ON SUCCESSIVE COEFFICIENTS
OF ODD UNIVALENT FUNCTIONS

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Abstract. The relative growth of successive coefficients of odd univalent functions is investigated. We prove that a conjecture of Hayman is true.

1. Introduction

Let $S$ be the class of functions $f(z) = z + a_2 z^2 + \ldots$ regular and univalent in $|z| < 1$ and let $S_2$ be the subclass of odd functions $f(z) = \sum_{k=0}^{\infty} b_k z^{2k+1}$ in $S$. In the investigation of the relative growth of successive coefficients of functions in $S_2$, Goluzin [1] has established the inequality

$$||b_n| - |b_{n-1}|| \leq A n^{-1/4} \ln n \quad (n = 2, 3, \ldots),$$

where $A$ is an absolute constant. In 1963, W. Hayman [2] formulated the conjecture that

$$||b_n| - |b_{n-1}|| \leq A(\varepsilon) n^{-1/2+\varepsilon}$$

for every $\varepsilon > 0$. Lucas [3] and Huke [4], coming close to this conjecture, proved respectively the estimates

$$||b_n| - |b_{n-1}|| = O(n^{-b}),$$

where $b = \sqrt{2} - 1$ and $b = 0.42667$. In this paper we shall prove Hayman’s conjecture. If $f \in S_2$, $f_2(z) = \sqrt{f(z^2)} = \sum_{k=0}^{\infty} b_k z^{2k+1}$, where $f$ is in $S$. Thus we only need to study the successive coefficients of the function $(f(z)/z)^{1/2} = \sum_{k=0}^{\infty} b_k z^k$, where $f$ is in $S$. Our main result is

**Theorem 1.1.** Let $f \in S$ and let the coefficients $b_k$ $(k = 0, 1, 2, \ldots)$ be defined by the expansion

$$\left(\frac{f(z)}{z}\right)^{1/2} = \sum_{k=0}^{\infty} b_k z^k.$$

Then for $n = 2, 3, \ldots$,

$$||b_n| - |b_{n-1}|| \leq A n^{-1/2} \log n,$$

where $A$ is an absolute constant.
The exponent $-\frac{1}{2}$ is sharp in Theorem 1.1, since
\[ f(z) = z(1 - z^4)^{\frac{1}{4}} = \sum_{k=0}^{\infty} b_{2k+1} z^{2k+1} \]
is univalent in $|z| < 1$ and
\[ b_{4k+1} \sim k^{-\frac{1}{2}}/\sqrt{\pi}, k \to \infty, \]
while
\[ b_{4k-1} = 0. \]

2. An auxiliary result

Assume, as usual, that the logarithmic coefficients $2\gamma_k \ (k = 1, 2, \ldots)$ of the function $f(z) \in S$ are defined by the expansion
\[ \log(f(z)/z) = 2 \sum_{k=1}^{\infty} \gamma_k z^k. \] (5)

**Theorem 2.1.** Let $f \in S$. Let $r_1 = (1 - 1/n)^{\frac{1}{2}}, r_2 = (1 - 1/(2n))^{\frac{1}{2}}$ $(n = 2, 3, \ldots)$. If $t$ is chosen such that $|f(t)| = \max\{|f(z)| : |z| = r_2\}$ and if $z = r_1 e^{i\theta}$, then
\[ \int_{0}^{2\pi} \left| \frac{f(z)}{z} (1 - t^{-1} z) \right| d\theta \leq A \log n, \]
where $A$ is an absolute constant.

**Proof.** We consider the function $g(z) = 1/f(z^{-1}) \ (|z| > 1)$ in $S$ which satisfies
\[ 1/g(z) = f(z^{-1}) = z^{-1} + \sum_{n=2}^{\infty} a_n z^{-n}. \] (7)

Let $b_{kl} \ (k, l = 1, 2, \ldots)$ be the Grunsky coefficients of $g(z)$. We define
\[ \alpha_k = \alpha_k(s) = \sum_{l=1}^{\infty} b_{kl} s^{-l} \ (|s| > 1, k = 1, 2, \ldots), \] (8)
\[ h = h(z,s) = \frac{z - s}{g(z) - g(s)} = \sum_{\nu=0}^{\infty} \beta_{\nu}(s) z^{-\nu} \ (|z|, |s| > 1). \] (9)

It follows that for a fixed $s$ (see [7], p. 82),
\[ \frac{d}{dz} \log h = \frac{1}{z - s} - \frac{g'(z)}{g(z) - g(s)} = - \sum_{k=0}^{\infty} k\alpha_k(s) z^{-k-1} \] (10)
and
\[ \frac{d}{dz} h = \frac{1}{g(z) - g(s)} - \frac{(z - s) g'(z)}{(g(z) - g(s))^2} = - \sum_{\nu} \nu \beta_{\nu}(s) z^{-\nu-1}. \] (11)

The Grunsky inequality shows that (see [7], p. 83)
\[ \sum_{k=1}^{\infty} k|\alpha_k(r)|^2 \leq \sum_{k=1}^{\infty} \frac{r^{-2k}}{k} = \log \frac{1}{1 - r^{-2}}. \] (12)
Write $\beta_\nu = \beta_\nu(s)$. Let $|s| = \rho = r^{-1}_2$. Applying Milin’s theorem (see \cite{7}, p. 80), we obtain from (12) that for $n = 1, 2, \ldots$,

$$|\beta_n(r)|^2 \leq \exp\left(\sum_{k=1}^{n} k|\alpha_k|^2\right) \leq n^{-1} \exp\left(\sum_{k=1}^{\infty} k|\alpha_k|^2\right) \leq n^{-1} \frac{1}{1 - \rho^{-2}} \leq A,$$

where $A$ is an absolute constant. We obtain from (9), (10) and (11) that (see \cite{7}, p. 82)

$$\frac{d}{dz} \frac{z - s}{g(z)} = \left(1 - \frac{g(s)}{g(z)}\right) \frac{h}{dz} \log h - \frac{g(s)}{dz} \frac{1}{g(z)}.$$

We choose a fixed $s$ such that

$$|g(s)| = \min\{|g(z)| : |z| = \rho\} = \frac{1}{M(\frac{1}{\rho})},$$

where

$$M(\frac{1}{\rho}) = \max\{|f(z)| : |z| = \frac{1}{\rho}\}.$$

Write $z = re^{i\theta}$, $r \in [\rho, 2]$, $\theta \in [0, 2\pi]$ and $\lambda = r^{-1}_1$ ($n = 1, 2, \ldots$). We obtain from (14) that

$$\int_0^{2\pi} \int_0^2 \left| \frac{\lambda}{g(z)} \right| |h| |d\lambda| |d\theta| \leq \int_0^{2\pi} \int_0^2 \left| \frac{d}{dz} \frac{z - s}{g(z)} \right| |d\lambda| |d\theta| + \int_0^{2\pi} \int_0^2 |g(s)||h| |d\lambda| |d\theta|. \tag{17}$$

Now we estimate the two terms $I_1$ and $I_2$ on the right-hand side of (17).

(a) Since $|g(s)| \leq |g(z)|$, by (15) and (16), we obtain from Schwarz’s inequality and (9) and (10) that

$$|I_1| \leq 2 \int_0^{2\pi} \int_0^2 \left| \frac{h}{d\theta} d\lambda \right| |d\lambda| |d\theta| \leq 4\pi \int_0^2 \sum_{\nu=0}^{\infty} \left| \beta_\nu \right|^2 r^{-2\nu} \left(2^{\nu} \sum_{\nu=1}^{\infty} |\alpha_\nu|^2 r^{-2\nu} \right)^{\frac{1}{2}} \leq 4\pi \left[ 2 + \sum_{\nu=1}^{\infty} \frac{1}{2\nu - 1} \left| \beta_\nu \right|^2 r^{-2(\nu - 1)} \left(2^{\nu} \sum_{\nu=1}^{\infty} |\alpha_\nu|^2 r^{-2\nu} \right)^{\frac{1}{2}} \right]. \tag{18}$$

It follows from (12), (13) and (18) that

$$|I_1| \leq A \log \frac{1}{1 - \lambda^{-2}} \leq A \log n, \tag{19}$$

where $A$ is an absolute constant.
(b) We apply Schwarz’s inequality again to estimate \( I_2 \). It follows from (7) and (13) that

\[
|I_2| \leq |g(s)| \left[ \int_0^{2\pi} |h|^2 d\theta dr \right]^{1/2} \left[ \int_0^{2\pi} \frac{1}{|dz|} |g(z)|^2 d\theta dr \right]^{1/2} \\
\leq 2\pi |g(s)| \left[ 2 + 2 \sum_{\nu=1}^{\infty} \frac{1}{2\nu - 1} |\beta_\nu|^2 \lambda^{-2\nu+1} \right]^{1/2} \left[ \int_0^{2\pi} \sum_{\nu=1}^{\infty} \nu^2 |\alpha_\nu|^2 r^{-2\nu} dr \right]^{1/2} \\
\leq A |g(s)| \left[ \log \frac{1}{1 - \rho^{-2}} \right]^{1/2} \sum_{\nu=1}^{\infty} \frac{\nu}{2\nu - 1} |\alpha_\nu|^2 \lambda^{-2\nu+1} \rho^{-2},
\]

where \( A \) is an absolute constant. Because \( f(z) \) is univalent in \(|z| < 1\), we obtain from (15) and (16) that

\[
\sum_{\nu=1}^{\infty} \frac{\nu}{2\nu - 1} |\alpha_\nu|^2 \lambda^{-2\nu} \leq \frac{1}{\pi} \int_{|z| < 1/\rho} |f'(z)|^2 d\sigma \leq (M(\frac{1}{\rho}))^2 = |g(s)|^2.
\]

We obtain from (20) and (21) that

\[
|I_2| \leq A \log \frac{1}{1 - \rho^{-2}} \leq A \log n,
\]

where \( A \) is an absolute constant. The estimate obtained in (a) and (b) for the terms on the right-hand side of (17) shows that there exists an absolute constant \( A \) such that

\[
\int_0^{2\pi} \int_0^{2\pi} \left| \frac{d}{dz} g(z) \right| dr d\theta \leq A \log n.
\]

Let \( z_1 = 2e^{i\theta}, z_2 = \lambda e^{i\theta} \) and \( z = re^{i\theta}, r \in [\rho, 2] \). We obtain from (23) that

\[
\int_0^{2\pi} \frac{z_2 - s}{g(z_2)}|d\theta - \int_0^{2\pi} \frac{z_1 - s}{g(z_1)}|d\theta \leq \int_0^{2\pi} \int_0^{2\pi} \left| \frac{\partial}{\partial r} \frac{z - s}{g(z)} \right| dr d\theta \\
\leq \int_0^{2\pi} \int_0^{2\pi} \left| \frac{z_1 - s}{g(z_1)} \right| dr d\theta \leq A \log n.
\]

It is clear that

\[
\int_0^{2\pi} \left| \frac{z_1 - s}{g(z_1)} \right| d\theta \leq 2\pi M(\frac{1}{2})(|z_1| + |s|) \leq A,
\]

where \( A \) is an absolute constant. We obtain from (24) and (25) that

\[
\int_0^{2\pi} |f(z_2^{-1})||z_2^{-1} - s^{-1}| d\theta = \int_0^{2\pi} \int_0^{2\pi} \left| \frac{z_2 - s}{g(z_2)} \right| dr d\theta \leq A \log n,
\]

where \( A \) is an absolute constant. The inequality (26) implies for \( z = r_1 e^{i\theta}, |t| = r_2 \) and \( |f(t)| = \max\{|f(z)|: |z| = r_2\} \) that

\[
\int_0^{2\pi} |f(z)||z - t| d\theta \leq A \log n.
\]
3. Proof of Theorem 1.1

Proof. We define two functions and give their power series expansions (we choose single-valued brunch with value 1 at $z = 0$) when $|z| < r_2$:

(28) \[ u(z) = (f(z)/z)^{1/2}(1 - t^{-1}z)^{1/2} = \sum_{k=0}^{\infty} d_k z^k, \]

(29) \[ v(z) = u(z)(1 - t^{-1}z)^{1/2} = (f(z)/z)^{1/2}(1 - t^{-1}z) = \sum_{k=0}^{\infty} c_k z^k, \]

where $|t| = r_2 = [1 - 1/(2n)]^{1/2}$ and $|f(t)| = \max\{|f(z)| : |z| = r_2\}$. Theorem 2.1 shows that

(30) \[ \sum_{k=0}^{\infty} |d_k|^2r_1^{2k} = \frac{1}{2\pi} \int_0^{2\pi} |u(r_1e^{i\theta})|^2d\theta \leq A \log n. \]

Hence, we obtain from (30) that

(31) \[ \sum_{k=0}^{\infty} |c_k|^2r_1^{2k} = \frac{1}{2\pi} \int_0^{2\pi} |v(r_1e^{i\theta})|^2d\theta \leq \frac{2}{2\pi} \int_0^{2\pi} |u(r_1e^{i\theta})|^2d\theta \leq A \log n, \]

where $A$ is an absolute constant. Let

(32) \[ \log v(z) = \sum_{k=1}^{\infty} A_k z^k \]

for $|z| < r_2$. Then it follows from (29) that

(33) \[ A_k = \gamma_k - t^{-k}/k \quad (k = 1, 2, \ldots). \]

It is clear that $v'(z) = v(z)[\log v(z)]'$. Comparing coefficients, we obtain the recursion formula

(34) \[ n c_n = \sum_{k=1}^{n} k A_k c_{n-k}. \]

Applying Schwarz’s inequality to the recursion formula (34), we obtain that

\[ n^2|c_n|^2 r_1^{2n} \leq \sum_{k=1}^{n} k^2 |A_k|^2 r_1^{2k} \sum_{k=0}^{n} |c_k|^2 r_1^{2k} \leq n \sum_{k=1}^{n} k |A_k|^2 r_1^{2k} \sum_{k=0}^{n} |c_k|^2 r_1^{2k} \]

\[ \leq n \sum_{k=1}^{\infty} k |A_k|^2 r_1^{2k} \sum_{k=0}^{\infty} |c_k|^2 r_1^{2k}. \]

(35) \[ \leq n \sum_{k=1}^{\infty} k |A_k|^2 r_1^{2k} \sum_{k=0}^{\infty} |c_k|^2 r_1^{2k}. \]

It follows by (33) that

(36) \[ \sum_{k=1}^{\infty} k |A_k|^2 r_1^{2k} \leq 2 \left( \sum_{k=1}^{\infty} k |\gamma_k|^2 r_1^{2k} + \sum_{k=1}^{\infty} \frac{r_1}{r_2}^{2k} \right). \]

Since $r_1/r_2 < r_2$, it follows that

(37) \[ \sum_{k=1}^{\infty} k^{-1} \left( \frac{r_1}{r_2} \right)^{2k} < \sum_{k=1}^{\infty} k^{-1} r_2^{2k} = \log \frac{1}{1 - r_2^2} < \log 2n. \]
On the other hand, we have (see [6], pp. 283-290)

\[(38) \sum_{k=1}^{\infty} k|\gamma_k|^2 r_1^{2k} \leq \max\{\log |f(z)/z| : |z| = r_1\} \leq \log \frac{1}{(1-r_1)^2} < 2\log n.\]

Combining (37) and (38), we obtain from (36) that

\[(39) \sum_{k=1}^{\infty} k|A_k|^2 r_1^{2k} \leq 6\log 2n.\]

Thus, it follows from (31), (39) and (35) that

\[(40) |c_n| \leq An^{-1/2}\log n,\]

where \(A\) is an absolute constant. It follows from (29) that \(c_n = b_n - t^{-1}b_{n-1}.\)

Hence, we have

\[(41) ||b_n| - |b_{n-1}|| \leq |c_n| + (1 - \frac{1}{2n})^{-\frac{1}{2}} - 1)|b_{n-1}|.\]

It is well known that \(|b_n| < 1.17\ (n = 1, 2, \ldots)\) (see [5]). We obtain from (40) and (41) that

\[(42) ||b_n| - |b_{n-1}|| \leq An^{-\frac{1}{2}}\log n + O\left(\frac{1}{n}\right).\]

Finally, Theorem 1 follows from (42).

\[\Box\]

References


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