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BLOCKS WITH p-POWER CHARACTER DEGREES

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ABSTRACT. Let B be a p-block of a finite group G. If $\chi(1)$ is a p-power for all $\chi \in \mathrm{Irr}(B)$, then B is nilpotent.

1. Introduction

A classical result in character theory asserts that if all irreducible character degrees of a finite group G are powers of a fixed prime p, then G is p-nilpotent. (In fact, J. Thompson proved that if all non-linear irreducible characters of G have degree divisible by p, then G is p-nilpotent [8], and later M. Isaacs and S. D. Smith proved in [4] that it is enough to consider irreducible characters in the principal block of G.) In this note, we assume that the irreducible characters of a p-block B are powers of p, and we obtain that B is nilpotent.

Theorem A. *Let G be a finite group and let B be a p-block of G. If $\chi(1)$ is a power of p for every $\chi \in \mathrm{Irr}(B)$, then B is nilpotent. In particular, $|\mathrm{IBr}(B)| = 1$ and there is a height-preserving bijection from $\mathrm{Irr}(B)$ onto $\mathrm{Irr}(D)$, where D is a defect group of B.*

Unfortunately, we have been unable to find a proof of Theorem A that does not use the Classification of Finite Simple Groups.

2. Characters of prime power degree of simple groups

The prime power degree characters of quasi-simple groups were classified in [1] and [6]. The main purpose of this section is to prove that p-power degree p-blocks of simple groups have defect zero, although we take the opportunity to prove something more. We would like to mention too that some quasi-simple groups, however, have prime power degree blocks with more than one character. This happens, for instance, if $G = SL_2(q)$, where q is a Fermat or a Mersenne prime.

(2.1) Theorem. *Let p be a prime. Suppose that G is a simple group and suppose that $1 \neq \chi \in \mathrm{Irr}(G)$ has degree a power of p. Then χ has height zero in its p-block B. In particular, $\chi^0 \in \mathrm{IBr}(G)$. Furthermore, if χ does not have defect zero, then there exists $\psi \in \mathrm{Irr}(B)$ such that $\psi(1)$ is not a power of p.*

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We use a result of R. Brauer ([2]): if $\chi \in \mathrm{Irr}(G)$ and $p\chi(1)_p = |G|_p$, then the block of χ has defect 1. Also, we use the well-known fact that if $\chi \in \mathrm{Irr}(G)$ has degree p and G is simple, then $|G|_p = p$.

Proof of Theorem (2.1). If χ has defect zero, then there is nothing to prove. So we may assume that χ does not have defect zero. In particular, we may assume that if G is a simple group of Lie type of characteristic p, then χ is not the Steinberg character.

Theorem (1.1) in [6], together with the Conjecture at the end of Section 1 (finally proved in [1]), lists all the quasi-simple groups having a faithful irreducible character of prime power degree (except for the alternating groups A_5, A_6, A_7 and A_8). The prime power degree characters in A_5, A_6 and A_8 are of defect zero, while A_7 does not have prime power degree irreducible characters.

Since our group G is simple, the cases (8), (9), (10), (13) and (20) to (27) in Theorem (1.1) are not considered. Cases (1) and (12) have also been treated.

In case (19), $G = G_2(3)$ and χ has defect zero. In case (18), $G = U_3(3)$ and χ has also defect zero.

There are three cases to consider in case (17). If $G = A_9$ or $Sp_6(2)$, then $\chi(1) = 27$ while $|G|_3 = 81$. These characters live in blocks of defect 1 and have height zero. Also, they are the unique irreducible characters of degree 27 in the group, so the theorem is true in this case. In the third case, $G = {}^2F_4(2)'$ and χ has defect zero.

In the cases (11), (14) and (16), $\chi(1) = p$, and therefore χ has defect zero.

We have three cases in (15). If $G = M_{11}$ or $G = L_3(3)$, then χ has defect zero. In the case $G = M_{12}$, the defect is then 2, and there are 4 irreducible characters in the block of χ, two of degree 16 and two more of degree not a power of 2.

Next, we are going to consider cases (2) to (7). In all these cases, we shall see that χ has defect zero.

Case (2). Here, $G = L_2(q)$ and $\chi(1) = q \pm 1$, or q is odd and $\chi(1) = \frac{q\pm1}{2}$.

Suppose that $\chi(1) = q \pm 1$. If $q = 2^f$, then $|G| = q(q-1)(q+1)$. These three numbers are mutually coprime and therefore χ has defect zero. Suppose now that q is odd. Then

$$|G| = \frac{1}{2}q(q-1)(q+1).$$

If $\chi(1) = q + 1 = p^c$, then $p = 2$. Hence $(q-1)_2 = 2$, and therefore χ has defect zero. The same happens if $\chi(1) = q - 1$.

Suppose now that q is odd and $\chi(1) = \frac{q-1}{2} = p^c$. Recall that $L_2(q)$ has irreducible characters of degree $\frac{q-1}{2}$ if this number is odd. Thus p is odd. Then p divides $q-1$, p does not divide $q+1$, and thus χ has defect zero. Suppose now that q is odd and $\chi(1) = \frac{q+1}{2} = p^c$. Recall that $L_2(q)$ has irreducible characters of degree $\frac{q+1}{2}$ if $\frac{q-1}{2}$ is even. Thus p is odd, p divides $q+1$, p does not divide $q-1$, and thus χ has defect zero.

Case (3). Here, $G = L_n(q)$, $q > 2$, n is an odd prime, $(n, q-1) = 1$ and $\chi(1) = \frac{q^n-1}{q-1} = p^c$.

We have that

$$|G| = q^{\binom{n}{2}}(q-1)(q^2-1)\cdots(q^{n-1}-1)\chi(1)\,.$$

If $q = 2$ and $n = 6$, then $\chi(1)$ is not a prime power. So there exists l a Zsigmondy prime divisor of $q^n - 1$. Then $\chi(1)$ is a power of l, and χ has l-defect zero.

Case (4). In this case, we have that $G = U_n(q)$, n is an odd prime, $(n, q+1) = 1$ and $\chi(1) = \frac{q^n+1}{q+1}$.

We have that

$$|G| = q^{\binom{n}{2}}(q+1)(q^2-1)(q^3+1)\cdots(q^{n-1}-1)\chi(1)\,.$$

If $q = 2$ and $2n = 6$, then $U_3(2)$ is not simple. Since $2n > 2$, we have a Zsigmondy prime divisor l of $q^{2n}-1$. We have that l does not divide q^m-1 for $m < 2n$. Also, if l divides q^i+1 for $i < n$, then l divides $q^{2i}-1$ and this is a contradiction. Hence χ has l-defect zero.

Case (5). Here we have that $G = PSp_{2n}(q)$, $n > 1$, $q = r^k$ with r an odd prime, kn is a 2-power, and $\chi(1) = (q^n+1)/2$.

We have that

$$|G| = q^{n^2}(q^2-1)(q^4-1)\cdots(q^{2n-2}-1)(q^n-1)\chi(1)\,.$$

Now, we have that $q > 2$ and $2n > 2$, so let l be a Zsigmondy prime divisor of $q^{2n}-1$. In particular, l is odd. Then $\chi(1)$ is a power of l, and χ has l-defect zero.

Case (6). Here we have that $G = PSp_{2n}(3)$, $n > 1$ prime, and $\chi(1) = \frac{3^n-1}{2}$.

If $n = 2$, then $G = PSp_4(3)$ which does not have irreducible characters of degree 4. So we may assume that $n > 2$. Write $q = 3$. We have that

$$|G| = q^{n^2}(q^2-1)(q^4-1)\cdots(q^{2n-2}-1)(q^n+1)\chi(1)\,.$$

Now, let l be a Zsigmondy prime divisor of q^n-1. In particular, l does not divide $q-1 = 2$, and l is odd. Hence, $\chi(1)$ is a power of l. We have that the order of q modulo l is n. Now if l divides $q^{2i}-1$ for $i < n$, then $q^{2i} = 1 \bmod l$, and therefore n would divide $2i$. Hence, n divides i, a contradiction. Now, since l divides q^n-1, we have that l does not divide q^n+1, and this proves that χ has defect zero.

Finally, we are left with the case A_{p^d+1} and $\chi(1) = p^d$. Write $n = p^d+1$, and assume that $n > 6$. It is well known that in this case χ is the unique irreducible character of A_n of degree $n-1$. We have that χ has two extensions ψ_1, ψ_2 to S_n, where, for instance, ψ_1 corresponds to the partition $(p^d, 1)$ and $\psi_2 = \lambda\psi_1$, where λ is the sign character. The p-core of ψ_1 is the partition $(p, 1)$. Let B be the p-block of ψ_1. Hence the weight of B is $p^{d-1}-1$ and the defect group of B is a Sylow p-subgroup of S_{p^d-p}. Since $(p^d-p)!_p = (p^d)!_p/p^d$, this easily implies that ψ_1 has height zero. If p is odd, then this implies that χ has height zero. If $p = 2$, then B is the unique block covering the block of χ, and χ has height zero by Corollary (9.18) of [7]. $\square$

From Theorem (2.1), it follows that if G is a direct product of non-abelian simple groups and B is a p-block of G such that $\chi(1)$ is a power of p for every $\chi \in \mathrm{Irr}(B)$, then B has defect zero.

3. Nilpotent blocks and Theorem A

The nilpotent blocks were introduced by M. Broué and L. Puig in [3]. A block b of a finite group G is **nilpotent** if whenever Q is a p-subgroup of G and e is a block of $Q\mathbf{C}_G(Q)$ inducing b, then $\mathbf{N}_G(Q,e)/\mathbf{C}_G(Q)$ is a p-group.

If a block b has central defect group D, then b is nilpotent. This follows because if e is a block of $Q\mathbf{C}_G(Q)$ inducing b, then $Q \subseteq D$ (by Theorem (9.24) of [7], for instance).

First, we prove Theorem A in the easy case where the block has maximal defect.

(3.1) Lemma. *Suppose that B is a p-block of a finite group G with defect group D such that every $\chi \in \mathrm{Irr}(B)$ has p-power degree. If $D \in \mathrm{Syl}_p(G)$, then G has a normal p-complement.*

Proof. Let $\lambda \in \mathrm{Irr}(B)$ of height zero. Then $\lambda(1)$ is not divisible by p, and we conclude that λ is linear. Now, by using the linking relation in Theorem (3.19) of [7], for instance, we see that there is some block B^* of G such that $\mathrm{Irr}(B^*) = \{\bar{\lambda}\chi \mid \chi \in \mathrm{Irr}(B)\}$. Now, B^* contains the trivial character, and therefore B^* is the principal block. Also, all irreducible characters in B^* have p-power degree. Hence G has a normal p-complement by the Isaacs-Smith theorem ([4]). □

(3.2) Lemma. *Let G be a finite group and suppose that $G = \mathbf{E}(G)\mathbf{Z}(G)$, where $\mathbf{E}(G)$ is the layer of G. Let B be a faithful p-block of G. If $\chi(1)$ is a power of p for every $\chi \in \mathrm{Irr}(B)$, then the defect group of B is central in G.*

Proof. Let $E = \mathbf{E}(G)$. Write $\mathbf{Z}(G) = Z = Z_p \times Z_{p'}$, where Z_p is a Sylow p-subgroup of Z. Now, B covers a unique block b of E. This block is faithful and all of its irreducible ordinary characters have p-power degree. Suppose that $E < G$. Arguing by induction on $|G|$, we will have that b has a defect group contained in $\mathbf{Z}(E) \subseteq \mathbf{Z}(G)$. Now, let D be a defect group of B such that $D \cap E$ is a defect group of b. Since EZ_p has p'-index, we have that $D \subseteq EZ_p$. Since $Z_p \subseteq D$, we conclude that $D = (D \cap E)Z_p$ is central in G. So we may assume that $G = E$. Hence G is perfect and G/Z is the direct product of non-abelian simple groups. We have that B covers a single faithful irreducible character λ of $Z_{p'}$. If $\chi \in \mathrm{Irr}(B)$ lies over λ, then $\chi_{Z_{p'}} = \chi(1)\lambda$. Now, by taking determinants and using that G is perfect, we deduce that $\lambda^{\chi(1)} = 1$. Then $\lambda = 1$ and Z is a p-group. Now, there is a unique block $\bar{B}$ of G/Z contained in B. In fact, if Q is a defect group of B, then Q/Z is a defect group of $\bar{B}$. (See, for instance, Theorem (9.10) of [7].) By the comment after Theorem (2.1), $\bar{B}$ has defect zero, $Q = Z$ and the proof of the lemma is complete. □

The proof of the following result is quite straightforward, and we leave it for the reader to check.

(3.3) Lemma. *Suppose that B is a block of a finite group G, $N \triangleleft G$, $N \subseteq \ker(B)$, and $\bar{B}$ is the unique block of G/N such that $\mathrm{Irr}(B) = \mathrm{Irr}(\bar{B})$. If $\bar{B}$ is nilpotent, then B is nilpotent.*

Now we proceed to prove Theorem A of the Introduction.

(3.4) Theorem. *Let G be a finite group and let B be a p-block of G. If $\chi(1)$ is a power of p for every $\chi \in \mathrm{Irr}(B)$, then B is nilpotent.*

Proof. Suppose that the theorem is false. We choose a counterexample (G, B) minimizing first $|G : \mathbf{Z}(G)|$ and then $|G|$.

Step 1: The block B is quasi-primitive.

If B covers a block b of a normal subgroup N of G, then there is a block B' of the stabilizer $I_G(b)$ of b in G which is Morita equivalent to B with equivalent Brauer category, such that all irreducible character degrees for B' divide irreducible character degrees for B. If $I_G(b) < G$, then B' is nilpotent by minimality, and then B is nilpotent, contrary to assumption. Hence $I_G(b) = G$, so the claim follows.

Step 2: $\mathbf{O}_{p'}(G) = \mathbf{Z}(G)$.

Since the kernel of B is a p'-group, it is clear that B has trivial kernel by Lemma (3.3), minimality and the hypotheses. Furthermore, by Step 1, there is a single irreducible character of $\mathbf{O}_{p'}(G)$ covered by $\mathrm{Irr}(B)$. This irreducible character is clearly linear by Clifford's Theorem, so $\mathbf{O}_{p'}(G) \subseteq \mathbf{Z}(G)$. Let $Z_p = \mathbf{O}_p(\mathbf{Z}(G))$. Now, by Theorem (9.10) of [7], for instance, we have that B contains a unique block $\bar{B}$ of G/Z_p and this block is certainly a p-power degree block. Hence $\bar{B}$ is nilpotent by minimality, and we deduce that B is also nilpotent by Lemma 2 of [9]. Thus $Z_p = 1$.

Step 3: If $\mathbf{E}(G)$ is the layer of G, then $\mathbf{E}(G) \neq 1$.

Suppose that $\mathbf{E}(G) = 1$. Then $\mathbf{F}^*(G) = \mathbf{F}(G) = \mathbf{O}_p(G) \times \mathbf{Z}(G)$. Now, we claim that B has maximal defect. Suppose that D is the defect group of B. Then $\mathbf{O}_p(G) \leq D$. However, $D \in \mathrm{Syl}_p(\mathbf{C}_G(y))$ for some p-regular $y \in G$. Hence y centralizes $\mathbf{F}^*(G)$, so $y \in \mathbf{Z}(\mathbf{F}^*(G))$. Since y is p-regular, we have $y \in \mathbf{Z}(G)$. Since D is a Sylow p-subgroup of $\mathbf{C}_G(y)$, we have $D \in \mathrm{Syl}_p(G)$, as claimed. Now, G has a normal p-complement by Lemma (3.1), and B is certainly nilpotent in this case.

Step 4: The final contradiction.

Set $N = \mathbf{E}(G)\mathbf{Z}(G)$. Then B covers a G-stable p-power degree block of N, say b. By Lemma (3.2), we have that b has defect group $Z = \mathbf{O}_p(\mathbf{Z}(N))$.

Now, we invoke some of the results of Külshammer-Puig in [5] in this special situation. We first claim that the Külshammer-Puig 2-cocycle is trivial. There is a unique irreducible character μ in b which has Z in its kernel. Then μ may be realised by an RN-module, where R is a complete discrete valuation ring of characteristic 0 containing p'-roots of unity of all orders and with $R/J(R)$ of characteristic p. This RN-module is unique up to isomorphism. Let σ be the associated representation. As usual, we want to define $g\sigma$ for every $g \in G$, and if we assume (as we may) that this has finite order, then this is uniquely determined up to multiplication by a root of unity. Since $\mu(1)$ is a power of p, we may assume that $\det(g\sigma)$ is a p-power root of unity for all $g \in G$, in which case the associated factor set also consists of p-power roots of unity. Since, in this situation, the Külshammer-Puig 2-cocycle is the p'-part of the 2-cocycle of G/N associated to this factor set, we have the triviality of the Külshammer-Puig 2-cocycle as claimed.

Let $\tilde{G} = G/N$. Then from [5], there is a block $\hat{B}$ of a (not necessarily central) extension $\hat{G}$ of $\tilde{G}$ by Z which is Morita equivalent to B and has Brauer category equivalent to that of B.

Hence there are positive integers r and s, and there is a bijective correspondence between simple B-modules and simple $\hat{B}$-modules such that if U and V correspond, then

$$\dim_F(U)/\dim_F(V) = r/s.$$

Also (for the same r and s), there is a bijective correspondence between irreducible characters in B and irreducible characters in $\hat{B}$ such that if χ and ψ correspond, then

$$\chi(1)/\psi(1) = r/s\,.$$

If Z is trivial, then the irreducible character degrees in $\hat{B}$ are divisors of those of B, so that $\hat{B}$ is nilpotent by minimality, so suppose that $Z \neq 1$.

We will prove that the dimensions of all simple B-modules and all simple $\hat{B}$-modules are powers of p. Since we already know that all irreducible character degrees in B are powers of p, it follows that all irreducible characters in $\hat{B}$ are powers of p. By minimality, it follows that $\hat{B}$ is nilpotent, so B is nilpotent too, contrary to hypothesis.

Now Z may not be central in G, so that B may a priori dominate several blocks of $\tilde{G} = G/Z$. However, each such block covers the unique block of defect 0 of N/Z corresponding to b. Furthermore, each such block of $\tilde{G}$ has all its irreducible characters of degree a power of p.

Now each block of $\tilde{G}$ which is dominated by B is nilpotent by minimality, and the dimension of the unique simple module of such a block is the degree of an irreducible character of height 0 in that block, so is a power of p. Thus each simple B-module (which certainly has Z in its kernel) has dimension a power of p.

Each block of G/Z which is dominated by B covers a stable block of defect 0 of N/Z (whose unique simple module has dimension a power of p) and is thus Morita equivalent to a block of $(G/Z)/(N/Z) \cong G/N$. This last block has all its irreducible characters of degree a power of p, so is nilpotent by minimality and has a unique simple module whose dimension is a power of p. The dimension of the corresponding simple module for the block of G/Z is the product of this last dimension, with the dimension of the unique simple module in the covered block of N/Z, so is again a power of p.

But by consideration of central characters, we see that the blocks of G/Z which are dominated by B correspond bijectively via the above Morita equivalences to the blocks of G/N which are dominated by $\hat{B}$. (Recall that $\hat{G}$ is an extension of G/N by the p-group Z.) Hence all simple $\hat{B}$-modules have dimension a power of p, as required. $\square$

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