, $\pi\Big(\sum_{i=1}^n a_i\Big) >_{K_a} 0$, but $\pi\Big(f(a_1, ..., a_n)\Big) \leq_{K_a} 0$.

Since $\leq_{K_a}$ is the restriction of $\leq$ on K_a, we have $\pi(a_i) \geq 0$ for $i = 1, ..., n$, $\sum_{i=1}^{n} \pi(a_i) > 0$, but $f\big(\pi(a_1), ..., \pi(a_n)\big) \leq 0$. This contradicts the hypothesis on f. Hence, this claim is verified.

By Pólya's theorem and the above claim, there exists a natural number r such that all the coefficients of $(x_1 + ... + x_n)^r f$ are strictly positive. According to Definition 1.3, F satisfies Pólya's theorem with respect to $(K, \leq)$. This completes the proof. □

As an immediate consequence of Theorem 2.1, we may establish the following result:

Theorem 2.2. *Let $(F, \leq)$ be an ordered field. Then the following statements are equivalent:*

(1) *$(F, \leq)$ satisfies Pólya's theorem.*
(2) *For some natural number m with $m > 1$, $(F, \leq)$ satisfies Pólya's theorem on homogeneous polynomials in m variables.*
(3) *$(F, \leq)$ satisfies Pólya's theorem on binary homogeneous polynomials, i.e. the following statement is true:*
If $f(x, y)$ is a binary homogeneous polynomial in $F[x, y]$ such that $f(a, b) > 0$ for all nonnegative elements a, $b \in F$ with $a + b > 0$, then there exists a natural number r such that all the coefficients of $(x + y)^r f$ are strictly positive.
(4) *F is a real closed field, and $\leq$ is an archimedean ordering.*

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