THE POLYNOMIAL ANALOGUE OF A THEOREM OF RÉNYI

KENT E. MORRISON

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Abstract. Rényi’s result on the density of integers whose prime factorizations have excess multiplicity has an analogue for polynomials over a finite field.

Let \( n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \) be the prime factorization of a positive integer \( n \). Define the excess of \( n \) to be \( (\alpha_1 - 1) + \cdots + (\alpha_r - 1) \), which is the difference between the total multiplicity \( \alpha_1 + \cdots + \alpha_r \) and the number of distinct primes in the factorization. An integer with excess 0 is also said to be square-free. Let \( E_k \) denote the set of positive integers of excess \( k \), \( k = 0, 1, 2, \ldots \). Rényi proved that the set \( E_k \) has a density \( d_k \) and that the sequence \( \{d_k\} \) has a generating function given by

\[
\sum_{k \geq 0} d_k z^k = \prod_p \left( 1 - \frac{1}{p} \right) \left( 1 + \frac{1}{p - z} \right),
\]

where the product extends over the primes. Recall that the density of a set of positive integers \( E \) is the limit (if it exists)

\[
\lim_{n \to \infty} \frac{\# (E \cap \{1, 2, \ldots, n\})}{n},
\]

which is the limiting probability that an integer from 1 to \( n \) is in \( E \).

The set of square-free integers is \( E_0 \) and setting \( z = 0 \) in the generating function gives \( d_0 = \prod_p (1 - 1/p^2) \), which is the well-known result that the density of square-free integers is \( 1/\zeta(2) = 6/\pi^2 \). (This was first proved by Gegenbauer [2] in 1885. A clear, non-rigorous presentation is in [3].) By setting \( z = 1 \) one sees that \( \sum_k d_k = 1 \), so that the density is countably additive on the specific partition of \( \mathbb{Z}^+ \) given by the \( E_k \). Rényi’s proof appeared in [7], but an alternative proof was given by Kac in [4, pp. 64–71].

The aim of this paper is to derive an analogue of the generating function for polynomials in one variable over a finite field. Let \( \mathbb{F}_q \) be the field with \( q \) elements and let \( \mathbb{F}_q[x] \) be the polynomial ring. The prime elements of \( \mathbb{F}_q[x] \) are the irreducible monic polynomials. Let \( f \) be a monic polynomial with prime factorization \( f = \pi_1^{\alpha_1} \cdots \pi_r^{\alpha_r} \), and define the excess of \( f \) to be \( (\alpha_1 - 1) + \cdots + (\alpha_r - 1) \), just as for an integer. Let \( e_{n,k} \) be the number of monic polynomials of degree \( n \) and excess \( k \). Define

\[
d_{n,k} = \frac{e_{n,k}}{q^n},
\]

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which is the probability that a monic polynomial of degree \( n \) has excess \( k \). Note that \( d_{0,k} = 0 \) for \( k > 0 \). Then define the analogue of the density to be the limiting “probability” as the degree goes to infinity:

\[
d_k = \lim_{n \to \infty} d_{n,k}.
\]

Define \( D(z) = \sum_{k \geq 0} d_k z^k \) to be the ordinary power series generating function of the sequence \( \{d_k\} \). Let \( N_f = q^\deg f \) be the norm of the polynomial \( f \), which is the cardinality of the residue ring \( \mathbb{F}_q[x]/(f) \). The main result of this paper is the following theorem concerning \( D(z) \).

**Theorem 1.** The generating function \( D(z) \) has a factorization over the prime polynomials given by

\[
D(z) = \prod_{\pi} \left( 1 - \frac{1}{N\pi} \right) \left( 1 + \frac{1}{N\pi - z} \right).
\]

**Proof.** We begin with the geometric series

\[
\frac{1}{1 - qt} = \sum_{n \geq 0} q^n t^n,
\]

which is the generating function for the number of monic polynomials of degree \( n \). Then unique factorization in \( \mathbb{F}_q[x] \) allows us to factor the generating function formally:

\[
\frac{1}{1 - qt} = \prod_{\pi} \sum_{j \geq 0} t^j \deg \pi = \prod_{\pi} \frac{1}{1 - t^\deg \pi}.
\]

By grouping the primes of the same degree and letting \( \nu_i \) denote the number of primes of degree \( i \), we can rewrite the last line above as

\[
\frac{1}{1 - qt} = \prod_{i \geq 1} \left( \frac{1}{1 - t^i} \right)^{\nu_i}.
\]

From this it follows that

\[
1 - qt = \prod_{i \geq 1} (1 - t^i)^{\nu_i}
\]

as a formal power series. In the product on the right there is a finite number of terms for each power of \( t \) so that the coefficients make sense. In fact, the coefficient of \( t^n \) is 0 except for \( n = 0, 1 \). However, considered as a function of a complex variable \( t \), the product does not converge for all \( t \). It does converges absolutely for \( |t| < 1/q \). This follows from consideration of the series \( \sum \nu_i t^i \) and the fact that \( \nu_i \) is asymptotic to \( q^i/i \).

Next we define the two-variable generating function

\[
E(t, z) = \sum_{n,k} e_{n,k} t^n z^k.
\]

Modifying the factorization in (1), we see that

\[
E(t, z) = \prod_{\pi} (1 + t^{\deg \pi} + t^{2\deg \pi} z + \cdots + t^{i \deg \pi} z^{i-1} + \cdots).
\]
Note that the variable $z$ appears with a power that is equal to the excess multiplicity. That is, if $f = \prod_{\pi}^{\alpha_1 \deg \pi \ldots \deg \pi_{\alpha_r} z^{\alpha_1 - 1} \ldots z^{\alpha_r - 1}}$, then the product expansion of $E(t, z)$ has a term of the form $t^{\deg \pi} \ldots t^{\deg \pi} z^{\alpha_1 - 1} \ldots z^{\alpha_r - 1}$. Sum the geometric series in each factor to obtain the formal factorization

$$E(t, z) = \prod_{\pi} \left( 1 + \frac{t^{\deg \pi}}{1 - t^{\deg \pi} z} \right).$$

Group the irreducibles by degree to get

$$(3) E(t, z) = \prod_{\nu_i} \left( 1 + \frac{t_i z}{1 - t_i} \right)^{\nu_i}.$$  

Now the product on the right converges absolutely if and only if the series

$$(4) \sum_{i \geq 1} \nu_i \left| \frac{t_i z}{1 - t_i} \right|$$

converges. We claim that $(4)$ converges for $|t| < 1/q$ and $|z| < q$, because the denominators $|1 - t_i z|$ are bounded away from 0 and $\nu_i$ is asymptotic to $q^i / i$.

(Actually, it suffices that $\nu_i < q^i$.) From $(2)$ and $(3)$ we get

$$(1 - qt) E(t, z) = \prod_{i \geq 1} \left( 1 - t_i \right)^{\nu_i} \prod_{i \geq 1} \left( 1 + \frac{t_i}{1 - t_i z} \right)^{\nu_i}.$$  

On the domain where both products converge absolutely, we can combine the factors for each $i$ to get

$$(5) (1 - qt) E(t, z) = \prod_{i \geq 1} \left( 1 - t_i \right)^{\nu_i} \left( 1 + \frac{t_i}{1 - t_i z} \right)^{\nu_i}.$$  

By multiplying the factors together we can see that the absolute convergence of the infinite product depends on the convergence of the series

$$\sum_{i} \nu_i \left| \frac{t^{2i} z - t^{2i}}{1 - t_i} \right|.$$  

Then reasoning along the same lines as before, we see that this series converges for $|t^2| < q$ and $|z| < \sqrt{q}$. In particular, the product converges for $t = 1/q$, and so after carrying out the multiplication of the left side of $(5)$ we arrive at

$$\sum_{n,k} (e_{n,k} - q e_{n-1,k}) t^n z^k = \prod_{i \geq 1} \left( 1 - t_i \right)^{\nu_i} \left( 1 + \frac{t_i}{1 - t_i z} \right)^{\nu_i}.$$  

We evaluate this at $t = 1/q$ to get

$$\sum_{n,k} (e_{n,k} - q e_{n-1,k}) \frac{1}{q^n} z^k = \prod_{i \geq 1} \left( 1 - (1/q)^i \right)^{\nu_i} \left( 1 + \frac{(1/q)^i}{1 - (1/q)^i z} \right)^{\nu_i}.$$  

The coefficient of $z^k$ is the sum $\sum_{n \geq 1} (e_{n,k} / q^n - e_{n-1,k} / q^{n-1})$. This telescopes to give

$$\lim_{n \to \infty} \frac{e_{n,k}}{q^n} = \lim_{n \to \infty} d_{n,k},$$
which is the definition of $d_k$, and so we have

$$D(z) = \sum_k d_k z^k = \prod_{i \geq 1} \left( 1 - \left(\frac{1}{q}\right)^i \right)^{\nu_i} \left( 1 + \frac{\left(\frac{1}{q}\right)^i}{1 - \left(\frac{1}{q}\right)^i z} \right)^{\nu_i}.$$  

Finally, we write the product by indexing over the prime polynomials $\pi$ and note that the norm of $\pi$ is $N\pi = q^{\deg \pi}$. With this we have the generating function for $d_k$ in the form that is most directly analogous to Rényi’s theorem:

$$D(z) = \prod_{\pi} \left( 1 - \left(\frac{1}{q}\right)^{\deg \pi} \right) \left( 1 + \frac{\left(\frac{1}{q}\right)^{\deg \pi}}{1 - \left(\frac{1}{q}\right)^{\deg \pi} z} \right) = \prod_{\pi} \left( 1 - \frac{1}{N\pi} \right) \left( 1 + \frac{1}{N\pi - z} \right).$$

□

The coefficient $d_0$ is the limiting “probability” that a monic polynomial is square-free. To develop the analogy with the density of the square-free integers given by $d_0$ in Rényi’s generating function, we use the zeta function of $F_q[x]$ (i.e. the zeta function of the affine line over $F_q$)

$$\zeta(s) = \frac{1}{1 - q^{-s}},$$

which immediately comes from the definition

$$\zeta(s) = \sum_a \frac{1}{(Na)^s},$$

where the sum is over all ideals of $F_q[x]$ and the norm $Na$ is the cardinality of the residue ring $F_q[x]/a$. It has a factorization over the prime ideals (i.e. irreducible polynomials)

$$\zeta(s) = \prod_{\pi} \frac{1}{1 - (N\pi)^{-s}} = \prod_{i \geq 1} \left( \frac{1}{1 - q^{-is}} \right)^{\nu_i}.$$  

**Corollary 1.** $d_0 = \frac{1}{\zeta(2)} = 1 - \frac{1}{q}$

*Proof.* We have

$$d_0 = D(0) = \prod_{i \geq 1} \left( 1 - \frac{1}{q^{2i}} \right)^{\nu_i}.$$  

Then in (2) we may let $t = 1/q^2$, because the product converges for $|t| < 1/q$, to obtain

$$1 - \frac{1}{q} = \prod_{i \geq 1} \left( 1 - \frac{1}{q^{2i}} \right)^{\nu_i}.$$  

Note that the product is $1/\zeta(2)$.

□

Corollary 1 can be obtained as a special case of much more general results on square-free values of polynomials in one or more variables from the work of Ramsay [6] and Poonen [5]. It turns out that for $n \geq 2$, the value of $d_{n,0}$ is $1 - 1/q$. This can
be seen by finding the coefficients $e_{n,0}$, which count the number of monic, square-free polynomials of degree $n$. These polynomials can be counted directly; see, for example, [1].

**Corollary 2.** The number of square-free monic polynomials of degree $n \geq 2$ is $q^n - q^{n-1}$.

**Proof.** The generating function $\sum_{n \geq 0} e_{n,0}t^n = E(t,0)$. From (3) we see that

$$E(t,0) = \prod_{i \geq 1} (1 + t^i)^{\nu_i}.$$  

Using (2) we see that

$$E(t,0)(1 - qt) = \prod_{i \geq 1} (1 + t^i)^{\nu_i}(1 - t^i)^{\nu_i} = \prod_{i \geq 1} (1 - t^{2i})^{\nu_i} = 1 - qt^2.$$  

Therefore,

$$E(t,0) = \frac{1 - qt^2}{1 - qt},$$

from which it follows that $e_{n,0} = q^n - q^{n-1}$ for $n \geq 2$. \hfill \Box

From the expression

$$D(z) = \prod_{i \geq 1} \left(1 - \frac{1}{q^i}\right)^{\nu_i} \left(1 + \frac{1}{q^i - z}\right)^{\nu_i}$$

we can see that $D(z)$ has poles at $z = q^i$ of multiplicity $\nu_i$. In particular the pole at $z = q$ has multiplicity $q - 1$. Elementary analysis of the singularity there, along the lines of Kac [4] in his discussion of Rényi’s result, enables us to describe the asymptotic behavior of the $d_k$ as $k \to \infty$.

**Corollary 3.** As $k$ goes to infinity, $d_k$ is asymptotic to

$$A \frac{k^{q-2}}{q^k},$$

where the constant $A$ is given by

$$A = \frac{1}{(q-2)!} \left(\frac{1}{q} - \frac{1}{q^2}\right)^{q-1} \prod_{i \geq 2} \left(1 - \frac{1}{q^i}\right)^{\nu_i} \left(1 - \frac{1}{q^i - q}\right)^{\nu_i}.$$  

One may contrast this asymptotic result with the classical case of Rényi. Although the generating functions have clearly analogous form, the generating function for the number-theoretic version has only a simple pole $z = 2$, which is the pole of smallest absolute value. The asymptotic analysis shows that

$$d_k \sim \frac{\delta}{2k},$$

where

$$\delta = \frac{1}{4} \prod_{p \geq 3} \frac{(p-1)^2}{p(p-2)}.$$
The referee has observed that Theorem 1 of this article can be extended naturally to function fields over finite fields by using $S$-zeta functions and their residues at $t = 1/q$. Let $K$ be a function field over the constant field $\mathbf{F}_q$. Let $S$ be a finite, non-empty set of places on $K$ and let $\mathcal{O}_{K,S}$ denote the ring of $S$-integers of $K$. Then for every integer $k \geq 0$, the density $d_{k,S}$ of ideals in $\mathcal{O}_{K,S}$ with excess $k$ exists, and the following analytic identity holds:

$$\sum_{k \geq 0} d_k z^k = \prod_{v \in S} \left(1 - \frac{1}{N_v}\right) \left(1 + \frac{1}{N_v - z}\right),$$

where for each place $v$ on $K$ the norm $N_v = q^{\deg v}$ is the cardinality of the residue field at $v$.

Finally, the referee has pointed out that by generalizing Kac’s proof of Rényi’s theorem [4, pp. 64–71], there should also be an analogue of the theorem for the density of ideals with excess $k$ in the ring of algebraic integers (or ring of $S$-integers) of any number field.

REFERENCES


Department of Mathematics, California Polytechnic State University, San Luis Obispo, California 93407

E-mail address: kmorriso@calpoly.edu