SUMS OF SQUARES IN OCTONION ALGEBRAS

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Abstract. Sums of squares in composition algebras are investigated using methods from the theory of quadratic forms. For any integer \( m \geq 1 \) octonion algebras of level \( 2^m \) and of level \( 2^m + 1 \) are constructed.

Introduction

The investigation of sums of squares is a classical number-theoretic problem and goes back to Diophantes, Fermat, Lagrange and Gauss who studied how to express integers as sums of squares. The notion of level of a field seems to have been introduced by Artin and Schreier [AS]. It was later generalized to commutative rings (see Pfister [Pf] and Dai, Lam and Peng [DLP] for lists of references) and then to noncommutative rings, in particular to division rings and hence quaternion algebras over fields, for instance by Leep [Le] and Lewis [L3].

As mentioned already by Lewis [L1], the definition of level makes sense not just for associative unital rings. However, there seems to be nothing in the literature about this problem in a nonassociative setting. It turns out that much of the existing theory on sums of squares in noncommutative rings can be effortlessly transferred to quadratic algebras with a scalar involution. The best known among these are certainly the octonion algebras. We investigate the level of composition algebras over arbitrary rings, extending results on sums of squares in finite-dimensional division algebras (which are finite-dimensional over the center) by Leep, Shapiro and Wadsworth [LSW], and on the level of quaternion algebras over fields of characteristic not two by Koprowski [Ko], and Lewis [L2], [L3]. Furthermore, we construct octonion algebras of level \( 2^m \) (indeed, even octonion algebras of level \( 2^m \), where \( -1 \) is not a sum of \( 2^m \) squares of pure octonions), and of level \( 2^m + 1 \), for any integer \( m \geq 1 \) using arguments relying on function fields of quadratic forms as in Laghribi and Mammone [LM]. We do not know if other integers can also appear as a level of an octonion algebra (this seems to be still an open question for quaternion algebras as well). The aim of this paper is to give a first insight in how easily many, by now well-known results on sums of squares and levels, can be transferred to a nonassociative setting.

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1. Preliminaries

For the convenience of the reader, we summarize the main facts about the algebras needed in this paper:

Let $R$ be a unital commutative associative ring, and $A$ a unital nonassociative $R$-algebra. The term "$R$-algebra" always refers to a unital nonassociative algebra which are finitely generated projective as $R$-modules. We write $A^2$ for the set of squares of elements in $A$ and $\Sigma A^2$ for the set of all sums of squares of elements in $A$. The smallest positive integer $m$ such that $-1$ is a sum of $m$ squares in $A$ is called the level of $A$, denoted $s(A)$. If there is no such integer, we set $s(A) = \infty$.

Associativity in $A$ is measured by the associator $[x, y, z] = (xy)z - x(yz)$ and commutativity by the commutator $[x, y] = xy - yx$. Define the nucleus of $A$ by $\text{Nuc}(A) = \{x \in A | [x, A, A] = \{A, x, A\} = \{A, A, x\}\}$ and the commutator by $\text{Comm}(A) = \{x \in A | [x, A] = 0\}$. A map $\tau$ is called an involution on $A$ if it is an anti-automorphism of period 2. If $2$ is an invertible element in $R$, we have $A = \text{Sym}(A, \tau) \oplus \text{Skew}(A, \tau)$ with $\text{Skew}(A, \tau) = \{x \in A | \tau(x) = -x\}$ the set of skew-symmetric elements and $\text{Sym}(A, \tau) = \{x \in A | \tau(x) = x\}$ the set of symmetric elements in $A$ with respect to $\tau$. An involution is called scalar if all norms $\tau(x)x$ and all traces $\tau(x)+x$ are scalars in $R$. For a scalar involution, $n_A(x) = \tau(x)x$ (resp. $t_A(x) = \tau(x)+x$) is a quadratic (resp. a linear form) on $A$, whenever $a1_A = 0$ implies $a = 0$, for every $a \in R$ [M] p. 86). Thus we will assume this whenever we refer to an algebra $A$ with a scalar involution.

An $R$-algebra $A$ is called quadratic in case there exists a quadratic form $n : A \to R$ such that $n(1_A) = 1$ and $x^2 - n(1_A)x + n(x)1_A = 0$ for all $x \in A$, where $n(x, y)$ denotes the induced symmetric bilinear form $n(x, y) = n(x+y) - n(x) - n(y)$. The form $n$ is uniquely determined, usually denoted by $n_A$, and is called the norm of the quadratic algebra $A$.

Let $A$ be a quadratic $R$-algebra with a scalar involution $\sigma$ and norm form $n_A(x) = x\sigma(x)$ of rank greater than 2. Put $F = \text{Skew}(A, \tau)$. If $2$ is an invertible element in $R$, then $A = R \oplus F$ and $n_A = (1) \perp n_0$ with $n_0 = n_A|_F$. The multiplication in $A$ can be described by

$$(a, u)(b, v) = (ab - B(u, v), av + bu + u \times v),$$

for $a, b \in R$ and $u, v \in F$. Here, $\times : F \times F \to F$ is a skew-symmetric $R$-bilinear map, and $B : F \times F \to R$ the symmetric bilinear form defined by $B(u, v) = \frac{1}{2}n_A(u, v)$. The scalar involution on $A$ is given by $\sigma : A \to A$, $\sigma(a, u) = (a, -u)$.

An $R$-algebra $C$ is called a composition algebra if it carries a quadratic form $n : C \to R$ satisfying the following two conditions: (i) its induced symmetric bilinear form $n(x, y) = n(x+y) - n(x) - n(y)$ is nondegenerate, i.e. determines an $R$-module isomorphism $C \to C^{\vee} = \text{Hom}_R(C, R)$; (ii) $n$ permits composition; that is, $n(xy) = n(x)n(y)$ for all $x, y \in C$.

Composition algebras are quadratic alternative algebras. More precisely, a quadratic form $n$ of the composition algebra satisfying (i) and (ii) above agrees with its norm as a quadratic algebra and thus is unique. It is called the norm of the composition algebra $C$ and is often denoted by $n_C$. Composition algebras only exist in ranks 1, 2, 4 or 8. Those of rank 2 are exactly quadratic étale $R$-algebras, those of rank 4 exactly the quaternion algebras. The ones of rank 8 are called octonion algebras.
A composition algebra over $R$ is called split if it contains a composition subalgebra isomorphic to $R \oplus R$ (see [P] for an explicit description of all possible split composition algebras). A composition algebra $C$ has a canonical involution $\overline{\cdot}$ given by $\overline{a} = t_C(x)1_C - x$, where $t_C : C \to R$ is the trace given by $t_C(x) := n(1_C,x)$. This involution is scalar.

Let $A$ be a quadratic $R$-algebra with scalar involution $\ast$ and let $\mu \in R$ be invertible. Then the $R$-module $A \oplus A$ becomes a quadratic $R$-algebra via the multiplication
\[(u,v)(u',v') = (uu' + \mu v' \ast v, v'u + \mu v' \ast v')\]
for $u, u', v, v' \in A$, with involution $(u,v)^\ast = (u^\ast, -v)$. It is called the (classical) Cayley-Dickson doubling of $A$, and is denoted by $\text{Cay}(A, \mu)$. The new involution $\ast$ is a scalar involution on $\text{Cay}(A, \mu)$ with norm $n_{\text{Cay}(A,\mu)}((u,v)) = n_A(u) - \mu n_A(v)$. The Cayley-Dickson doubling process depends on the scalar $\mu$ only up to an invertible square. By repeated application of the Cayley-Dickson doubling process starting from a composition algebra $C$ over $R$ we obtain either again a composition algebra (if the rank of the new algebra is less than or equal to 8), or a generalized Cayley-Dickson algebra of rank $\geq 16$. The latter are no longer alternative, but still flexible (i.e., $x(yz) = (xy)z$, for all elements $x, y \in A$) with a scalar involution [M].

Over fields, the classical Cayley-Dickson process generates all possible composition algebras. Over rings, a more general version is required, which yields all those composition algebras containing a composition subalgebra of half their rank. This generalized Cayley-Dickson doubling process is due to Petersson [P]: Let $D$ be a composition algebra over $R$ of rank $\leq 4$ over $R$ with canonical involution $\overline{\cdot}$. Let $P$ be a finitely generated projective right $D$-module of rank one, with a nondegenerate $-\overline{\cdot}$-hermitian form $h : P \times P \to D$ (i.e., a biadditive map $h : P \times P \to D$ with $h(ws, w't) = \overline{sh}(w, w')t$ and $h(w, w') = \overline{h(w', w)}$ for all $s, t \in D, w, w' \in P$, and where $P \to \overline{P}', w \mapsto h(w, \cdot)$ is an isomorphism of right $D$-modules). The $R$-module $C = D \oplus P$ is an $R$-algebra via the multiplication
\[(u,w)(u',w') = (uu' + h(w', w), w' \cdot u + w \cdot u')\]
for $u, u' \in D, w, w' \in P$, with $\cdot$ denoting the right $D$-module structure of $P$. It is called $\text{Cay}(D, P, h)$. Its norm is given by $n((u, w)) = n_D(u) - h(w, w)$. $D$ itself is canonically a (free) right $D$-module of rank one, equipped with a nondegenerate hermitian form $h_0 : D \times D \to D, (w, w') \mapsto \overline{w}w$. For any $\mu \in R^\times$, we obtain in this special case the “classical” doubling $\text{Cay}(D, \mu) = \text{Cay}(D, D, \mu h_0)$.

2. Some classical results in a nonassociative setting

As a first step we consider some elementary cases where every element in the algebra is a sum of squares. Of course, rings of characteristic 2 will always play a special role; for instance, let $A$ be an $R$-algebra with a scalar involution. Then
\[\Sigma A^2 \subset \{ x \in A | \text{tr}_A(x) \in k^2 \},\]
for any ring $R$ of characteristic 2, since in that case $t_A(x)^2 = t_A(x^2)$ holds for the trace map $t_A$. From now on, we will exclusively deal with rings where 2 is an invertible element. The proof of [LSW] 1.1 easily generalizes as follows:

2.1. Lemma. Let $A$ be an algebra over $R$ where $R$ can be viewed as a subring of $A$, and where $R \subset \text{Nuc}(A) \cap \text{Comm}(A) = \text{Center}(A)$. If $-1 \in \Sigma R^2$ (e.g. if $R$ is a
field which is not formally real), then
\[ A = \Sigma A^2. \]

**Proof.** The proof is exactly as given in [LSW 1.1]: Let \(-1 = \sum_{i=1}^{m} x_i^2\) in \(R\), with \(x_i \in R\). For every \(a \in A\) we have
\[
a = \left(\frac{a+1}{2}\right)^2 - \left(\frac{a-1}{2}\right)^2 = \left(\frac{a+1}{2}\right)^2 + \sum_{i=1}^{m} \left(\frac{x_i(a-1)}{2}\right)^2 \in \Sigma A^2.
\]
\[ \square \]

### 2.2. Examples.

(i) ([LSW 1.2]) Every element in the split quaternion algebra \(D = \text{Mat}_2(R)\) is a sum of 3 squares.

(ii) Let \(a \in R^\times\). Every element in the split octonion algebra \(C = \text{Cay}(D, a)\) is a sum of 6 squares and \(s(C) \leq 3\). In particular, every element in Zorn’s algebra of vector matrices \(\text{Zor}(R)\) is a sum of 6 squares. (This follows directly from (i): Each element \(x \in C\) can be written as \(x = (u, v)\) with \(u, v \in D = \text{Mat}_2(R)\). Since both \(u\) and \(v\) are sums of 3 squares in \(D\) this implies the assertion.)

(iii) Let \(C\) be a composition algebra over \(R\) (resp., any \(R\)-algebra \(A\) with a scalar involution such that \(R \subseteq \text{Center}(A)\)). If there exists an invertible element \(u \in \text{Skew}(C, \ast)\) such that \(nc(u) \in R^{\times 2}\), then \(s(C) = 1\). (Since \(nc(u) = -u^2\) write \(-1 = \frac{1}{nc(u)} u^2 = (a^{-1}u)^2\) if \(nc(u) = a^2\) with \(a \in R^\times\).) This condition is equivalent to \(C = \text{Cay}(T, P, N)\) with \(T = \text{Cay}(R, -1)\) if \(C\) is a quaternion algebra. It is satisfied for any octonion algebra \(C\) containing a quadratic étale algebra isomorphic to \(T = \text{Cay}(R, -1)\).

### 2.3. Lemma.

Let \(k\) be a field of characteristic not 2. Then any split composition algebra \(A\) over \(k\) of dimension greater than 2 has \(s(C) = 1\).

**Proof.** If \((a, b)_F\) is a split quaternion algebra, then the form \((a, b, -ab)\) is isotropic, and thus there are elements \(x_i \in F\), not all zero, such that \(-1 = ax_1^2 + bx_2^2 - abx_3^2\). Hence \(-1 = (x_1 i + x_2 j + x_3 k)^2\) with \(1, i, j, k\) a standard basis for \((a, b)_F\). This implies the assertion for split octonions. \(\square\)

We call a quadratic form \(q\) over a ring \(R\) isotropic if there exists an element \(x\) such that \(q(x) = 0\), and weakly isotropic if its multiple \(m \times q = q \perp \perp \perp q\) is isotropic, for some integer \(m\). It is well known that zero is a nontrivial sum of squares in a central simple algebra over a field of characteristic not 2 if and only if the trace form of the algebra is weakly isotropic [L2]. This turns out to be true in a more general context. Again the trace form is defined to be the quadratic form \(tr_A : A \rightarrow R, x \rightarrow t_A(x^2)\), where \(t_A\) is the trace \(t_A(x) = x + \bar{x}\) of an algebra \(A\) with scalar involution \(\ast\).

### 2.4. Proposition.

(i) Let \(A\) be any \(R\)-algebra with a scalar involution (e.g. a composition algebra). Then 0 is a nontrivial sum of squares in \(A\) if and only if the trace form \(tr_A\) is a weakly isotropic quadratic form.

(ii) ([LSW 2.4]) Let \(k\) be a formally real SAP field (e.g. a formally real algebraic extension of \(\mathbb{Q}\), or a field of transcendence degree \(\leq 1\) over a real closed field). Then 0 is a nontrivial sum of squares in every composition algebra over \(k\) of dimension greater than 2.
Proof. (i) If $0 = \sum_{i=1}^{m} x_i^2$ with $x_i \in A$ not all zero, then $0 = \sum_{i=1}^{m} tr_A(x_i^2)$ and hence $tr_A$ is weakly isotropic. Conversely, if $tr_A$ is weakly isotropic, then there are $x_i \in A$ not all zero such that $0 = \sum_{i=1}^{m} tr_A(x_i^2) = x_1^2 + \overline{x}_1^2 + \ldots + x_m^2 + \overline{x}_m^2$, and thus $0$ is a nontrivial sum of squares in $A$.

(ii) This is straightforward, since in the above situation, every trace form of a composition algebra of dimension greater than 2 is weakly isotropic [LSW 2.3]. \[ \square \]

2.4. Proposition. Let $k_0$ be a formally real field, and let $k = k_0(x_1, x_2, x_3)$ be a purely transcendental field extension of $k_0$. Then $C = \text{Cay}(k, x_1, x_2, x_3)$ is a composition division algebra over $k$ and by Springer’s theorem, $t_C$ is strongly anisotropic; hence $0$ is not a nontrivial sum of squares in $C$ by 2.4 (i).

There is a hermitian analogue of 2.4 (i) (cf. [Se] and [U] for corresponding results for central simple algebras with involutions, [PU] for results on the hermitian level of composition algebras): Define the involution trace form of an algebra $A$ with scalar involution by $t_r : C \to R, x \to t_A(\tau(x)x)$ whenever $\tau$ is any involution on $A$. Instead of sums of squares, we now look at sums of hermitian squares, i.e. sums of elements of the type $\tau(x)x$ with $x \in A$.

2.6. Theorem. Let $C$ be a composition algebra over a ring $R$ and let $\tau$ be any involution on $C$. Then $0$ is a nontrivial sum of hermitian squares $\tau(x)x$ in $C$ if and only if the involution trace form $t_r$ is a weakly isotropic quadratic form.

Proof. If $0 = \sum_{i=1}^{m} \tau(x_i)x_i$ with $x_i \in C$ not all zero, then it follows that $0 = \sum_{i=1}^{m} tr_C(\tau(x_i)x_i)$ and hence $t_r$ is weakly isotropic. Conversely, we know that $\tau \circ \tau = -\tau \circ \tau$, for any involution $\tau$ on $C$ [Pu1]. Hence $t_r(x) = \tau(x)x + \overline{x}\tau(\overline{x})$. If $t_r$ is weakly isotropic, then there are $x_i \in C$ not all zero such that $0 = \sum_{i=1}^{m} tr_C(x_i) = \sum_{i=1}^{m} (\tau(x_i)x_i + \overline{x_i}\tau(\overline{x_i})).$ Put $y_i = \tau(\overline{x_i}).$ Then $0 = \sum_{i=1}^{m} (\tau(x_i)x_i + \tau(y_i)y_i)$ is a nontrivial sum of hermitian squares in $C$.

This proof works for any quadratic $R$-algebra with a scalar involution as long as $\tau$ commutes with it.

The next result is well known for fields and is proved in [LSW, Theorem D] for division algebras which are finite-dimensional over their center (and thus in particular for quaternion algebras as well). The proof given there easily generalizes to octonion algebras.

2.7. Theorem. Let $k$ be a field of characteristic not 2, and let $C$ be a composition division algebra over $k$ of dimension greater than 2. The following are equivalent:

(i) Zero is a nontrivial sum of squares in $C$.
(ii) $-1 \in \Sigma C^2$.
(iii) $C = \Sigma C^2$.

Proof. The only nontrivial step is to prove that (i) implies (iii). Suppose that $0$ is a nontrivial sum of squares in $C$. Without loss of generality assume that $k$ is formally real (otherwise 2.1 applies and we are done). Thus $k$ has characteristic zero. Put $V = \{ x \in C | x = x^2 \in \Sigma C^2 \}$. Then $V$ is a $k$-subspace of $C$ which is invariant under all the automorphisms of $C$. Thus $V$ must be $C, 0, k1$ or $\text{Skew}(C, -)$ by [W Theorem 7]. By assumption there are $y_i \in C$ such that $0 = y_1^2 + \ldots + y_m^2$. These elements cannot all lie in $k$, since $k$ is formally real; thus assume $y_i \notin k$. Moreover, $y_i \in V$ since $-y_i^2 = y_1^2 + \ldots + y_m^2$. Therefore $V \not\subset k$. Assume that $V = \text{Skew}(C, -)$. Then $1 \notin V$ since $t_C(1) \neq 0$ and hence also $-1 \notin \Sigma C^2$. If two elements $a, b \in V$...
commute, then \( ab = \frac{1}{2}((a + b)^2 + (-a^2) + (-b^2)) \in \Sigma C^2 \). Therefore we look at a subspace of \( V \) whose elements commute: Let \( T \) be a maximal subfield of \( C \), i.e. \( C = \text{Cay}(T, d, e) \). Then \( t_C(x) = t_{T/k}(x) \) for all elements \( x \in T \), where \( t_{T/k} \) is the field trace of the field extension \( T/k \). Define \( W = T \cap V = \{x \in T | t_{T/k}(x) = 0 \} \). Since \( T = k(\sqrt{c}) \) for some element \( c \in k \) it follows that \( W = k \cdot \sqrt{c} \), implying that \(-1 = \sqrt{c}(\frac{1}{2}\sqrt{c}) \in W \subset \Sigma C^2 \), a contradiction. Thus \( V \) must be \( C \). This, however, means that \( a = \frac{1}{2}(((a + 1)^2 + (-a)^2 + (-1)) \in \Sigma C^2 \), for all \( a \in C \). \( \square \)

Obviously, the proof of the above theorem generalizes as follows to algebras over rings:

2.8. Theorem. Let \( C \) be a composition algebra over \( R \) of rank greater than 2, satisfying the following two conditions:

(1) \( 0, C, R_1 \) and \( \text{Skew}(C, \sigma) \) are the only invariant submodules relative to \( \text{Aut}(C) \).

(2) \( C \) contains a quadratic étale \( R \)-algebra isomorphic to a classical Cayley-Dickson doubling \( \text{Cay}(R, a) \).

Then the following are equivalent:

(i) \( -1 \) is a nontrivial sum of squares in \( C \).

(ii) \( -1 \in \Sigma C^2 \).

(iii) \( C = \Sigma C^2 \).

The following two statements generalize \[L, W, A, \] Theorem A] and a result in \[Ko\].

2.9. Corollary. For a composition algebra \( C \) over a field \( k \) of characteristic not 2 the following are equivalent:

(i) The trace form \( t_C \) is weakly isotropic.

(ii) \( -1 \in \Sigma C^2 \).

(iii) \( C = \Sigma C^2 \).

Let \( A \) be a quadratic \( R \)-algebra with a scalar involution \( \sigma \) and norm form \( n_A(x) = x\sigma(x) \) of rank greater than 2. Recall that \( A = R \oplus F \) and \( n_A(x) = \{1\} \perp n_0 \) with \( n_0 = n_{A|F} \) for \( F = \text{Skew}(A, \sigma) \), if \( 2 \in R^* \). The multiplication is given by \((a, u)(b, v) = (ab - B(u, v), av + bu + u \times v)\), for \( a, b \in R \) and \( u, v \in F \) with \( \times : F \times F \to R \) a skew-symmetric \( R \)-bilinear map, and \( B : F \times F \to R \) is given by \( B(u, v) = \frac{1}{2}n_A(u, v) \).

2.10. Lemma. Let \( A \) be a quadratic \( R \)-algebra with scalar involution \( \sigma \) (e.g. a composition algebra) of rank greater than 2. Then \(-1\) is a sum of \( m \) squares of “pure” elements in \( C \), i.e. elements in \( \text{Skew}(A, \sigma) \), if and only if \(-1 \in D(m \times (-n_0))\). If \( R \) is a field, this is equivalent to the form \( \{1\} \perp m \times (-n_0) \) being isotropic.

Proof. If \(-1 = u_1^2 + ... + u_m^2 \) with \( u_i \in \text{Skew}(A, \sigma) \), then \(-1 = -n_0(u_1) - ... - n_0(u_m) \) and it follows that \(-1 \) is represented by \( m \times (-n_0) \). Conversely, \(-1 = -n_0(u_1) - ... - n_0(u_m) \) implies that \(-1 = u_1^2 + ... + u_m^2 \). \( \square \)

2.11. Lemma. Let \( A \) be a quadratic \( R \)-algebra with scalar involution \( \sigma \) (e.g. a composition algebra) of rank greater than 2. Then \( s(A) \leq m \) implies that \(-1 \in D(m \times \{1\} \perp (-n_0)) \). In particular, if \( R \) is a field, then \( s(C) \leq m \) implies that \( (m + 1) \times \{1\} \perp m \times (-n_0) \) is isotropic.

Proof. Obviously, \((a, u)^2 = (a^2 - B(u, u), 2au)\) for all \( a \in R, u \in F \). Hence \( s(A) \leq m \) implies \(-1 = \sum_{i=1}^m (a_i, u_i)^2 = \sum_{i=1}^m a_i^2 - \sum_{i=1}^m B(u_i, u_i), \sum_{i=1}^m a_i^2 \). Thus
of level 2

\[ \sum_{i=1}^{m} a_i u_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} a_i^2 - \sum_{i=1}^{m} B(u_i, u_i) = -1. \]
In particular, the quadratic form \( m \times (1) \perp (-n_0) \) over \( R \) represents \(-1\). \(\square\)

We easily rephrase [Le] Theorem 2.2 for generalized Cayley-Dickson algebras and octonion algebras:

2.12. Proposition. Let \( k_0 \) be a field of characteristic not 2, and let \( A = \text{Cay}(k_0, a_1, \ldots, a_d) \) with \( d \geq 2 \) be a composition algebra or a generalized Cayley-Dickson algebra, with norm \( n_A = (1) \perp n_0 \). If the quadratic form \((2^m + 1) \times (1) \perp (2^m - 1) \times (-n_0)\) is isotropic over \( k_0 \), then \( s(A) \leq 2^m \).

The proof is analogous to the one given in [Le], since all arguments use quadratic forms only and rely on the fact that the forms \( 2^m \times (1) \) and \( n_A \) are Pfister forms.

3. OCTONION ALGEBRAS OF LEVEL \( 2^m + 1 \) AND OF LEVEL \( 2^m \)

Laghribi and Mannone [LM] presented examples of quaternion division algebras of level \( 2^m \) and \( 2^m + 1 \) for any integer \( m \geq 1 \). (The existence of such quaternion division algebras was already proved by Lewis [L3].) Their method of proof can be generalized to obtain examples of octonion division algebras of levels \( 2^m \) and \( 2^m + 1 \) for any integer \( m \geq 1 \).

Let \( s \geq 1 \) be an integer, \( k_0 \) a formally real field, and \( k = k_0(x, y, z) \) the rational function field in three variables over \( k_0 \). Define \( C = \text{Cay}(k, x, y, z) \) and \( \widetilde{\psi}_s = \langle 1 \rangle \perp s \times (-n_0) \). Since \( \widetilde{\psi}_s \) is isotropic over its function field \( k(\widetilde{\psi}_s) \), we know that \(-1\) is a sum of \( s \) squares of pure octonions in \( C \otimes_k k(\widetilde{\psi}_s) \), and in particular that \( s(C \otimes_k k(\widetilde{\psi}_s)) \leq s \) by 2.10. As in the analogous situation for quaternion algebras considered in [LM], we are able to show more when \( s = 2^m + 1 \).

3.1. Theorem. Let \( m \geq 1 \) be an integer, and let \( k = k_0(x, y, z) \) be the rational function field in three variables over a formally real field \( k_0 \). Let \( C = \text{Cay}(k, x, y, z) \) with \( n_C = (1) \perp n_0 \) and put

\[ \widetilde{\psi}_m = \langle 1 \rangle \perp (2^m + 1) \times (-n_0). \]

Then \( C = \text{Cay}(k, x, y, z) \otimes_k k(\widetilde{\psi}_m) \) is an octonion division algebra of level \( 2^m + 1 \).

For the proof we need two results which are analogous to [LM 1.2, 1.4]:

3.2. Proposition. Let \( k = k_0(x_1, \ldots, x_d) \) be the rational function field in \( d \) variables over a formally real field \( k_0 \), \( d \geq 2 \). Consider the \( d \)-fold Pfister form \( n = \langle \langle x_1, \ldots, x_d \rangle \rangle = (1) \perp n_0 \) over \( k \), with \( n_0 \) denoting its pure part.

(i) For any integers \( m, l \geq 1 \) the quadratic form \( m \times (1) \perp l \times (-n_0) \) is anisotropic over \( k \).

(ii) Let \( \varphi \) be a quadratic form over \( k \) of dimension greater than or equal to \( 2^d + 1 \) or of dimension \( 2^d \) with \( \det \varphi \neq 1 \). Then \( n \otimes_k k(\varphi) \) stays anisotropic over \( k(\varphi) \).

Proof. (i) We use induction on \( d \). The induction beginning \( (d = 2) \) is given by [LM 1.2]. Now let \( d \geq 3 \) and assume that the quadratic form \( m \times (1) \perp l \times (-n_0) = m \times (1) \perp l \times (-n_0) \times d(l \times \langle \langle x_1, \ldots, x_d-1 \rangle \rangle) \) is isotropic over \( k \), with \( n_0 \) the pure part of \( \langle \langle x_1, \ldots, x_d-1 \rangle \rangle \). By Springer’s theorem, this means that the form \( m \times (1) \perp l \times (-n_0) \) must be isotropic over \( k(x_1, \ldots, x_d-1) \) (since the form \( l \times \langle \langle x_1, \ldots, x_d-1 \rangle \rangle \) is anisotropic), contradicting our induction hypothesis.

(ii) The proof is completely analogous to the one given in [LM 1.4] and will be omitted here. \(\square\)
3.3. Proposition (cf. [LM 2.3]). Under the above assumptions, the quadratic form 
\( \tilde{\varphi}_m = (2^m + 1) \times (1) \perp 2^m \times \langle x, y, -xy, z, -xz, -yz, yxz \rangle \) stays anisotropic over 
k(\psi_m).

Proof. Using the notation from [LM], put 
\[ \psi_m = (1) \perp (2^m + 1) \times \langle x, y, -xy \rangle. \]
Then \( \psi_m \) is a subform of \( \tilde{\psi}_m \). Since \( \psi_m \) is isotropic over its function field \( k(\psi_m) \), so is 
\( \tilde{\psi}_m \); thus there exists a \( k \)-place from \( k(\tilde{\psi}_m) \) to \( k(\psi_m) \) by Knebusch [K, Theorem 3.3]. Now 
\[ \tilde{\varphi}_m = (2^m + 1) \times (1) \perp 2^m \times \langle x, y, -xy, z, -xz, -yz, yxz \rangle = \varphi_m \perp z(2^m \times (1, -x, -y, xy)) \]
over \( k_0(x, y, z) \), where \( \varphi_m = (2^m + 1) \times (1) \perp 2^m \times \langle x, y, -xy \rangle \)
as in [LM 2.3]. If \( \tilde{\varphi}_m \) is isotropic over \( k(\psi_m) \), then \( \tilde{\varphi}_m \) is isotropic over \( k(\psi_m) \). It 
follows that \( \varphi_m \) or \( 2^m \times \langle 1, -x, -y, xy \rangle \) is isotropic over \( k_0(x, y)(\psi_m) \). However, by 
[LM 2.3], \( \varphi_m \) never is. Put \( \alpha_m = (2^m + 1) \times (1, -x) \). Then \( \alpha \alpha_m \) is a subform of \( \psi_m \).
If \( 2^m \times \langle 1, -x, -y, xy \rangle \) is isotropic over \( k_0(x, y)(\psi_m) \), it must then also be isotropic 
over \( k_0(x, y)(\alpha_m) \) [K, Theorem 3.3]. This in turn implies that the quadratic form 
\( 2^m \times \langle 1, -x \rangle \) is isotropic over \( k_0(x)(\alpha_m) \), a contradiction to [LM 2.2]. □

Proof of Theorem 3.1. Let \( C_m = Cay(k, x, y, z) \otimes_k k(\tilde{\psi}_m) \) with \( m \geq 1 \). By 3.2, 
\( C_m \) is a division algebra and \( s(C_m) \leq 2^m + 1 \). If \( s(C_m) < 2^m + 1 \), then the form 
\( \tilde{\varphi}_m = (2^m + 1) \times (1) \perp 2^m \times \langle x, y, -xy, z, -xz, -yz, yxz \rangle \) becomes isotropic over 
k(\tilde{\psi}_m), contradicting 3.3. □

Note that the following remark made in [LM] applies here as well: Let 
\[ \tilde{\varphi}_s = s \times (1) \perp (s - 1) \times \langle x, y, -xy, z, -xz, -yz, yxz \rangle \]
and let \( \psi_s = (1) \perp s \times \langle x, y, -xy, z, -xz, -yz, yxz \rangle \). For each integer \( s \) for which the quadratic form 
\( \tilde{\varphi}_s = s \times (1) \perp (s - 1) \times \langle x, y, -xy, z, -xz, -yz, yxz \rangle \)
stays anisotropic over \( k(\tilde{\psi}_s) \), whenever \( \tilde{\varphi}_s \) and \( \tilde{\psi}_s \) are anisotropic, we are able to 
construct an octonion algebra of level \( s \) in a similar way as before in 3.1. Again, 
there indeed are integers \( s \) for which the quadratic form \( \tilde{\varphi}_s \) becomes isotropic over 
k(\tilde{\psi}_s), for instance \( s = 2^m \) with \( m \geq 2 \) (since [LM 2.5] can be generalized to our 
situation accordingly).

If we take the generalized Cayley-Dickson algebra \( A = Cay(k, x_1, ..., x_d, k) \), \( d \geq 4 \), 
over the rational function field \( k = k_0(x_1, ..., x_d) \), then this is a quadratic algebra 
with scalar involution. Its norm is exactly the form \( n_A = \langle x_1, ..., x_d \rangle \) in 3.2. We 
know that \( A \) is a division algebra if and only if \( n_A \) is anisotropic, and \( A \) contains 
no subalgebra of dimension 3 [3 Satz 5]. If again \( \tilde{\psi}_s = (1) \perp s \times (-n_0) \), the 
same argument as used above shows that \( -1 \) is a sum of \( s \) squares of elements in 
Skew(\( A \otimes_k k(\tilde{\psi}_s), \sigma \)). In particular, \( s(A \otimes_k k(\tilde{\psi}_s)) \leq s \). Moreover, if desired, 
the proof of 3.1 can be adapted accordingly to show that \( s(A \otimes_k k(\tilde{\psi}_s)) \) with \( s = 2^m + 1 \) 
is a generalized Cayley-Dickson algebra of level \( 2^m + 1 \).

3.4. Theorem. Let \( m \geq 1 \) be an integer, let \( k = k_0(x, y, z) \) be the rational function 
field in three variables over a formally real field \( k_0 \), and put 
\[ \lambda_m = (2^m + 1) \times (1) \perp (2^m - 1) \times (-n_0). \]
Then \( C_m = Cay(k, x, y, z) \otimes_k k(\lambda_m) \) is an octonion division algebra of level \( 2^m \).

This shows the existence of octonion algebras of level \( 2^m \). For the proof, we need 
the equivalent of [LM 3.4] as follows.
3.5. **Proposition.** The quadratic form \( \widetilde{\gamma}_m = 2^m \times \langle 1 \rangle \perp (2^m - 1) \times (-n_0) \) stays anisotropic over \( k(\widetilde{\lambda}_m) \).

**Proof.** The quadratic form \( \lambda_m = (2^m + 1) \times \langle 1 \rangle \perp (2^m - 1) \times \langle x, y, -xy \rangle \) is a subform of \( \lambda_m \). Hence there exists a \( k \)-place from \( k(\lambda_m) \) to \( k(\lambda_m) \). Now \( \widetilde{\gamma}_m = \gamma_m \perp z((2^m - 1) \times \langle 1, -x, -y, xy \rangle) \) with \( \gamma_m = 2^m \times \langle 1 \rangle \perp (2^m - 1) \times \langle x, y, -xy \rangle \). Assume that \( \widetilde{\gamma}_m \) is isotropic over \( k(\widetilde{\lambda}_m) \). Then it must also be isotropic over \( k(\lambda_m) \). Hence \( \gamma_m \) or \( (2^m - 1) \times \langle 1, -x, -y, xy \rangle \) is isotropic over \( k_0(x,y)(\lambda_m) \). However, \( \gamma_m \) never is [LM, 3.4]. Put \( \mu_m = (2^m + 1) \times \langle 1 \rangle \). Then \( \mu_m \) is a subform of \( \lambda_m \), and thus there exists a \( k_0(x,y) \)-place from \( k_0(x,y)(\lambda_m) \) to \( k_0(x,y)(\mu_m) \). This implies that the quadratic form \( (2^m - 1) \times \langle 1, -x, -y, xy \rangle \) is isotropic over \( k_0(x,y)(\mu_m) \), and in turn that the form \( (2^m - 1) \times \langle 1, -x \rangle \) is isotropic over \( k_0(x)(\mu_m) \), contradicting [LM, 3.3]. \( \square \)

**Proof of Theorem 3.4.** Let \( C_m = \text{Cay}(k, x, y, z) \otimes_k k(\lambda_m) \) with \( m \geq 1 \). This is a division algebra by 3.2 (ii). Moreover, \( s(C_m) \leq 2^m \) (2.12). In case \( s(C_m) < 2^m \) it follows that the form \( \widetilde{\gamma}_m \) becomes isotropic over \( k(\lambda_m) \), a contradiction to 3.5. \( \square \)

Again, the same idea can be used to construct examples of generalized Cayley-Dickson algebras of level \( 2^m \); for example, taking \( A = \text{Cay}(k, x_1, \ldots, x_d) \), then \( s(A \otimes_k k(\lambda_m)) = 2^m \) where \( \lambda_m = (2^m + 1) \times \langle 1 \rangle \perp (2^m - 1) \times (-n_0) \) is a generalized Cayley-Dickson algebra of level \( 2^m \).

We end with the analogue of [LM, 3.5], which reproves [Ko]:

3.6. **Proposition.** Under the same assumptions as in 3.4, \(-1\) is not a sum of \( 2^m \) squares of pure octonions in \( C_m = \text{Cay}(k, x, y, z) \otimes k(\lambda_m) \).

**Proof.** Put \( \bar{\theta} = \langle 1 \rangle \perp 2^m \times (-n_0) = \theta \perp z(2^m \times \langle 1, -x, -y, xy \rangle) \) with \( \theta = \langle 1 \rangle \perp 2^m \times \langle x, y, -xy \rangle \) as in [LM]. Assume that \( \bar{\theta} \) is isotropic over \( k(\lambda_m) \). By the same argument as in the proof of 3.5 this implies that \( \bar{\theta} \) is isotropic over \( k(\lambda_m) \), which in turn means that the forms \( \theta \) or \( 2^m \times \langle 1, -x, -y, xy \rangle \) are isotropic over \( k_0(x,y)(\lambda_m) \). However, this is a contradiction as seen in the proof of 3.5, since \( \theta \) is anisotropic over \( k_0(x,y)(\lambda_m) \) by [LM, 3.5]. \( \square \)

We thus have even constructed examples of octonion algebras of level \( 2^m \), where \(-1\) is not a sum of squares of pure octonions. Of course, the same argument can be applied to generalized Cayley-Dickson algebras, implying that in the algebra \( \text{Cay}(k, x_1, \ldots, x_d) \otimes_k k(\lambda_m) \) constructed above, \(-1\) is not a sum of \( 2^m \) squares of pure elements as well.

**References**


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