BOUNDS ON THE NUMBER OF HOLOMORPHIC MAPS
OF COMPACT RIEMANN SURFACES

MASAHARU TANABE

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Abstract. We give bounds on the number of nonconstant holomorphic maps
of compact Riemann surfaces of genera $g > 1$.

1. Introduction

Let $X$ be a compact Riemann surface of genus $g (> 1)$. De Franchis \[F\] stated
the following:

Theorem of de Franchis. (a) For a fixed compact Riemann surface $Y$ of genus
$g > 1$, the number of nonconstant holomorphic maps $X \to Y$ is finite.

(b) There are only finitely many compact Riemann surfaces $Y_i$ of genus $g > 1$
which admit a nonconstant holomorphic map from $X$.

The second statement (b) is often attributed to Severi. After knowing the finite-
ess of maps, we may ask if there exists an upper bound depending only on some
topological invariant, for example, the genus $g$. Related to the statement (a),
Martens \[Mr\] showed if one fixes a Riemann surface $Y$, then the number of all
nonconstant holomorphic maps from $X$ to $Y$ is less than $(cg)^{2g^2}$ for some constant
$c > 1$ independent of $g$. Recently, the author \[T2\] showed that the bound is smaller
than $(cg)^{2g}$ for some constant $c$. Now, we consider a bound for holomorphic maps
when $Y$ is not fixed, that is, we estimate the number of all nonconstant holomorphic
maps from $X$ to other Riemann surfaces. Let $f_i : X \to Y_i$ be nonconstant holomor-
phic maps for $i = 1, 2$. We say that $f_1$ and $f_2$ are isomorphic if and only if there
is a conformal map $h : Y_1 \to Y_2$ such that $h \circ f_1 = f_2$. Let $I_i(X)$ denote the set
of all isomorphic classes of nonconstant holomorphic maps into compact Riemann
surfaces of genus $\gamma > 1$, and denote $\mathcal{I}(X) = \bigcup_{\gamma > 1} I_\gamma(X)$. By the theorem of
de Franchis, we see that $\sharp \mathcal{I}(X)$ is finite. In 1983 Howard and Sommese \[H-S\] first
showed that there is a bound on $\sharp \mathcal{I}(X)$ depending only on $g$. In 1986 Kani \[K\]
gave a bound of morphisms from a smooth projective curve defined over a field $K$,
depending only on the genus of the curve. In particular, when $K = \mathbb{C}$, he showed that

$$\sharp \mathcal{I}(X) < (g - 1)2^{2g^2 - 2}(2^{2g^2} - 1),$$

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and in 1990 Alzati and Pirola [A-P] gave a better bound
\[ \sharp \mathcal{I}(X) < \exp\{ (4/3)(g^2 - 1) \log 3 + |\log_2 g| \log (84g) + \log (12\sqrt{2}) \}. \]

Here, we will improve them.

**Theorem.** Let $X$ be a compact Riemann surface of genus $g > 1$. Then the number $\sharp \mathcal{I}(X)$ satisfies
\[ \sharp \mathcal{I}(X) < (2g)^{4g} \times 2^{2g-3} \times (2g - 1)^{g-1} (2g - 3)(g - 2)(g - 1). \]

We will show this by estimating the possible number of pull backs of harmonic differentials in $H^1(Y, \mathbb{Z})$. We may see that $H^2(Y, \mathbb{Z})$ is the lattice of the dual Jacobian variety $J(Y)$. Thus, we will use theory of homomorphisms of Jacobians and theory of lattices from the geometry of numbers.

Let $M(g) = \max_X \{ \sharp \mathcal{I}(X) \}$, where the maximum is taken over all Riemann surfaces $X$ of genus $g$. It is an interesting problem to determine the exact rate of growth of $M(g)$. By the Theorem, we see
\[ M(g) \leq (cg)^{5g} \]
for some constant $c$, while all the previously given ones (by Kani and Alzati-Pirola) guarantee $M(g) \leq c g^2$. On the other hand, Kani [K] also constructed a sequence $F_1, F_2, \ldots, F_n, \ldots$ of function fields over $\mathbb{K}$ of genus $g_{F_1} < g_{F_2} < \ldots < g_{F_n} < \ldots$, such that the number of separable subfields of $F_n/\mathbb{K}$ is larger than $\exp (c(\log(g_{F_n}))^2)$ for some constant $c > 0$ (independent of $n$). This implies that $M(g)$ cannot be bounded by any polynomial in $g$.

**2. Notation and Lemmata**

In this paper, all of the Riemann surfaces will be compact and of genera greater than 1. First, we recall some notions from complex tori. Let $V$ be a complex vector space and $\Gamma$ a lattice in $\mathbb{C}$. The quotient $T = V/\Gamma$ is called a complex torus. Denote by $\widehat{T} = V^*/\hat{\Gamma}$ the dual where $V^*$ is the space of $\mathbb{C}$-antilinear functionals on $V$ and $\hat{\Gamma} = \{ l \in V^* : \text{Im}(l) \subseteq \mathbb{Z} \}$ is the dual lattice of $\Gamma$. Let $f$ be a homomorphism between two complex tori $T = V/\Gamma$ and $T' = V'/\Gamma'$. Then, there is a unique $\mathbb{C}$-linear map $F : V \rightarrow V'$ with $F(\Gamma) \subseteq \Gamma'$ inducing $f$. We call $F$ the analytic representation of $f$, and the restriction $F|_\Gamma$ the rational representation of $f$. For the analytic representation $F : V \rightarrow V'$ of a homomorphism $f : T \rightarrow T'$, the dual map $\overline{f} : V^* \rightarrow V'^*$ associating to an antilinear functional $l \in V'^*$ the antilinear functional $l \circ F \in V^*$ induces a homomorphism $\overline{f} : \widehat{T} \rightarrow \widehat{T}'$, since $F(\hat{\Gamma}) \subseteq \hat{\Gamma}'$. We call $\overline{f}$ the dual map of $f$. Let $X$ and $Y$ be compact Riemann surfaces of genera $g$ and $\gamma$, respectively. Denote by $\mathcal{H}$ the space of holomorphic differentials on $X$. Set $\Omega = \text{Hom}(\mathcal{H}, \mathbb{C})$. The Jacobian variety $J(X) := \Omega/H_1(X, \mathbb{Z})$ is a complex torus of dimension $g$, and considering $\mathcal{H}$ of $\mathbb{C}$-antilinear forms on $\Omega$, we will denote by $J(\widehat{X}) = \overline{\mathcal{H}}/H^1(X, \mathbb{Z})$ the dual. In order to describe $J(X)$ in terms of period matrices, choose basis $\lambda_1, \ldots, \lambda_{2g}$ of $H_1(X, \mathbb{Z})$ and $\omega_1, \ldots, \omega_g$ of $\mathcal{H}$. Let $l_1, \ldots, l_g$ be the basis of $\Omega$ dual to $\omega_1, \ldots, \omega_g$, i.e., $l_i(\omega_j) = \delta_{ij}$ (Cronecker’s delta). Considering
\[ \lambda_j \text{ as a linear form on } \mathcal{H}, \text{ we have } \lambda_j = \sum_{k=1}^{g} (\int_{\lambda_j} \omega_k)l_k \text{ for } j = 1, \ldots, 2g. \]

Hence

\[
\Pi_X = \begin{pmatrix}
\int_{\lambda_1} \omega_1 & \cdots & \int_{\lambda_2} \omega_1 \\
\vdots & & \vdots \\
\int_{\lambda_1} \omega_g & \cdots & \int_{\lambda_2} \omega_g
\end{pmatrix}
\]

is a period matrix for \( J(X) \) with respect to these bases. Similarly, we define \( J(Y), \ \hat{J}(\hat{Y}) \) and \( \Pi_Y \) for \( Y \). Let \( f : J(X) \to J(Y) \) be a homomorphism. In terms of matrices, \( f \) can be expressed as

\[
A\Pi_X = \Pi_Y M
\]

with \( A \in M(\gamma, g; \mathbb{C}), \ M \in M(2\gamma, 2g; \mathbb{Z}) \) (we denote by \( M(m, n; K) \) the set of \( m \times n \) matrices with \( K \)-coefficients). Conversely, if there are matrices \( A \in M(\gamma, g; \mathbb{C}) \) and \( M \in M(2\gamma, 2g; \mathbb{Z}) \) such that \( A\Pi_X = \Pi_Y M \), then these matrices are matrix representations of some homomorphism \( J(X) \to J(Y) \). We also call \( A \) the analytic representation of \( f \), and \( M \) the rational representation of \( f \) (with respect to the bases). There is a canonical principal polarization on \( J(X) \). Fix a homology basis \( \lambda_1, \ldots, \lambda_{2g} \) of \( H_1(X, \mathbb{Z}) \) with an intersection matrix

\[
J = \begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix}
\]

where each entry is \( g \times g \) sized. Considered as a \( \mathbb{R} \)-vector space, \( \lambda_1, \ldots, \lambda_{2g} \) is a basis for \( \Omega \). Denote by \( E \) the alternating form on \( \Omega \) with the matrix \( J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \) with respect to the basis \( \lambda_1, \ldots, \lambda_{2g} \) for \( \Omega \) and define a hermitian form \( H : \Omega \times \Omega \to \mathbb{C} \) by

\[
H(u, v) = E(\dot{i}u, v) + iE(u, v).
\]

We denote by \( \langle \zeta, v \rangle, \zeta \in \mathcal{H}, v \in \Omega \) the value of \( \zeta \) at \( v \). The \( \mathbb{C} \)-linear map \( \phi_E : \Omega \to \mathcal{H} \) such that \( H(u, v) = \langle \phi_E(u), v \rangle \) induces a homomorphism, also denoted by \( \phi_E \),

\[
\phi_E : J(X) \to \hat{J}(\hat{X}).
\]

The matrix \( -J = J^{-1} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \) is the rational representation of \( \phi_E \) with respect to the homology basis \( \lambda_1, \ldots, \lambda_{2g} \) for \( H_1(X, \mathbb{Z}) \) above and the dual basis \( \alpha_1, \ldots, \alpha_{2g} \) for \( H^1(X, \mathbb{Z}) \), i.e., \( \alpha_j(\lambda_k) = \delta_{jk} \). Also, we can transport \( E \) to \( \mathcal{H} \) by defining

\[
E'(\phi_E(u), \phi_E(v)) = E(u, v).
\]

We will obtain a bound for \( \mathcal{H}(X) \) by counting the possible numbers of pull backs of holomorphic differentials. Thus, we will mainly deal with the dual Jacobian varieties and the endomorphisms of them. By an underlying real structure for a \( g \)-dimensional complex torus \( T = V/\Gamma \), we mean the real torus \( \mathbb{R}^{2g}/\mathbb{Z}^{2g} \) together with a map \( \mathbb{R}^{2g}/\mathbb{Z}^{2g} \to T \) induced by a linear map \( \mathbb{R}^{2g} \ni x \mapsto \Pi x \in \mathbb{C}^g \) where \( \Pi \) is a period matrix. For Jacobian varieties, it is known that the real part \( E(iu, v) \) of the hermitian \( H(u, v) \) is symmetric positive. Here, we define an inner product.

**Definition.** On \( \mathbb{R}^{2g} \), which is over the underlying real structure for \( \hat{J}(\hat{X}) \), we define an inner product by

\[
(x, y)_X = E'(i\Pi x, \Pi y),
\]
then

For an arbitrary Jacobians. Put \( f \) holomorphic map

\[
\text{Lemma 2. Let } x \text{ for an arbitrary vector } \hat{\phi}.
\]

Proof. The assumption means that there exist holomorphic differentials \( \phi_1 \) on \( Y_1 \) and \( \phi_i \) on \( Y_i \) such that their pull backs satisfy \( f_1 \gamma \phi_1 = f_i \gamma \phi_i \).

Then, for a zero \( p_{01} \) of \( \phi_1 \), the number of possible \( f_1^{-1}(p_{01}) \) (counting multiplicities) that can occur is at most \( (2g_d - 2) \). After determining \( \phi = f_1 \gamma \phi_1 \) and \( f_1^{-1}(p_{01}) \), we can show that there are at most \( (2g_d - 2) \) possible isomorphic classes of holomorphic maps of degree \( d \) as follows.
Let \( f_i : X \rightarrow Y_i \) be holomorphic maps \((i = 1, 2)\). Suppose that there are holomorphic differentials \( \phi_1 \) and \( \phi_2 \) on \( Y_1 \) and \( Y_2 \), respectively, with \( f_1^* \phi_1 = f_2^* \phi_2 \), and there is a zero \( p_{01} \) (resp. \( p_{02} \)) of \( \phi_1 \) (resp. \( \phi_2 \)) satisfying \( f_1^{-1}(p_{01}) = f_2^{-1}(p_{02}) \).

We put \( \phi = f_1 \phi_1 = f_2 \phi_2 \). Let \( \tilde{p}_0 \in f_1^{-1}(p_{01}) = f_2^{-1}(p_{02}) \). Take a sufficiently small neighbourhood \( U_{\tilde{p}_0} \) (resp. \( U_{p_{0i}} \)) of \( \tilde{p}_0 \) (resp. \( p_{0i} \)) so that there is no zero of \( \phi \) (resp. \( \phi_i \)) on \( U_{\tilde{p}_0} \) (resp. \( U_{p_{0i}} \)) except \( \tilde{p}_0 \) (resp. \( p_{0i} \)), and that \( f_i(U_{\tilde{p}_0}) \subset U_{p_{0i}} \) \((i = 1, 2)\). We may take a local coordinate \( z \) (resp. \( z_i \)) on \( U_{\tilde{p}_0} \) (resp. \( U_{p_{0i}} \)) such that \( z(\tilde{p}_0) = 0 \) (resp. \( z_i(p_{0i}) = 0 \)) and the differential is written as

\[
\phi = z^m dz \quad \text{(resp. } \phi_i = z_i^n dz_i).\]

Recalling that \( f_1^{-1}(p_{01}) = f_2^{-1}(p_{02}) \), we see \( n_1 = n_2 \) and we will denote it by \( n \) for brevity. We take two real lines \( \gamma_i : [0, a) \rightarrow U_{p_{0i}} \) with \( \gamma_i(t) = t \in \mathbb{R} \) in the local coordinates \( z_i \) \((i = 1, 2)\). For an arbitrary \( \tilde{p} \in U_{\tilde{p}_0} \setminus \{ \tilde{p}_0 \} \),

\[
\int_{0}^{\tilde{p}} z^m dz = \int_{0}^{f_1(\tilde{p})} z_1^n dz_1 = \int_{0}^{f_2(\tilde{p})} z_2^n dz_2,
\]

hence the number of possible positions for the set of lifts of \( \gamma_1 \) (thus also those of \( \gamma_2 \)) in \( U_{\tilde{p}_0} \) is at most \( m + 1 \leq 2g - 1 \). Accordingly, the total number of possible positions for the set of all the lifts of \( \gamma_1 \) is at most \((2g - 1)^d \). Let \( \{ p_{0j} \}_{j=1}^{N} = f_1^{-1}(p_{01})(= f_2^{-1}(p_{02})) \). Suppose that, for every \( p_{0j} \in f_1^{-1}(p_{01}), U_{p_{0j}} \cap f_1^{-1}(\gamma_1) = U_{p_{0j}} \cap f_2^{-1}(\gamma_2) \), that is, the set of lifts of \( \gamma_1 \) coincide with that of \( \gamma_2 \). Then, it is easy to see that we can define a local conformal map \( h : f_1(U_{p_{0j}}) \rightarrow f_2(U_{p_{0j}}) \) such that \( h \circ f_1 \mid _{U_{p_{0j}}} = f_2 \mid _{U_{p_{0j}}} \). We want to extend it to a global conformal map from \( Y_1 \) to \( Y_2 \), and actually it is possible. Indeed, for an arbitrary point \( p \in Y_1 \), we will draw a curve \( c \) from \( p_{01} \) to \( p \) avoiding branch points of \( f_1 \) other than possibly at \( p_{01} \) and \( p \). Let \( \tilde{c} \) and \( \tilde{c}' \) be two lifts of \( c \) by \( f_1 \). Then, we see that \( f_2(\tilde{c}) = f_2(\tilde{c}') \) since \( h \circ f_1 \) is well-defined near \( p_{0j} \) \((j = 1, \ldots, N)\). This implies that \( h \) is well-defined on \( Y_1 \). It is easy to see that \( h \) is invertible.

\[
\square
\]

3. A Bound for \( I(X) \)

Now, we will recall the notion of successive minima which is a basic tool in the geometry of numbers (see e.g. [3]). In \( n \)-dimensional real vector space, let \( \Lambda \) be a lattice, that is, the set of all points

\[
x = u_1 a_1 + \cdots + u_n a_n \in \Lambda
\]

with integers \( u_1, \ldots, u_n \), and fixed linearly independent vectors \( a_1, \ldots, a_n \). Let \( F(x) \) be a distance function, namely, \( F(x) \) is a non-negative and continuous function with \( F(tx) = t F(x) \) \((t \geq 0)\). The \( k \)-th successive minimum \( \lambda_k \) of the distance function \( F \) with respect to the lattice is the lower bound of the numbers \( \lambda \) such that \( \{||x|| < \lambda \} \) contains \( k \) linearly independent lattice points. In this paper, we take \( \Lambda = \{x \in \mathbb{Z}^{2g}\} \), and the distance function \( F(x) = ||x|| = \sqrt{(x,x)_X} \).

Notation. Let \( a_1, a_2, \ldots, a_{2g} \) be linearly independent points of the lattice such that \( ||a_k|| = \lambda_k \) for \( 1 \leq k \leq 2g \), where \( \lambda_k \) is the \( k \)-th successive minimum.

We estimate the possible number of pull backs of an element in \( H^1(X, \mathbb{Z}) \), that is, the lattice of the dual Jacobian variety, to count the number of holomorphic maps from \( X \). For this purpose, matrix representations are effective. In the following,
rational representations of endomorphisms are considered to be matrix representations with respect to the homology basis such that the alternating form $E$ is expressed as $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ with them.

**Proposition.** Let $f_i : X \to Y_i$ be nonconstant holomorphic maps, and $F_i$ the rational representations of the endomorphisms associated with $f_i$ $(i = 1, 2)$. Suppose that, for some $k < 2g$,

\[
\begin{align*}
\{ \{F_1 a_1 = \cdots = F_1 a_{k-1} = 0, \\
\{F_2 a_1 = \cdots = F_2 a_{k-1} = 0,
\end{align*}
\]

and that there exists some integer $l > (2g - 2)^2$ such that $F_1 a_k \equiv F_2 a_k \pmod{l}$ holds. Then $F_1 a_k = F_2 a_k$. If, in addition, $Y_1$ and $Y_2$ are of the same genus $\gamma$, then the assumption $l > (2g - 2)^2$ can be replaced by $l > (2g - 2)^2/(\gamma - 1)^2$.

**Remark.** Here, $k$ may be equal to 1. In this case, assumption (5) is not needed.

**Proof.** Let $D = F_1 - F_2$. Then, $D$ is the rational representation of some endomorphism of $J(X)$. It suffices to show that

\[
\tag{6} \int D^t D a_k = 0.
\]

Indeed, using formula (2) in Lemma 1, we have

\[
\int (F_1 a_k, F_2 a_k)_X = (\int D^t D a_k, a_k)_X,
\]

and because of the property of inner products, (6) implies that $\int D a_k = 0$ which means $F_1 a_k = F_2 a_k$.

To show (6), we estimate the norm of the vector $\int D^t D a_k$. First, we note that $\int D^t D x, a_1, \ldots, a_{k-1}$ are linearly independent for any vector $x \in \mathbb{R}^{2g}$ if $\int D^t D x$ is not zero. Indeed, using (2) again, we see $(\int D^t D x, a_j)_X = (\int D x, a_j)_X = 0$ for $j = 1, \ldots, k - 1$ by the assumption. Thus, $\int D^t D x, a_1, \ldots, a_{k-1}$ are linearly independent. By the assumption $\int D a_k \equiv 0 \pmod{l}$, the vector $\int D^t D a_k$ can be written in the form $\int D^t D a_k = l \times n$, where $n \in \mathbb{Z}^{2g}$. Thus, if it is not 0, then

\[
\tag{7} \| \int D^t D a_k \| \geq l \lambda_k,
\]

since the $n$ is a point of the lattice and not in the span of $\{a_1, \ldots, a_{k-1}\}$. Next, we will give an upper bound for $\| \int D^t D a_k \|$. Set $\tilde{a} = \int D a_k$. By an easy calculation, we see that $F_i' = F_i$ for $i = 1, 2$. Thus,

\[
\| \int D^t \tilde{a} \| \leq \| \int F_1 \tilde{a} \| + \| \int F_2 \tilde{a} \| \leq d_1 \| \tilde{a} \| + d_2 \| \tilde{a} \|,
\]

where $d_i$ is the degree of $f_i$ $(i = 1, 2)$. The first inequality is just the triangle inequality, and the second one is obtained by (4) in Lemma 2. Using Lemma 2 again, we have

\[
\| \tilde{a} \| = \| (\int F_1 - \int F_2) a_k \| \leq \| \int F_1 a_k \| + \| \int F_2 a_k \| \leq \| a_k \| (d_1 + d_2).
\]

Therefore, we have

\[
\| \int D^t D a_k \| \leq \| a_k \| (d_1 + d_2)^2 = (d_1 + d_2)^2 \lambda_k.
\]

By the Riemann-Hurwitz formula, $d_i \leq g - 1$ and the right-hand side of (8) is $\leq 2^2 (g - 1)^2 \lambda_k$. Using the inequality (7), we see that $\int D^t D a_k$ must be 0 since
\( l > 2^2(g-1)^2 \). If, in addition, \( Y_1 \) and \( Y_2 \) are of the same genus \( \gamma \), then \( d_i \leq (g-1)/(\gamma-1) \) and the right-hand side of (8) is \( \leq 2^2(g-1)^2/(\gamma-1)^2\lambda_k \). Now, the proof is completed since (6) is shown.

We will say that \( F_i \in M(2g, 2g; \mathbb{Z}) \) is of the \( k \)-th type if \( ^tF_1 a_1 = \cdots = ^tF_1 a_{k-1} = 0 \) and \( ^tF_0 a_k \neq 0 \).

**Proof of the Theorem.** To make the calculation easier, first, we fix the genus \( \gamma \) of target surfaces and the degree \( d \) of maps. Then we note that the number of possible types for associated endomorphisms \( F_i \) is at most \( 2g-2\gamma+1 \) since the rank of each \( F_i \) is \( 2\gamma \). For a fixed endomorphism \( F_i \) of \( k \)-th type, the number of possible \( k \)-th rows is at most \( ((2g-2)/\gamma+1)^{2g} \) by the Proposition. If \( F_1 \) and \( F_2 \) have the same \( k \)-th row, then \( ^tF_1 \) and \( ^tF_2 \) have the same \( k \)-th column. Put \( f_i : J(\mathcal{X}) \to J(Y_i) \) the homomorphism induced by \( f_i \) and \( F_i \) the rational representation of \( f_i \) (\( i = 1, 2 \)). We can write \( f_i = ^tF_i \gamma F_i J_i F_i \) for \( i = 1, 2 \). Let \( e_k \) be a vector in \( \mathbb{R}^{2g} \) whose \( k \)-th entry is 1 and others are zero. Then, we see \( ^tF_1 \gamma F_1 J_i e_k = ^tF_2 \gamma F_2 J_i e_k \) and this means \( x_i := J_i \gamma F_1 J_i J_i x_i \in H^1(Y_i, \mathbb{Z}) \) (\( i = 1, 2 \)) satisfy \( ^tF_1 x_1 = ^tF_2 x_2 \). This implies that \( ^tF_1 (\tilde{\Pi}_1 x_1) = ^tF_2 (\tilde{\Pi}_2 x_2) \), where \( \tilde{\Pi}_1 \) (resp. \( \tilde{\Pi}_2 \)) is the period matrix for \( J(Y_1) \) (resp. \( J(Y_2) \)). Applying Lemma 3, we see the number of all isomorphic classes of nonconstant holomorphic maps into compact Riemann surfaces of genus \( \gamma > 1 \) smaller than \( g \) satisfies,

\[
\sharp \mathcal{I}_\gamma(\mathcal{X}) < \sum_{d>1} (2g-2\gamma+1) \times \left\{ \left( \frac{2g-2}{\gamma-1} \right)^2 + 1 \right\}^{2g} \times \left( \frac{2g-2}{d} \right) \times (2g-1)^d
\]

\[
< \left\{ \left( \frac{2g-2}{\gamma-1} \right)^2 + 1 \right\}^{2g} \times 2^{2g-2} \times (2g-1)^{\frac{d+1}{2}} \times (2g-2\gamma+1)(g-\gamma)/(\gamma-1).
\]

Now, we obtain (1):

\[
\sharp \mathcal{I}(\mathcal{X}) = \sum_{g>\gamma>1} \sharp \mathcal{I}_\gamma(\mathcal{X}) < (2g)^{4g} \times 2^{2g-3} \times (2g-1)^{g-1}(2g-3)(g-2)(g-1).
\]

□

**References**


Department of Mathematics, Tokyo Institute of Technology, Ohokayama, Meguro, Tokyo, 152-8551, Japan
E-mail address: tanabe@math.titech.ac.jp