PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 10, Pages 2817–2820
S 0002-9939(05)07905-0
Article electronically published on March 24, 2005

CENTRALIZER SIZES AND NILPOTENCY CLASS IN LIE ALGEBRAS AND FINITE p-GROUPS

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(Communicated by Jonathan I. Hall)

Abstract. In this work we solve a conjecture of Y. Barnea and M. Isaacs about centralizer sizes and the nilpotency class in nilpotent finite-dimensional Lie algebras and finite p-groups.

1. Introduction

By a well-known result of N. Ito [3], a finite group with only two conjugacy class sizes (one of which must be 1) is nilpotent. Recently K. Ishikawa [2] showed that the nilpotency class of such a group is at most 3. This result was an open question for many years. A simplification of Ishikawa's argument was given by M. Isaacs, and his proof is presented in [1]. Note that the number of different conjugacy class sizes in a finite group G is equal to the number of different orders of centralizers of elements of G. This observation allows us to consider possible analogs of Ito's and Ishikawa's theorems for Lie algebras: What can be said about a finite-dimensional Lie algebra L if the centralizer subalgebras $C_L(x)$ have just two different dimensions as x runs over the elements of L? This question was considered by Y. Barnea and M. Isaacs in [1]. In particular, when L is nilpotent they proved that, as in Ishikawa's Theorem, the nilpotency class of L is at most 3.

If L is a finite-dimensional non-abelian Lie algebra, then denote by $\mathrm{cd}(L)$ the set of dimensions of centralizers of noncentral elements of L. Denote by $\Delta(L)$ the difference $\max(\mathrm{cd}(L)) - \min(\mathrm{cd}(L))$. Similarly, if G is a non-abelian finite p-group, denote by $\mathrm{cs}(G)$ the set of orders of centralizers of noncentral elements of G and let $\Delta(G)$ be the difference $\log_p \max(\mathrm{cs}(G)) - \log_p \min(\mathrm{cs}(G))$. Thus, $\Delta(G) = 0$ corresponds exactly to the assumption in Ishikawa's theorem. This observation led Barnea and Isaacs to conjecture the following generalization of Ishikawa's result:

Conjecture ([1, Conjectures E and F])**.**

(1) *Let L be a nilpotent Lie algebra. Then the nilpotence class of L is bounded in terms of $\Delta(L)$.*
(2) *Let G be a finite p-group. Then the nilpotency class of G is bounded in terms of $\Delta(G)$.*

Received by the editors May 19, 2004.
2000 *Mathematics Subject Classification.* Primary 20D15; Secondary 17B30.
Key words and phrases. p-groups, nilpotent Lie algebras, conjugacy class sizes.
This work was partially supported by the MCYT Grants BFM2001-0201, BFM2001-0180, FEDER and the Ramón y Cajal Program.

In this paper we prove this conjecture. In the following we will write $[N,_k M]$ for $[N, M, \dots, M] = [\dots[N, M], \dots, M]$, where M appears k times. If L is a nilpotent Lie algebra we denote by L^i the ith term of the lower central series (i.e. $L^i = [L,_{i-1} L]$). We denote by $\mathrm{cl}(L)$ the nilpotency class of L. We use a similar notation for p-groups: G^i denotes the ith term of the lower central series of a p-group G and $\mathrm{cl}(G)$ denotes the nilpotency class of G. Now we are ready to state our results.

Theorem 1.1. *Let L be a nilpotent finite-dimensional Lie algebra. Then*

$$L^{\Delta(L)+3} \subseteq Z(L) + L'' \text{ and } \dim(L'' + Z(L)/Z(L)) \leq \Delta(L)(2\Delta(L)+1).$$

In particular, $\mathrm{cl}(L) \leq 2\Delta(L)^2 + 2\Delta(L) + 3$.

Theorem 1.2. *Let G be a finite p-group. Then*

$$G^{\Delta(G)+3} \subseteq Z(G)G'' \text{ and } |G''Z(G)/Z(G)| \leq p^{\Delta(G)(2\Delta(G)+1)}.$$

In particular, $\mathrm{cl}(G) \leq 2\Delta(G)^2 + 2\Delta(G) + 3$.

As a consequence of Theorem 1.1 and [1, Theorems C and D] we obtain the following corollary.

Corollary 1.3. *Let L be a finite-dimensional complex Lie algebra. Then L contains a nilpotent ideal N such that $\dim(L/N)$ and the nilpotency class of N are bounded in terms of $\Delta(L)$.*

Finally we would like to pose two further questions for future investigations.

Conjecture A. *Corollary* 1.3 *is stil true when L is a Lie algebra over an arbitrary field.*

Conjecture B. *Let L be a nilpotent Lie algebra. Then the derived length of L is bounded in terms of $|\mathrm{cd}(L)|$.*

2. Proofs of the main results

Proof of Theorem 1.1. Let m be the nilpotency class of L (so that $L^m \neq 0$ and $L^{m+1} = 0$) and assume that $m > 3$ (otherwise the theorem is trivial). We have $[L^{m-2}, L, L] = L^m > 0$, and thus $[L^{m-2}, L] \not\subseteq Z(L)$. Since the set $\{r \in L^{m-2} \mid [r, L] \subseteq Z(L)\}$ is a proper subalgebra of L^{m-2}, there exists an element $u \in L^{m-2}$ that does not lie in this set. Then the set $R = \{t \in L \mid [u, t] \in Z(L)\}$ is a proper subalgebra of L. Choose $v \in L \setminus R$ and write $x = [u, v]$, so that $x \in L^{m-1}$. Recall that $[L^i, L^j] \subseteq L^{i+j}$ for all superscripts $i, j \geq 1$, and thus $[L^2, v, u] \subseteq [L^3, L^{m-2}] \subseteq L^{m+1} = 0$.

Let $S = [L, v, u]$ and $T = [L^{m-1}, v]$. We have

$$[L, x] = [L, [u, v]] \subseteq [L, v, u] + [L, u, v] \subseteq S + [L^{m-1}, v] = S + T$$

and thus

$$\dim(S) + \dim(T) \geq \dim([L, x]) = \dim(L) - \dim(C_L(x)). \tag{2.1}$$

Notice that $L' + C_L(v) \subseteq \ker(\mathrm{ad}\, u \circ \mathrm{ad}\, v)$. Hence

$$\dim S \leq \dim L - \dim(L' + C_L(v)).$$

Thus, using (2.1), we obtain that

$$\dim L - \dim(L' + C_L(v)) + \dim L^{m-1} - \dim C_{L^{m-1}}(v) \geq \dim(L) - \dim(C_L(x)).$$

Since $\dim L^{m-1} - \dim C_{L^{m-1}}(v) = \dim(C_L(v) + L^{m-1}) - \dim C_L(v)$, we have

$$\dim(L' + C_L(v)) - \dim(C_L(v) + L^{m-1}) \leq \dim(C_L(x)) - \dim(C_L(v)).$$

Therefore, we obtain that

$$\begin{aligned}\dim L' - \dim((C_L(v) + L^{m-1}) \cap L') &= \dim(L' + C_L(v)) - \dim(C_L(v) + L^{m-1}) \\ &\leq \dim(C_L(x)) - \dim(C_L(v)) \leq \Delta(L).\end{aligned}$$

Hence, if we denote L/L^m by $\bar{L}$, we obtain that for any $\bar{v} \in \bar{L} \setminus \bar{R}$,

$$\dim(\bar{L}'/C_{\bar{L}'}(\bar{v})) \leq \Delta(L).$$

If $y \in \bar{L} \setminus \bar{R}$ and $z \in \bar{L}'$, then $y + z \in \bar{L} \setminus \bar{R}$. Thus, we obtain that

$$\dim(\bar{L}'/C_{\bar{L}'}(z)) \leq \dim(\bar{L}'/(C_{\bar{L}'}(y) \cap C_{\bar{L}'}(z + y))) \leq 2\Delta(L).$$

By [4], $\dim \bar{L}'' \leq \Delta(L)(2\Delta(L) + 1)$.

Now, let $\hat{L} = L/(L'' + Z(L))$ and put $D = \hat{L} \setminus \hat{R}$. Then D spans $\hat{L}$. If $y \in D$, then $[\hat{L}', y]$ is an ideal of $\hat{L}$ because the derived length of $\hat{L}$ is 2. On the other hand, we proved that $\dim[\hat{L}', y] \leq \Delta(L)$. Since $\hat{L}$ is nilpotent, $[\hat{L}', y,_{\Delta(L)} \hat{L}] = 0$ and so $\hat{L}^{\Delta(L)+3} = 0$. □

Proof of Theorem 1.2. Let $G = G^1 > G^2 > \cdots > G^m > G^{m+1} = 1$ be the lower central series of G, where m is the nilpotency class, and assume that $m > 3$ (otherwise the theorem is trivial). We have $[G^{m-2}, G, G] = G^m > \{1\}$, and thus $[G^{m-2}, G] \not\subseteq Z(G)$. Since the set $\{r \in G^{m-2} \mid [r, G] \subseteq Z(G)\}$ is a proper subgroup of G^{m-2}, there exists an element $u \in G^{m-2}$ that does not lie in this set. Then the set $R = \{t \in G \mid [u, t] \in Z(G)\}$ is a proper subgroup of G. Choose $v \in G \setminus R$ and write $x = [u, v]$, so that $x \in G^{m-1}$. Recall that $[G^i, G^j] \subseteq G^{i+j}$ for all superscripts $i, j \geq 1$, and thus $[G^2, v, u] \subseteq [G^3, G^{m-2}] \subseteq G^{m+1} = 1$.

Next, we define the map $W : G \to G$ by means of $W(g) = [g, v, u]$ for all $g \in G$. Note that $[x_1, x_2, x_3, x_4] = 1$ if some x_i is u and some x_j is v. Hence for any $g, h \in G$,

$$W(gh) = [gh, v, u] = [[g, v][g, v, h][h, v], u] = [g, v, u][h, v, u] = W(g)W(h),$$

and so W is a homomorphism of groups.

Now let $y \in G$ be arbitrary. We have $u \in G^{m-2}$ and, of course, $v, y \in G^1$, and so the Witt identity applies and we have $[v, u, y][y, v, u][u, y, v] = 1$. Since $[u, v] = x$, it follows that

$$[y, v, u][u, y, v] = [y, [v, u]] = [x, y].$$

Hence $|G : \ker W||G^{m-1} : C_{G^{m-1}}(v)| \geq |G : C_G(x)|$. Since $G'C_G(v) \leq \ker W$ we obtain that

$$\frac{|G||G^{m-1}C_G(v)|}{|G'C_G(v)||C_G(v)|} \geq \frac{|G|}{|C_G(x)|},$$

and so

$$\frac{|G'|}{|(G^{m-1}C_G(v)) \cap G'|} = \frac{|G'C_G(v)|}{|G^{m-1}C_G(v)|} \leq \frac{|C_G(x)|}{|C_G(v)|} \leq p^{\Delta(G)}.$$

Denote G/G^m by $\bar{G}$. Then from the last inequality we obtain that

$$|\bar{G}' : C_{\bar{G}'}(\bar{v})| \leq p^{\Delta(G)}. \tag{2.2}$$

If $y \in \bar{G} \setminus \bar{R}$ and $z \in \bar{G}'$, then $yz \in \bar{G} \setminus \bar{R}$. Thus, we obtain that

$$|\bar{G}' : C_{\bar{G}'}(z)| \leq |\bar{G}' : (C_{\bar{G}'}(y) \cap C_{\bar{G}'}(zy))| \leq p^{2\Delta(G)}.$$

By [4], $|\bar{G}''| \leq p^{\Delta(G)(2\Delta(G)+1)}$.

Now, let $\hat{G} = G/(G''Z(G))$ and put $D = \hat{G} \setminus \hat{R}$. Then D generates $\hat{G}$. If $y \in D$, then $[\hat{G}', y]$ is a normal subgroup of $\hat{G}$ because the derived length of $\hat{G}$ is 2. On the other hand, by (2.2), $|[\hat{G}', y]| \leq p^{\Delta(G)}$. Since $\hat{G}$ is nilpotent, $[\hat{G}', y,_{\Delta(G)} \hat{G}] = 1$ and so $\hat{G}^{\Delta(G)+3} = 1$. □

Acknowledgments

This work was done during my visit to the Imperial College of London and I thank the Department of Mathematics and personally M. Liebeck for their hospitality. I also thank Y. Barnea for some helpful comments on a preliminary version of this work.

References

[1] Y. Barnea and I. M. Isaacs, *Lie algebras with few centralizer dimensions*, J. Algebra **259** (2003), 284–299. MR1953720 (2004a:17005)

[2] K. Ishikawa, *On finite p-groups which have only two conjugacy lengths*, Israel J. Math. **129** (2002), 119–123. MR1910937 (2004b:20032)

[3] N. Ito, *On finite groups with given conjugate types.* I, Nagoya Math. J. **6** (1953), 17–28. MR0061597 (15:851c)

[4] M. R. Vaughan-Lee, *Breadth and commutator subgroups of p-groups*, J. Algebra **32** (1974), 278–285. MR0364436 (51:690)

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