REAL $k$-FLATS TANGENT TO QUADRICS IN $\mathbb{R}^n$

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Abstract. Let $d_{k,n}$ and $\#_{k,n}$ denote the dimension and the degree of the Grassmannian $G_{k,n}$, respectively. For each $1 \leq k \leq n-2$ there are $2^{d_{k,n}} \cdot \#_{k,n}$ (a priori complex) $k$-planes in $\mathbb{P}^n$ tangent to $d_{k,n}$ general quadratic hypersurfaces in $\mathbb{P}^n$. We show that this class of enumerative problems is fully real, i.e., for $1 \leq k \leq n-2$ there exists a configuration of $d_{k,n}$ real quadrics in (affine) real space $\mathbb{R}^n$ so that all the mutually tangent $k$-flats are real.

Introduction

Understanding the real solutions of a system of polynomial equations is a fundamental problem in mathematics (see, e.g., [13] for some recent lines of research and applications). However, as pointed out in [3, p. 55], even for problem classes with a finite number of complex solutions (enumerative problems), the question of how many solutions can be real is still widely open. A class of enumerative problems is called fully real if there are general real instances for which all the (a priori complex) solutions are real.

One of us (Sottile) began a systematic study of this question in the special Schubert calculus [9, 10], a class of enumerative problems from classical algebraic geometry. This special Schubert calculus asks for linear subspaces of a fixed dimension meeting some given (general) linear subspaces (whose dimensions and number ensure a finite number of solutions) in $n$-dimensional complex projective space $\mathbb{P}^n$. For any given dimensions of the subspaces, this problem is fully real, i.e., there exist real linear subspaces for which each of the a priori complex solutions is real.

In particular, for $1 \leq k \leq n-2$ there are $d_{k,n} := (k+1)(n-k)$ real $(n-k-1)$-planes $U_1, \ldots, U_{d_{k,n}}$ in $\mathbb{P}^n$ with

$$\#_{k,n} := \frac{1! \cdots k!((k+1)(n-k))!}{(n-k)!(n-k+1)! \cdots n!}$$

real $k$-planes meeting $U_1, \ldots, U_{d_{k,n}}$. Here, $d_{k,n}$ and $\#_{k,n}$ are the dimension and the degree of the Grassmannian $G_{k,n}$, respectively (see [9, 10]). These were the first results showing that a large class of non-trivial enumerative problems is fully real.
Recently, Vakil [14] has shown that any Schubert problem on a Grassmannian is fully real.

We continue this line of research by considering \( k \)-flats tangent to quadratic hypersurfaces (hereafter quadrics). This is also motivated by recent investigations in computational geometry (see [6, 11, 12]). It was shown in [12] that \( 2n - 2 \) general spheres in affine real space \( \mathbb{R}^n \) have at most \( 3 \cdot 2^{n-1} \) common tangent lines in \( \mathbb{C}^n \), and that there exist spheres for which all the a priori complex tangent lines are real. The present paper addresses the following question: What is the maximum number of real \( k \)-flats simultaneously tangent to \( d_{k,n} \) general quadrics in \( \mathbb{R}^n \) (respectively in \( \mathbb{P}^n \mathbb{R} \))? As this problem may be formulated as the complete intersection of \( d_{k,n} \) quadratic equations on the Grassmannian of \( k \)-planes in \( \mathbb{P}^n \), the expected number of complex solutions is the product of the degrees of the equations with the degree of the Grassmannian, i.e., \( 2^{d_{k,n}} \cdot \#_{k,n} \). We show that the problem is fully real:

**Theorem 1.** Let \( 1 \leq k \leq n - 2 \). Given \( d_{k,n} \) general quadrics in \( \mathbb{P}^n \) there are \( 2^{d_{k,n}} \cdot \#_{k,n} \) complex \( k \)-planes that are simultaneously tangent to all \( d_{k,n} \) quadrics. Furthermore, there is a choice of quadrics in \( \mathbb{R}^n \) for which all the \( k \)-flats are real, distinct, and lie in the affine space \( \mathbb{R}^n \).

When \( k = 1 \), we have \( d_{1,n} = 2(n - 1) \) and \( \#_{1,n} \) is the Catalan number \( \#_{1,n} = \frac{1}{n} \binom{2n-2}{n-1} \). Table 1 exhibits the large discrepancy between the number of lines tangent to spheres and the number of lines tangent to general quadrics. When \( n = 3 \) this discrepancy was accounted for by Aluffi and Fulton [1].

**Table 1.**

<table>
<thead>
<tr>
<th>( n )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
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<td>( 3 \cdot 2^{n-1} )</td>
<td>12</td>
<td>24</td>
<td>48</td>
<td>96</td>
<td>192</td>
<td>384</td>
<td>768</td>
</tr>
<tr>
<td>( 2^{d_{1,n}} \cdot #_{1,n} )</td>
<td>32</td>
<td>320</td>
<td>3584</td>
<td>43008</td>
<td>540672</td>
<td>7028736</td>
<td>93716480</td>
</tr>
</tbody>
</table>

In Section 1 we review some facts on Plücker coordinates of \( k \)-planes in projective space. In Section 2 we combine recent results in the real Schubert calculus with classical perturbation arguments adapted to the real numbers to prove Theorem 1. Since the proof for general \((k, n)\) is non-constructive, we give a symbolic, constructive proof for the case \((k, n) = (1, 3)\) in Section 3.

1. Preliminaries

We review the well-known Plücker coordinates of \( k \)-dimensional linear subspaces (hereafter \( k \)-planes) in complex projective space \( \mathbb{P}^n \) (see, e.g., [H]). Let \( U \) be a \( k \)-plane in \( \mathbb{P}^n \) which is spanned by the columns of an \((n + 1) \times (k + 1)\)-matrix \( L \). For every subset \( I \subseteq \{0, \ldots, n\} \) of size \( k + 1 \) let \( p_I \) be the \((k + 1) \times (k + 1)\)-subdeterminant of \( L \) given by the rows in \( I \) and let \( N := \binom{n + 1}{k + 1} - 1 \). Then \( p := (p_I)_{I \subseteq \{0, \ldots, n\}, |I| = k + 1} \in \mathbb{P}^N \) is the Plücker coordinate of \( U \). The set of all \( k \)-planes in \( \mathbb{P}^n \) is called the Grassmannian of \( k \)-planes in \( \mathbb{P}^n \) and is denoted by \( \mathbb{G}_{k,n} \). If the indices are written as ordered tuples, then the Plücker coordinates are skew-symmetric in the indices. \( \mathbb{G}_{k,n} \) is in 1-1-correspondence with the set of vectors in...
P\(^N\) satisfying the Plücker relations, i.e.,

\[ \sum_{i=1}^{k+1} (-1)^i p_{i_1...i_{k+1}} = 0 \]

for every \( I = \{i_1, \ldots, i_{k+1}\} \), \( J = \{j_1, \ldots, j_k\} \subset \{0, \ldots, n\} \) of strictly ordered index sets (where ‘‘ over an index means that it is omitted). See, e.g., [4, Theorem VII.5.I]. By Schubert’s results \([7]\), the dimension of \( \mathbb{G}_{k,n} \) is \( d_{k,n} = (k+1)(n-k) \) and its degree is \( \#_{k,n} \).

If an \((n-k-1)\)-plane \( V \) is given as the intersection of the \( k+1 \) hyperplanes
\[ \sum_{i=0}^{n} v_{i}^{(0)} x_{i} = 0, \ldots, \sum_{i=0}^{n} v_{i}^{(k)} x_{i} = 0, \]
then the dual Plücker coordinate \( q = (q_{I})_{I \subset \{0,\ldots,n\}, |I|=k+1} \in \mathbb{P}^{k,n} \) of \( V \) is defined by the \((k+1) \times (k+1)\)-subdeterminants of the matrix with columns \( v^{(0)}, \ldots, v^{(k)} \).

A \( k \)-plane \( U \) intersects an \((n-k-1)\)-plane \( V \) in \( \mathbb{P}^{n} \) if and only if the dot product of the Plücker coordinate \( p \) of \( U \) and the dual Plücker coordinate \( q \) of \( V \) vanishes, i.e., if and only if
\[ \sum_{I \subset \{0,\ldots,n\}, |I|=k+1} p_{I} q_{I} = 0 \]
(see, e.g., [4, Theorem VII.5.1]).

We use Plücker coordinates to characterize the \( k \)-planes tangent to a given quadric in \( \mathbb{P}^{n} \) (see \([11]\)). We identify a quadric \( x^{T}Qx = 0 \) in \( \mathbb{P}^{n} \) with its symmetric \((n+1) \times (n+1)\)-representation matrix \( Q \). Further, for \( r \in \mathbb{N} \) let \( \wedge^{r} \) denote the \( r \)-th exterior power of matrices

\[ \wedge^{r} : \mathbb{C}^{m \times n} \rightarrow \mathbb{C}^{{m \choose r} \times {n \choose r}} \]
(see \([11]\)). The row and column indices of the resulting matrix are subsets of cardinality \( r \) of \( \{1, \ldots, m\} \) and \( \{1, \ldots, n\} \), respectively. For \( I \subset \{1, \ldots, m\} \) with \(|I| = r \) and \( J \subset \{1, \ldots, n\} \) with \(|J| = r \), \((\wedge^{r}A)_{I,J}\) is the subdeterminant of \( A \) whose rows are indexed by \( I \) and whose columns are indexed by \( J \). If a \( k \)-plane \( U \subset \mathbb{P}^{n} \) is spanned by the columns of an \((n+1) \times (k+1)\)-matrix \( L \), then the \((k+1) \times 1\)-matrix \( \wedge^{k+1} L \), considered as a vector in \( \mathbb{P}^{n} \), is the Plücker coordinate of \( U \).

Recall the following algebraic characterization of tangency: A \( k \)-plane \( U \) is tangent to a quadric \( Q \) if the restriction of the quadratic form to \( U \) is singular (which includes the case \( U \subset Q \)). When the quadric is smooth, this implies that \( U \) is tangent to the quadric in the usual geometric sense.

**Proposition 2** (Proposition 5.5.3 of \([11]\)). A \( k \)-plane \( U \subset \mathbb{P}^{n} \) is tangent to a quadric \( Q \) if and only if the Plücker coordinate \( p_{U} \) of \( U \) satisfies
\[ p_{U}^{T} (\wedge^{k+1} Q) p_{U} = 0. \]

A \( k \)-flat in affine real space \( \mathbb{R}^{n} \) is a \( k \)-dimensional affine subspace in \( \mathbb{R}^{n} \). Throughout the paper we assume that \( \mathbb{R}^{n} \) is naturally embedded in the real projective space \( \mathbb{P}_{\mathbb{R}}^{n} \) via \((x_{1}, \ldots, x_{n}) \mapsto (1, x_{1}, \ldots, x_{n}) \in \mathbb{P}_{\mathbb{R}}^{n}\).

2. Proof of the main theorem

We first illustrate the essential geometric idea underlying our constructions for \((k, n) = (1, 3)\), which is the first nontrivial case. Here, Theorem \([1]\) states that there
exists a configuration of four quadrics in $\mathbb{R}^3$ with 32 distinct real common tangent lines.

By (2), the set of lines meeting four given lines in $\mathbb{P}^3$ is the intersection of four hyperplanes on the Grassmannian $G_{1,3}$, and hence there are at most two or infinitely many common lines meeting $\ell_1, \ldots, \ell_4$. If $e_1$ and $e_2$ are opposite edges in a tetrahedron $\Delta \subset \mathbb{R}^3$, then the lines underlying $e_1$ and $e_2$ are the two common transversals of the four lines underlying the other four edges (see Figure 1).

Consider the lines $\ell_1, \ldots, \ell_4$ as (degenerate) infinite circular cylinders with radius $r = 0$. When the radius is slightly increased, then the cylinders intersect pairwise in the regions (combinatorially) given by the four vertices of $\Delta$, and the common tangents roughly have the direction of $e_1$ or $e_2$. Since the neighborhood of a vertex is divided into four regions by the two cylinders, and since each region contains common tangents, this gives $4 \cdot 4$ tangents close to the direction of $e_1$ and $4 \cdot 4$ tangents close to the direction of $e_2$ (see Figure 1).

For the general case, let $1 \leq k \leq n - 2$. By Section 1, the number of $k$-planes in $\mathbb{P}^n$ simultaneously meeting $d_{k,n}$ general $(n-k-1)$-planes is $\#_{k,n}$. We begin with a configuration of $d_{k,n}$ real $(n-k-1)$-flats $U_1, \ldots, U_{d_{k,n}}$ in $\mathbb{R}^n$ having $\#_{k,n}$ real $(n-k-1)$-flats simultaneously meeting $U_1, \ldots, U_{d_{k,n}}$. We then argue that we can replace each of these $(n-k-1)$-flats by a real quadric such that for each of the $k$-flats, there are $2^{d_{k,n}}$ nearby real $k$-flats tangent to each quadric.

**Proposition 3.** For $1 \leq k \leq n - 2$, there exists a configuration of $d_{k,n}$ real $(n-k-1)$-flats $U_1, \ldots, U_{d_{k,n}}$ in $\mathbb{R}^n$ such that there exist exactly $\#_{k,n}$ real $k$-flats simultaneously meeting $U_1, \ldots, U_{d_{k,n}}$.

**Proof.** The corresponding statement for real projective space $\mathbb{P}^n$ was proven for $k = 1$ in [9, Theorem C] and for $k \geq 2$ in [10]. We deduce the affine counterpart above simply by removing a real hyperplane that contains none of the $(n-k-1)$-flats or any of the transversal $k$-flats. □
For $k = 1$, the purely existential statement in [9] and Proposition 3 was improved by Eremenko and Gabrielov [2] who gave the following explicit construction of such a collection of $(n-2)$-flats. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$, $\gamma(s) := (1, s, s^2, \ldots, s^{n-1})^T$ be the moment curve in $\mathbb{R}^n$. For each $s \in \mathbb{R}$, set $U(s)$ to be

$$U(s) := \text{affine span}(\gamma(s), \gamma'(s), \ldots, \gamma^{(n-3)}(s)).$$

Geometrically, $U(s)$ is the $(n-2)$-flat osculating the moment curve at the point $\gamma(s)$. By [2], for any distinct $s_1, \ldots, s_{2n-2} \in \mathbb{R}$, the $(n-2)$-flats $U(s_1), U(s_2), \ldots, U(s_{2n-2})$ have exactly $\#_{1,n} = C_{n-1}$ common real transversals, where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the $n$-th Catalan number. For general $k$, it is only known that there exist distinct $s_1, \ldots, s_{dk,n} \in \mathbb{R}$ such that there are $\#_{k,n}$ distinct real $k$-flats meeting the osculating $(n-k-1)$-flats to the moment curve at $s_1, \ldots, s_{dk,n}$ [10]. The conjecture on total reality in [8] §1 and §4 conjectures that any choice of distinct $s_1, \ldots, s_{dk,n} \in \mathbb{R}$ implies the reality of all transversal subspaces.

**Definition.** Suppose that $1 \leq k \leq n-2$, and let $U \subset \mathbb{R}^n$ be a $k$-flat and $r > 0$. The $k$-cylinder $Cy(U, r)$ is the set of points having Euclidean distance $r$ from $U$.

This quadratic hypersurface is smooth in $\mathbb{R}^n$ but its extension to $\mathbb{P}^n$ is singular. A $k'$-flat $V \subset \mathbb{R}^n$ is tangent to $Cy(U, r)$ if and only if its Euclidean distance to $U$ is $r$.

We will use the following basic property of intersection multiplicities [3] p. 1].

**Proposition 4.** Let $A$ be an algebraic curve in $\mathbb{P}^n$, and let $x$ be a singular point on $A$. For any hyperplane $H \subset \mathbb{P}^n$ such that $x$ is an isolated point in $A \cap H$, the intersection multiplicity of $A$ and $H$ in $x$ is greater than 1.

**Theorem 5.** Let $1 \leq k \leq n-2$, and let $U_1, U_2, \ldots, U_{dk,n}$ be $(n-k-1)$-flats in $\mathbb{R}^n$ having exactly $\#_{k,n}$ common transversal $k$-flats, all real. For each $i = 0, 1, \ldots, dk,n$, there exist $r_1, \ldots, r_i > 0$ such that there are exactly $2^i \cdot \#_{k,n}$ distinct $k$-flats, each of them real, that are simultaneously tangent to each of the $(n-k-1)$-cylinders $Cy(U_j, r_j)$, $j = 1, \ldots, i$, and also meet each of the $(n-k-1)$-flats $U_{i+1}, \ldots, U_{dk,n}$.

The case of $i = dk,n$ implies Theorem 1.

**Proof.** We induct on $i$, with the case of $i = 0$ being the hypothesis of the theorem.

Suppose that $i \leq dk,n$ and that there exist $r_1, \ldots, r_{i-1} > 0$ such that there are exactly $2^{i-1} \cdot \#_{k,n}$ distinct $k$-flats $V_1, \ldots, V_{2^{i-1} \#_{k,n}}$ which are simultaneously tangent to $Cy(U_j, r_j)$ for each $j = 1, \ldots, i-1$, and meet each of $U_1, \ldots, U_{dk,n}$, and each of these $k$-flats is real.

Now we drop the condition that the $k$-flats meet $U_i$. Let $A \subset \mathbb{G}_{k,n}$ be the curve of $k$-flats that are tangent to the cylinders $Cy(U_j, r_j)$ for $j = 1, \ldots, i-1$ and that also meet each of the $(n-k-1)$-flats $U_{i+1}, \ldots, U_{dk,n}$. Since $A$ is the intersection of $i-1$ quadrics (the tangency conditions) with $dk,n - i$ hyperplanes (conditions to meet the remaining $U_j$) on the Grassmannian, it has degree at most $2^{i-1} \#_{k,n}$. Since its intersection with the hyperplane defined by $U_i$ consists of $2^{i-1} \#_{k,n}$ points, we conclude that the degree of $A$ is $2^{i-1} \#_{k,n}$ and (by Proposition 4) that each of these points is a smooth point of $A$.

Let $V \in \{V_1, \ldots, V_{2^{i-1} \#_{k,n}}\}$. Since $V$ is a smooth real point of the real curve $A \subset \mathbb{G}_{k,n}$ (i.e., $V \subset \mathbb{P}^n$), the real points of $A$ contain a smooth arc $\alpha$ containing $V$ with $\alpha \cap \{(V_1, \ldots, V_{2^{i-1} \#_{k,n}}) \setminus V\} = \emptyset$. Let $\varphi : (-\delta, \delta) \to \alpha$ be a smooth parametrization
of the arc $\alpha$ with $\varphi(0) = V$. Such a parametrization exists, for example, by the Implicit Function Theorem.

Thus, for $t \in (-\delta, \delta) \setminus \{0\}$, the real $k$-flat $\varphi(t)$ does not meet $U_i$ and so it has a positive Euclidean distance $d(t)$ from $U_i$. Since $d(t)$ is a continuous function of $t$, for $\rho \in \mathbb{R}$ with $0 < \rho < \min\{d(-\delta/2), d(\delta/2)\}$ there are at least two distinct real $k$-flats in $\alpha$ whose Euclidean distance to $U_i$ is $\rho$.

In this way, we obtain $2^{i-1} \cdot \#_{k,n}$ such arcs, each containing one of $V_1, \ldots, V_{2^{i-1} \cdot \#_{k,n}}$. We may assume that these arcs are pairwise disjoint. Let $0 < r_i$ be small enough to ensure that each arc contains two $k$-flats having Euclidean distance $r_i$ from $U_i$. This gives at least $2 \cdot 2^{i-1} \cdot \#_{k,n}$ real $k$-planes in $A$ whose Euclidean distance to $U_i$ is $r_i$. Since $2^i \cdot \#_{k,n}$ is the maximum number of $k$-flats with this property, there are exactly $2^i \cdot \#_{k,n}$ distinct $k$-flats tangent to $\text{Cy}(U_j, r_j)$ for $j = 1, \ldots, i$ and that also meet each of the $(n-k-1)$-flats $U_{i+1}, \ldots, U_{dk,n}$.

Since the number of real $k$-flats will not change under a small perturbation of the $k$-cylinders $\text{Cy}(U_j, r_j)$, we may replace them by quadrics which are smooth in $\mathbb{P}^n$. Let $\text{sign}(Q)$ denote the signature of a quadric $Q \subset \mathbb{P}^n$.

**Corollary 6.** Let $1 \leq k \leq n - 2$. For

$$(s_1, \ldots, s_{dk,n}) \in \begin{cases} \{ (n-1, n-3, \ldots, 2k-n+1)^{d_{k,n}} \} & \text{if } k \geq n/2, \\ \{ (n-1, n-3, \ldots, 2 \cdot \left( \frac{n-1}{2} - \left\lfloor \frac{n-1}{2} \right\rfloor \right)^{d_{k,n}} \} & \text{if } k < n/2 \end{cases}$$

there exist smooth quadrics $Q_1, \ldots, Q_{dk,n} \subset \mathbb{P}^n_{\mathbb{R}}$ with $|\text{sign}(Q_i)| = s_i, 1 \leq i \leq dk,n$, such that the $\#_{k,n}$ (complex) common tangent $k$-flats to $Q_1, \ldots, Q_{dk,n}$ are all real, distinct, and lie in the affine space $\mathbb{R}^n$.

**Proof.** Since the absolute value of the signature of an $(n-k-1)$-cylinder is $k$, the proof immediately follows from the possible perturbations of the quadratic form in $\mathbb{P}^n$ of the type

$$-r^2 x_0^2 + x_1^2 + \cdots + x_{k+1}^2.$$ 

\[ \square \]

We conjecture that the reality statement holds for signatures not covered by Corollary 6.

**Conjecture 7.** Let $1 \leq k \leq n - 2$. For

$$(s_1, \ldots, s_{dk,n}) \in \{ (n-1, n-3, \ldots, 2 \cdot \left( \frac{n-1}{2} - \left\lfloor \frac{n-1}{2} \right\rfloor \right)^{d_{k,n}} \}$$

there exist smooth quadrics $Q_1, \ldots, Q_{dk,n} \subset \mathbb{P}^n_{\mathbb{R}}$ with $|\text{sign}(Q_i)| = s_i, 1 \leq i \leq dk,n$, such that the $\#_{k,n}$ (complex) common tangent $k$-flats to $Q_1, \ldots, Q_{dk,n}$ are all real, distinct, and lie in the affine space $\mathbb{R}^n$.

The first case of this conjecture which is not covered by Corollary 6 is when $k = 3$ and $n = 5$ and the signature is zero, that is, for 3-flats tangent to 8 smooth quadrics in $\mathbb{P}^5_{\mathbb{R}}$, with at least one having signature zero. We remark that an argument perturbing cylinders to singular quadrics gives an analog to Corollary 6 concerning $k$-flats tangent to singular quadrics. We omit its complicated formulation.
3. A constructive proof for lines in dimension 3

Our proof of Theorem 1 was non-constructive. We close this paper by providing a constructive proof in the first nontrivial case, \((k,n) = (1,3)\), i.e., the real lines tangent to four quadrics in 3-space. In order to realize the tetrahedral configuration of Figure 1 in \(\mathbb{P}^3\), let \(\ell_1, \ldots, \ell_4\) be given by the following equations:

\[
\ell_1 : x_0 = x_3 = 0; \quad \ell_2 : x_0 = x_1 = 0; \quad \ell_3 : x_1 = x_2 = 0; \quad \ell_4 : x_2 = x_3 = 0.
\]

The two common transversal lines are given by \(x_2 = x_4 = 0\) and by \(x_1 = x_4 = 0\).

For parameters \(\alpha, \beta \in \mathbb{R}\), consider the four quadrics

\[
Q_1 : x_0^2 + x_3^2 - \beta(x_1^2 + x_2^2) = 0, \\
Q_2 : x_0^2 + x_1^2 - \beta(x_2^2 + x_3^2) = 0, \\
Q_3 : x_1^2 + x_2^2 - \alpha(x_0^2 + x_3^2) = 0, \\
Q_4 : x_2^2 + x_3^2 - \alpha(x_0^2 + x_1^2) = 0.
\]

For \(\alpha = \beta = 0\), the four quadrics become the corresponding lines in \(\mathbb{P}^3\). For small \(\alpha, \beta > 0\), these quadrics are deformations of the lines with rank 4 and signature 0—smooth ruled surfaces.

**Theorem 8.** Let \((\alpha, \beta) \in \mathbb{R}^2\) satisfy

\[
\alpha \beta(1 - \alpha \beta)(1 - \beta^2)(1 - \alpha^2)((1 - \alpha)^2(1 - \beta)^2 - 16 \alpha \beta) \neq 0.
\]

Then there are 32 distinct (possibly complex) common tangent lines to \(Q_1, \ldots, Q_4\).

If \(0 < \alpha, \beta < 3 - 2\sqrt{2}\), then each of these 32 tangent lines is real.

**Proof.** Since the quadrics only contain monomials of the form \(x_i^2\), the tangent equations of \(Q_1, \ldots, Q_4\) only contain monomials of the form \(p_{ij}^2\). Hence, the four tangent equations give the following system of linear equations in \(p_{01}^2, \ldots, p_{23}^2\):

\[
\begin{pmatrix}
-\beta & -\beta & 1 & \beta^2 & -\beta & -\beta \\
1 & -\beta & -\beta & -\beta & \beta^2 & -\beta \\
-\alpha & -\alpha & \alpha^2 & 1 & -\alpha & -\alpha \\
\alpha^2 & -\alpha & -\alpha & -\alpha & -\alpha & 1
\end{pmatrix}
\begin{pmatrix}
p_{01}^2 \\
p_{02}^2 \\
p_{03}^2 \\
p_{12}^2 \\
p_{13}^2 \\
p_{23}^2
\end{pmatrix} = 0.
\]

Permute the variables into the order \((p_{02}, p_{13}, p_{03}, p_{12}, p_{01}, p_{23})\). For \(\alpha, \beta \in \mathbb{R}\) satisfying

\[
\alpha \beta(1 - \alpha \beta)(1 + \beta)(1 + \alpha) \neq 0,
\]

Gaussian elimination yields the following system:

\[
\begin{pmatrix}
-\beta & -\beta & (1 - \alpha)(1 - \beta) & 0 & 0 & 0 \\
0 & 0 & \alpha & -\beta & 0 & 0 \\
0 & 0 & 0 & -\beta & \alpha & 0 \\
0 & 0 & 0 & 0 & \alpha & -\beta
\end{pmatrix}
\begin{pmatrix}
p_{02}^2 \\
p_{13}^2 \\
p_{03}^2 \\
p_{12}^2 \\
p_{23}^2 \\
p_{01}^2
\end{pmatrix} = 0.
\]
Together with the Plücker equation (1), this gives the following system of equations:

\begin{align}
(6) & \quad -\beta p_{02}^2 - \beta p_{13}^2 + (1 - \alpha)(1 - \beta)p_{03}^2 = 0, \\
(7) & \quad p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0, \\
(8) & \quad \alpha p_{01}^2 = \alpha p_{03}^2 = \beta p_{12}^2 = \beta p_{23}^2.
\end{align}

For $\alpha, \beta$ satisfying (3) as well as $(1 - \alpha)(1 - \beta) \neq 0$, we distinguish the following three disjoint cases.

**Case 1:** $p_{02} = 0$.

Since $p_{13} = 0$ would imply that all components are zero and hence contradict $(p_{01}, \ldots, p_{23})^T \in P^5$, we can assume $p_{13} = 1$. Then (6) and (8) imply

$$\frac{\alpha^2}{(1 - \alpha)(1 - \beta)} = \frac{\alpha\beta}{(1 - \alpha)(1 - \beta)} \neq 0.$$

Since (7) implies $p_{01}p_{23} = -p_{03}p_{12}$, only 8 of the $2^4 = 16$ sign combinations for $p_{01}, p_{03}, p_{12}, p_{23}$ are possible. Namely, the 8 (complex) solutions for $p_{01}, p_{03}, p_{12}, p_{23}$ are

$$\left( p_{01}, p_{03}, p_{12}, p_{23} \right)^T = \frac{1}{\sqrt{(1 - \alpha)(1 - \beta)}} \left( \gamma_{01} \sqrt{\beta}, \gamma_{03} \sqrt{\beta}, \gamma_{12} \sqrt{\alpha}, -\gamma_{01} \gamma_{03} \gamma_{12} \sqrt{\alpha} \right)^T$$

with $\gamma_{01}, \gamma_{03}, \gamma_{12} \in \{-1, 1\}$. Hence, for $\alpha, \beta \in \mathbb{R}$ satisfying (3), this case gives 8 distinct common tangents.

**Case 2:** $p_{13} = 0$.

This case is symmetric to case 1. Setting $p_{02} = 1$, the resulting 8 solutions for the variables $p_{01}, p_{03}, p_{12}, p_{23}$ are the same ones as in (9).

**Case 3:** $p_{02}p_{13} \neq 0$.

Without loss of generality, we can assume $p_{02} = 1$. Solving (7) for $p_{13}$ and substituting this expression into (8) yields

$$-\beta - \beta p_{01}^2 p_{23} - \beta p_{03}^2 p_{12} - 2\beta p_{01}p_{03}p_{12}p_{23} + (1 - \alpha)(1 - \beta)p_{03}^2 = 0.$$

We use (8) to write this in terms of $p_{01}$. This is straightforward for the squared terms, but for the other terms, we observe that, by (8), $p_{01}p_{23} = \pm p_{03}p_{12}$ and since $p_{02}p_{13} \neq 0$, the Plücker equation (1) implies these have the same sign. This gives the quartic equation in $p_{01}$,

$$-\beta + (1 - \alpha)(1 - \beta)p_{01}^2 - 4\alpha p_{01}^4 = 0,$$

whose discriminant is

$$(1 - \alpha)^2(1 - \beta)^2 - 16\alpha \beta.$$

Hence, for $\alpha, \beta \in \mathbb{R}$ satisfying (3), and for which this discriminant does not vanish, there are two different solutions for $p_{01}^2$. For each of these two solutions for $p_{01}^2$, there are 8 distinct solutions for $p_{01}, p_{03}, p_{12}, p_{23}$, namely

$$\left( p_{01}, p_{03}, p_{12}, p_{23} \right)^T = \sqrt{p_{01}^2} \left( \gamma_{01}, \gamma_{03}, \gamma_{12}, \gamma_{01}\gamma_{03}\gamma_{12} \right)^T$$

with $\gamma_{01}, \gamma_{03}, \gamma_{12} \in \{-1, 1\}$. Since $p_{13}$ is uniquely determined by $p_{01}, p_{02}, p_{03}, p_{12}$, case 3 gives 16 distinct common tangents.

In order to determine when all solutions are real, suppose first that $\alpha = \beta$. Then the discriminant (10) becomes $(\alpha^2 - 6\alpha + 1)(\alpha + 1)^2$, and its smallest positive root is
\( \alpha_0 := 3 - 2\sqrt{2} \approx 0.17157 \). In particular, for \( 0 < \alpha < \alpha_0 \), the discriminant in case 3 is positive and both solutions for \( p_{01}^2 \) are positive. Thus, for \( 0 < \beta = \alpha < \alpha_0 \), the solutions of all three cases are distinct and real. Next, fix \( 0 < \alpha < \alpha_0 \) and suppose that \( 0 < \beta < \alpha \). Then the discriminant (10) is positive: for fixed \( 0 < \alpha < \alpha_0 \), the discriminant (10) is decreasing in \( \beta \) for \( 0 < \beta < \alpha \) and positive when \( \beta = \alpha \). This concludes the proof of Theorem 8. \( \square \)

Figure 2 illustrates the construction and the 32 tangents for \( \alpha = 1/10 \) and \( \beta = 1/20 \).

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\textbf{References}


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