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JORDAN ISOMORPHISMS OF TRIANGULAR RINGS

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ABSTRACT. We investigate Jordan isomorphisms of triangular rings and give a sufficient condition under which they are necessarily isomorphisms or anti-isomorphisms. As corollaries we obtain generalizations of two recent results: the one concerning Jordan isomorphisms of triangular matrix algebras by Beidar, Brešar and Chebotar, and the one concerning Jordan isomorphisms of nest algebras by Lu.

1. Introduction

Let $\mathcal{T}$ and $\mathcal{S}$ be rings. A bijective additive map $\varphi : \mathcal{T} \to \mathcal{S}$ is called a *Jordan isomorphism* if $\varphi(xy+yx) = \varphi(x)\varphi(y)+\varphi(y)\varphi(x)$ for all $x, y \in \mathcal{T}$. Isomorphisms and anti-isomorphisms are obvious examples, and the usual goal is to describe a Jordan isomorphism through these two examples. This problem has a long history; the initial results were obtained already in the 40s and the 50s [1, 2, 9, 10, 11, 12, 17]. From a classical theorem of Herstein [9] (together with a technical improvement by Smiley [17]) it follows that every Jordan isomorphism between prime rings of characteristic not 2 is either an isomorphism or an anti-isomorphism. The situation where the rings are semiprime is more involved, but also well understood [3, 5, 6], and so the problem is now interesting for rings containing nonzero nilpotent ideals.

We denote by $\mathcal{T}_r(R)$ the ring of all $r \times r$ upper triangular matrices over a ring R. In 1998, Molnár and Šemrl [14] proved that automorphisms and antiautomorphisms are the only Jordan automorphisms of $\mathcal{T}_r(F)$, where F is a field containing at least three elements. This result was generalized by Beidar, Brešar and Chebotar [4] who proved that every Jordan isomorphism of $\mathcal{T}_r(C)$ onto an arbitrary algebra over C is either an isomorphism or an anti-isomorphism, provided that C is a unital 2-torsionfree commutative ring whose only idempotents are 0 and 1. Further, recently Lu [13] proved that every Jordan isomorphism between nest algebras is either an isomorphism or an anti-isomorphism. Our aim in this paper is to unify and generalize these results. In Section 2 we shall introduce the concept of an indecomposable triangular ring and give some examples of such rings; in particular, algebras $\mathcal{T}_r(C)$ from [4] and nontrivial nest algebras are such examples. In Section 3 we shall prove our main result stating that every Jordan isomorphism from a

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2-torsionfree unital indecomposable triangular ring onto another ring is either an isomorphism or an anti-isomorphism.

2. Indecomposable triangular rings

We fix some notation first. Let A and B be rings and let M be an (A,B)-bimodule which is faithful both as a left A-module and as a right B-module. Note that

$$Tri(A,B,M)=\left\{\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \mid a\in A, b\in B, m\in M\right\}$$

is a ring under the usual matrix operations. Following Cheung [7] we shall call $Tri(A,B,M)$ a *triangular ring*.

We shall consider A, B, and M as subsets of $\mathcal{T}=Tri(A,B,M)$, i.e. we shall identify them by their copies inside $\mathcal{T}$. Note that $AB=BA=MA=BM=M^2=0$.

We recall that a ring R is said to be unital if it contains unity, which will be denoted by 1_R, and to be 2-torsionfree if it does not contain a nonzero element a such that $2a=0$. Suppose that $\mathcal{T}$ is a unital ring. Note that $1_{\mathcal{T}}$ is necessarily of the form

$$1_{\mathcal{T}}=\begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}$$

where unities 1_A and 1_B of A and B also satisfy $1_Am=m=m1_B$ for all $m\in M$.

In what follows, we write submodule in short for (A,B)-subbimodule of M. We shall say that a ring R is an *indecomposable triangular ring* if it is isomorphic to a triangular ring $Tri(A,B,M)$ such that M cannot be written as a direct sum of two nonzero submodules. We will give two examples of such rings in Theorems 2.1 and 2.7.

Theorem 2.1. *Suppose that R is a unital ring which contains no nontrivial central idempotents. Then $\mathcal{T}_r(R)$ is an indecomposable triangular ring for every $r\geq 2$.*

Proof. Pick any positive integers s,t such that $r=s+t$. Set $A=\mathcal{T}_s(R), B=\mathcal{T}_t(R)$, and $M=M_{s\times t}(R)$. Note that $\mathcal{T}_r(R)\cong Tri(A,B,M)$.

The matrix units of A, B and M will be denoted by e'_{ij}, e''_{ij} and e_{ij} respectively. Suppose $M=P\oplus Q$, where P,Q are submodules of M. Write $e_{s1}=p+q$ where $p\in P$ and $q\in Q$. Then $p+q=e_{s1}=e'_{ss}e_{s1}e''_{11}=e'_{ss}(p+q)e''_{11}=e'_{ss}pe''_{11}+e'_{ss}qe''_{11}$. Since $e'_{ss}\in A$ and $e''_{11}\in B$ we have $p-e'_{ss}pe''_{11}=-q+e'_{ss}qe''_{11}\in P\cap Q=0$ and so $p=e'_{ss}pe''_{11}=\alpha e_{s1}$ and $q=e'_{ss}qe''_{11}=\beta e_{s1}$ for some $\alpha,\beta\in R$ such that $\alpha+\beta=1_R$. Let $x\in R$. Note that, on the one hand,

$$\begin{aligned} xe_{s1}=(xe'_{ss})e_{s1} &= xe'_{ss}(\alpha e_{s1}+\beta e_{s1}) \\ &= x\alpha e_{s1}+x\beta e_{s1} \end{aligned}$$

and, on the other hand,

$$xe_{s1}=e_{s1}(xe''_{11}) = (\alpha e_{s1}+\beta e_{s1})xe''_{11}=\alpha xe_{s1}+\beta xe_{s1}.$$

Since $xe'_{ss}\in A$ and $xe''_{11}\in B$, we have $x\alpha=\alpha x$ and $x\beta=\beta x$ for all $x\in R$, i.e. α,β lie in the center of R. If $r\in\alpha R\cap\beta R$, then $re_{s1}\in P\cap Q=0$ and so $r=0$. Therefore $R=\alpha R\oplus\beta R$ as ideals. From $\alpha\beta,\beta\alpha\in\alpha R\cap\beta R=0$ and $1=\alpha+\beta=(\alpha+\beta)^2=\alpha^2+\beta^2$ it follows that $\alpha^2=\alpha$ and $\beta^2=\beta$ and so α,β are central idempotents of R. By the assumption, we have either $\alpha=0$ or $\beta=0$, say $\beta=0$, and so $e_{s1}\in P$. Let $x\in R$, $1\leq i\leq s, 1\leq j\leq t$. Since $xe'_{is}\in A$

and $e''_{1j} \in B$, we have $xe_{ij} = xe'_{is}e_{s1}e''_{1j} \in P$ and so $P = M$. Thus, $\mathcal{T}_r(R)$ is an indecomposable triangular ring. $\square$

A nest $\mathcal{N}$ is a totally ordered set of closed subspaces of a Hilbert space H such that $0, H \in \mathcal{N}$ and $\mathcal{N}$ is closed under arbitrary intersections and closed linear spans of its elements. By $\mathcal{B}(H)$ we denote the algebra of all bounded linear operators on H. The nest algebra $\mathcal{T}(\mathcal{N})$ associated to $\mathcal{N}$ is a subalgebra of $\mathcal{B}(H)$ consisting of those operators that leave N invariant for every $N \in \mathcal{N}$, i.e. $\mathcal{T}(\mathcal{N}) = \{T \in \mathcal{B}(H) \mid TN \subseteq N \text{ for all } N \in \mathcal{N}\}$. We follow [8] for the following notation and definitions. Let $N \in \mathcal{N}$. By $N^\perp$ we denote the orthogonal complement of N, by E_N we denote the orthogonal projection of H onto N and by $x \otimes y^*$, for elements $x, y \in H$, we denote the rank one operator $w \longmapsto \langle w, y\rangle x$ for all $w \in H$. Further, we define $N_- = \sup\{N' \in \mathcal{N} \mid N' \subset N\}$, $N_+ = \inf\{N' \in \mathcal{N} \mid N \subset N'\}$ and $N' \ominus N = N' \cap N^\perp$ for $N \subseteq N' \in \mathcal{N}$.

Suppose that $\mathcal{N}$ is nontrivial, i.e. $\mathcal{N}$ contains other spaces than 0 and H. The associated nest algebra $\mathcal{T}(\mathcal{N})$ is then called nontrivial. We now fix $N_0 \in \mathcal{N}$ in the following way: $N_0 = 0_+$ if $0_+ \neq 0$, and if $0_+ = 0$, then let N_0 be just any element in $\mathcal{N}$ different from $0, H$. We denote E_{N_0} by E; hence $E_{N_0^\perp} = 1 - E$. Let $A = \{T \in \mathcal{B}(H) \mid T = ETE\}$, $B = \{T \in \mathcal{B}(H) \mid T = (1-E)T(1-E)\}$ and $M = \{T \in \mathcal{B}(H) \mid T = ET(1-E)\}$. Let $T \in \mathcal{T}(\mathcal{N})$. Note that $(1-E)TE = 0$. Consider the map $T \mapsto \begin{pmatrix} ETE & ET(1-E) \\ 0 & (1-E)T(1-E) \end{pmatrix}$. We can see that it is an isomorphism and so $\mathcal{T}(\mathcal{N}) \cong Tri(A, B, M)$. We remark that this was noted, in a somewhat different form, in [7, Proposition 5].

We remark that $x \otimes y^* \in \mathcal{T}(\mathcal{N})$ if there exists $N \in \mathcal{N}$ such that $x \in N$ and $y \in N_-^\perp$ (see [8, Lemma 2.8] or [15, Lemma 3.3]). In what follows, N always denotes some element of $\mathcal{N}$. According to the definitions of A and B, we have

Lemma 2.2. (i) *If $N \subseteq N_0$, then $x \otimes y^* \in A$ for all $x \in N, y \in N_0 \ominus N_-$.*

(ii) *If $N \supseteq N_0$, then $x \otimes y^* \in B$ for all $x \in N \ominus N_0$, $y \in N_-^\perp$.*

Lemma 2.3. *Suppose $0_+ \neq 0$. Then $x \otimes y^* \in A$ for all $x, y \in N_0$.*

Proof. Note that $(0_+)_- = 0$ if $0_+ \neq 0$. Since $N_0 = 0_+$ in this case, $(N_0)_- = 0$. By substituting N by $N_0 = 0_+$ in Lemma 2.2(i), we have the desired result. $\square$

Lemma 2.4. *Suppose $0_+ = 0$. If $0 \neq x \in N_0$, then there exists $0 \neq N \in \mathcal{N}$, $N \subset N_0$ such that $x \notin N$.*

Proof. Since $x \neq 0$ and $0_+ = 0$, we have $x \notin 0_+ = \inf\{N \in \mathcal{N} \mid N \neq 0\}$ and so the desired result follows. $\square$

Lemma 2.5. *Let L be a nonzero submodule of M. Then there exist rank one elements in L.*

Proof. Let $0 \neq T \in L$. Then there exists $v \in N_0^\perp$ such that $0 \neq u = Tv \in N_0$. If $0_+ \neq 0$, then by Lemma 2.3 we have $u \otimes u^* \in A$ and so $0 \neq (u \otimes u^*)T \in L$ which is a rank one element. If $0_+ = 0$, then by Lemma 2.4 we have $u \notin N$ for some $0 \neq N \in \mathcal{N}$ and $N \subset N_0$. Let $x = E_{N^\perp}u \neq 0$ and choose $0 \neq w \in N$. From Lemma 2.2(i), we have $w \otimes x^* \in A$, and so $0 \neq (w \otimes x^*)T \in L$ which is an element of rank one. $\square$

Lemma 2.6. *Let L be a submodule of M and $0 \neq u \otimes v^* \in L$ for some $u \in N_0$, $v \in N_0^\perp$.*

(i) *If $u \notin N$ for some $N \subset N_0$, then $w \otimes v^* \in L$ for all $w \in N$.*
(ii) *If $v \in N$ for some $N \supset N_0$, then $u \otimes w^* \in L$ for all $w \in N_-^\perp$.*
(iii) *$u \otimes (E_{N^\perp} v)^* \in L$ for all $N \supseteq N_0$.*

Proof. (i) Let $x = E_{N^\perp} u$. Note that $x \neq 0$. Since $w \otimes x^* \in A$ by Lemma 2.2(i), we have $(w \otimes x^*)(u \otimes v^*) = ||x||^2 (w \otimes v^*) \in L$ and so $w \otimes v^* \in L$. (ii) By Lemma 2.2(ii) we have $v \otimes w^* \in B$ and so $(u \otimes v^*)(v \otimes w^*) = ||v||^2 (u \otimes w^*) \in L$, which in turn implies $u \otimes w^* \in L$. (iii) Let $x = E_{N^\perp} v$. Our claim trivially holds if $x = v$. In case $x \neq v$, then $y = E_N v \neq 0$. It follows from Lemma 2.2(ii) that $y \otimes x^* \in B$ and so $(u \otimes v^*)(y \otimes x^*) = ||y||^2 (u \otimes x^*) \in L$, which in turn implies $u \otimes x^* \in L$. □

Theorem 2.7. *Nontrivial nest algebras are indecomposable triangular rings.*

Proof. We have $\mathcal{T}(\mathcal{N}) \cong Tri(A, B, M)$ where A, B, M are as above. Suppose that M can be decomposed into a direct sum of two nonzero submodules P and Q. We show that this will lead to a contradiction. First, for $w \in H$, we define N_w as the least element of $\mathcal{N}$ containing w, i.e. $N_w = \inf\{N \in \mathcal{N} \mid w \in N\}$. Suppose $0 \neq u \otimes v^* \in P$ and $0 \neq u' \otimes v'^* \in Q$ for some $u, u' \in N_0$ and $v, v' \in N_0^\perp$. If $0_+ \neq 0$, then by Lemma 2.3 we have $u \otimes u'^* \in A$ and so $(u \otimes u'^*)(u' \otimes v'^*) = ||u'||^2 (u \otimes v'^*) \in Q$, which in turn implies $u \otimes v'^* \in Q$. If $0_+ = 0$ and if $N_u = N_{u'}$, then by Lemma 2.4 we have $u \notin N$ for some $0 \neq N \in \mathcal{N}$ and $N \subset N_0$. Choose $0 \neq w \in N$. By Lemma 2.6(i) we have $0 \neq w \otimes v^* \in P$ and $N_w \subseteq N \neq N_u = N_{u'}$. That is, we have that if $0 \neq u \otimes v^* \in P$ and $0 \neq u' \otimes v'^* \in Q$, then

$$u \otimes v'^* \in Q \text{ in case } 0_+ \neq 0; \tag{1}$$

and

$$\text{there exists } 0 \neq w \otimes v^* \in P \text{ such that } N_w \neq N_{u'} \text{ in case } 0_+ = 0. \tag{2}$$

Next, suppose $u = u'$ and $N_v \neq N_{v'}$, say $N_v \subset N_{v'}$. Note that $v' \notin N_v$ and so $E_{N_v^\perp} v' \neq 0$. It follows from Lemma 2.6(ii) and (iii) that $0 \neq u \otimes (E_{N_v^\perp} v')^* \in P \cap Q$, a contradiction. Therefore, we have

$$\text{if } 0 \neq u \otimes v^* \in P \text{ and } 0 \neq u \otimes v'^* \in Q, \text{ then } N_v = N_{v'}. \tag{3}$$

Suppose $N_u \neq N_{u'}$, say $N_u \subset N_{u'}$. Then by Lemma 2.6(i) we have $0 \neq u \otimes v'^* \in Q$. By the result in (3), we have $N_v = N_{v'}$. That is, we have

$$\text{if } 0 \neq u \otimes v^* \in P \text{ and } 0 \neq u' \otimes v'^* \in Q, \text{ then either } N_u = N_{u'} \text{ or } N_v = N_{v'}. \tag{4}$$

In case $0_+ \neq 0$, by (1) we have $0 \neq u \otimes v'^* \in Q$ and so $N_v = N_{v'}$ by (3). Suppose $0_+ = 0$. If $N_u \neq N_{u'}$, then by (4) we have $N_v = N_{v'}$. If $N_u = N_{u'}$, then by (2) there exists $0 \neq w \otimes v^* \in P$ and $N_w \neq N_{u'}$ and so, by (4) again, we have $N_v = N_{v'}$. Therefore, it follows that

$$\text{if } 0 \neq u \otimes v^* \in P \text{ and } 0 \neq u' \otimes v'^* \in Q, \text{ then } N_v = N_{v'}. \tag{5}$$

Suppose that $N_v \neq H$. Choose $0 \neq w \in N_v^\perp$. Since by Lemma 2.6(ii) we have $0 \neq u \otimes w^* \in P$, then by (5) we have $N_w = N_{v'}$. However, it follows from $w \notin N_v$ that $N_w \neq N_v = N_{v'}$, a contradiction. Therefore, we have

$$\text{if } 0 \neq u \otimes v^* \in P, \text{ then } N_v = H. \tag{6}$$

Suppose that $v \notin N^\perp$ for some $N_0 \subseteq N \neq H$. Let $x = E_N v \neq 0$ and $y = E_{N^\perp} v$. From Lemma 2.6(iii) it follows that $u \otimes y^* \in P$ and so $0 \neq u \otimes x^* = u \otimes v^* - u \otimes y^* \in P$.

By (6), we have $N_x = H$, but $N_x \subseteq N \neq H$, a contradiction. Therefore $v \in H_-^\perp$. By Lemma 2.6(ii), we have $u' \otimes v^* \in Q$. Suppose $N_u \neq N_{u'}$, say $N_u \subset N_{u'}$. Since $u' \notin N_u$, from Lemma 2.6(i) it follows that $0 \neq u \otimes v^* \in P \cap Q$, a contradiction. Therefore we have

$$\text{if } 0 \neq u \otimes v^* \in P \text{ and } 0 \neq u' \otimes v'^* \in Q, \text{ then } u' \otimes v^* \in Q \text{ and } N_u = N_{u'}. \tag{7}$$

If $0_+ = 0$, then by (2) there exist $0 \neq w \otimes v^* \in P$ such that $N_w \neq N_{u'}$; this contradicts (7) and so we have

$$0_+ \neq 0. \tag{8}$$

Now by Lemma 2.5, there exist $0 \neq u \otimes v^* \in P$ and $0 \neq u' \otimes v'^* \in Q$ for some $u, u' \in N_0$ and $v, v' \in N_0^\perp$. By (7) we have $u' \otimes v^* \in Q$. Now, by (8), $0_+ \neq 0$ and so it follows from (1) that $0 \neq u \otimes v^* \in P \cap Q$, a contradiction. This completes the proof. □

3. Jordan isomorphisms

Theorem 3.1. *Let $\mathcal{T}$ be a 2-torsionfree unital indecomposable triangular ring. Then every Jordan isomorphism from $\mathcal{T}$ onto another ring is either an isomorphism or an anti-isomorphism.*

Proof. In the proof we use some ideas from [4].

We may assume that $\mathcal{T} = Tri(A, B, M)$ where M cannot be decomposed. Let φ be a Jordan isomorphism from $\mathcal{T}$ onto a ring $\mathcal{S}$. Since $\mathcal{T}$ is 2-torsionfree, so is $\mathcal{S}$, and so φ clearly satisfies $\varphi(x^2) = \varphi(x)^2$ for all $x \in \mathcal{T}$. We denote $xy + yx$ by $x \circ y$. Since $2xyx = x \circ (x \circ y) - x^2 \circ y$, we see that φ also satisfies $\varphi(xyx) = \varphi(x)\varphi(y)\varphi(x)$ for all $x, y \in \mathcal{T}$. This obviously yields $\varphi(xyz + zyx) = \varphi(x)\varphi(y)\varphi(z) + \varphi(z)\varphi(y)\varphi(x)$ for all $x, y, z \in \mathcal{T}$. In what follows we shall often use these identities without explicit mention.

Let $e = \varphi(1_A)$ and $f = \varphi(1_B)$. Obviously, e and f are idempotents in $\mathcal{S}$. From $e \circ f = \varphi(1_A \circ 1_B) = 0 = \varphi(1_A 1_B 1_A) = efe$ it follows that $ef = 0 = fe$. Further, let $a \in A$, $\varphi(a) = \varphi(1_A a 1_A) = e\varphi(a)e$ and $2\varphi(a) = \varphi(1_A \circ a) = e \circ \varphi(a)$ imply that $\varphi(a) = e\varphi(a) = \varphi(a)e$, i.e. $e = 1_{\varphi(A)}$. Similarly we see that $f = 1_{\varphi(B)}$ and $e + f = \varphi(1_\mathcal{T}) = 1_\mathcal{S}$.

Let $a \in A, b \in B, m \in M$. It follows from $1_A b 1_A = 0 = 1_A m 1_A$ that $e\varphi(b)e = 0 = e\varphi(m)e$. Similarly, $f\varphi(a)f = 0 = f\varphi(m)f$. Therefore $\varphi(A) = e\varphi(\mathcal{T})e = e\mathcal{S}e$ and $\varphi(B) = f\varphi(\mathcal{T})f = f\mathcal{S}f$ are subrings of $\mathcal{S}$. Since $\varphi(A)\varphi(B) = 0 = \varphi(B)\varphi(A)$, we have

$$\varphi(a)\varphi(m)\varphi(b) = \varphi(a)(\varphi(m) \circ \varphi(b)) = \varphi(a)\varphi(m \circ b) = \varphi(a)\varphi(mb)$$

and

$$\varphi(b)\varphi(m)\varphi(a) = \varphi(b)(\varphi(a) \circ \varphi(m)) = \varphi(b)\varphi(a \circ m) = \varphi(b)\varphi(am).$$

Comparing $\varphi(amb) = \varphi(amb+bma) = \varphi(a)\varphi(m)\varphi(b)+\varphi(b)\varphi(m)\varphi(a) = \varphi(a)\varphi(mb) + \varphi(b)\varphi(am)$ with $\varphi(amb) = \varphi(am \circ b) = \varphi(am) \circ \varphi(b) = \varphi(am)\varphi(b) + \varphi(b)\varphi(am)$, we have $\varphi(a)\varphi(mb) = \varphi(am)\varphi(b)$. Further, since $a \circ mb = amb = am \circ b$, we have $\varphi(mb)\varphi(a) = \varphi(b)\varphi(am)$, that is,

$$\begin{aligned} \varphi(a)\varphi(mb) &= \varphi(am)\varphi(b), \\ \varphi(mb)\varphi(a) &= \varphi(b)\varphi(am) \end{aligned}$$

for all $a \in A, b \in B, m \in M$. In particular, we have

$$\begin{aligned} \varphi(a)\varphi(m) = \varphi(am)f \quad &, \quad \varphi(m)\varphi(a) = f\varphi(am), \\ \varphi(m)\varphi(b) = e\varphi(mb) \quad &, \quad \varphi(b)\varphi(m) = \varphi(mb)e, \\ e\varphi(m) = \varphi(m)f \quad &, \quad \varphi(m)e = f\varphi(m) \end{aligned} \tag{9}$$

for all $a \in A, b \in B, m \in M$.

We claim first $e\varphi(M) \subseteq \varphi(M)$. Let $m \in M$ and write $e\varphi(m) = \varphi(a)+\varphi(b)+\varphi(n)$ for some $a \in A, b \in B, n \in M$. Multiplying this relation from the right by e and from the left by f separately and using $e\varphi(m)e = 0, ef = 0 = fe$, we have $0 = \varphi(a) + \varphi(n)e$ and $0 = \varphi(b) + f\varphi(n)$. Since, by (9), $\varphi(n)e = f\varphi(n)$, we have $\varphi(a) = \varphi(b) = 0$ and so $e\varphi(m) = \varphi(n) \in \varphi(M)$, as desired. Therefore, we have $e\varphi(M) \subseteq \varphi(M)$ and $f\varphi(M) = (1_{\mathcal{S}} - e)\varphi(M) \subseteq \varphi(M)$ and so $\varphi^{-1}(e\varphi(M)) \subseteq M$ and $\varphi^{-1}(f\varphi(M)) \subseteq M$.

We claim next that M is a direct sum of $\varphi^{-1}(e\varphi(M))$ and $\varphi^{-1}(f\varphi(M))$, which are both submodules. Let $x \in \varphi^{-1}(e\varphi(M)) \subseteq M$. Therefore $xa = 0$ and it follows from (9) that $\varphi(x)\varphi(a) \subseteq e\varphi(M)\varphi(A) \subseteq ef\varphi(M) = 0$ and $\varphi(a)\varphi(x) \in e\varphi(M)$. Therefore $\varphi(ax) = \varphi(a \circ x) = \varphi(a) \circ \varphi(x) = \varphi(a)\varphi(x) \in e\varphi(M)$, i.e. $ax \in \varphi^{-1}(e\varphi(M))$, and so $\varphi^{-1}(e\varphi(M))$ is a left A-submodule of M. Similarly, $\varphi^{-1}(e\varphi(M))$ is a right B-submodule and $\varphi^{-1}(f\varphi(M))$ is a submodule. Therefore $M = \varphi^{-1}(e\varphi(M)) \oplus \varphi^{-1}(f\varphi(M))$ as submodules. According to our assumption we have either $f\varphi(M) = 0$ or $e\varphi(M) = 0$.

Case 1: Suppose $f\varphi(M) = 0$. Then $\varphi(m) = e\varphi(m) = \varphi(m)f$ for all $m \in M$, and so $\varphi(M)\varphi(M) = \varphi(M)fe\varphi(M) = 0$ and $\varphi(M)\varphi(A) = 0 = \varphi(B)\varphi(M)$. Hence, by (9)

$$\begin{aligned} \varphi(a)\varphi(m) = \varphi(am)f = \varphi(am), \\ \varphi(m)\varphi(b) = e\varphi(mb) = \varphi(mb), \end{aligned}$$

and so, since M is faithful both as a left A-module and as a right B-module, we see that $\varphi(M)$ is faithful as a left $\varphi(A)$-module as well as a right $\varphi(B)$-module. Hence, it follows from $\varphi(aa')\varphi(m) = \varphi((aa')m) = \varphi(a(a'm)) = \varphi(a)\varphi(a'm) = \varphi(a)\varphi(a')\varphi(m)$ and $\varphi(m)\varphi(bb') = \varphi(m(bb')) = \varphi((mb)b') = \varphi(mb)\varphi(b') = \varphi(m)\varphi(b)\varphi(b')$ that $\varphi(aa') = \varphi(a)\varphi(a')$ and $\varphi(bb') = \varphi(b)\varphi(b')$. Therefore,

$$\begin{aligned} & \varphi((a+b+m)(a'+b'+m')) = \varphi(aa'+bb'+am'+mb') \\ = \quad & \varphi(aa') + \varphi(bb') + \varphi(am') + \varphi(mb') \\ = \quad & \varphi(a)\varphi(a') + \varphi(b)\varphi(b') + \varphi(a)\varphi(m') + \varphi(m)\varphi(b') \\ = \quad & (\varphi(a)+\varphi(b)+\varphi(m))(\varphi(a')+\varphi(b')+\varphi(m')) \\ = \quad & \varphi(a+b+m)\varphi(a'+b'+m'); \end{aligned}$$

that is, φ is an isomorphism.

Case 2: Suppose $e\varphi(M) = 0$. Then $\varphi(m) = f\varphi(m) = \varphi(m)e$ for all $m \in M$, and so

$$\varphi(M)\varphi(M) = 0 = \varphi(A)\varphi(M) = \varphi(M)\varphi(B). \tag{10}$$

By (9), we have

$$\begin{aligned} \varphi(m)\varphi(a) = f\varphi(am) = \varphi(am), \\ \varphi(b)\varphi(m) = \varphi(mb)e = \varphi(mb), \end{aligned} \tag{11}$$

and so, $\varphi(M)$ is faithful as a left $\varphi(B)$-module as well as a right $\varphi(A)$-module. Therefore, it follows from $\varphi(m)\varphi(aa') = \varphi((aa')m) = \varphi(a(a'm)) = \varphi(a'm)\varphi(a) = \varphi(m)\varphi(a')\varphi(a)$ and $\varphi(bb')\varphi(m) = \varphi(m(bb')) = \varphi((mb)b') = \varphi(b')\varphi(mb) = \varphi(b')\varphi(b)\varphi(m)$ that $\varphi(aa') = \varphi(a')\varphi(a)$ and $\varphi(bb') = \varphi(b')\varphi(b)$; together with (10) and (11) we have that φ is an anti-isomorphism. □

Now we have the following two immediate results which generalize [4, Theorem] and [13, Theorem 15] respectively.

Theorem 3.2. *Let R be a 2-torsionfree unital ring and let $r \geq 2$ be an integer. Then R contains no nontrivial central idempotents if and only if every Jordan isomorphism of $\mathcal{T}_r(R)$ onto any ring is either an isomorphism or an anti-isomorphism.*

Proof. If R contains no nontrivial central idempotents, then the desired result follows from Theorem 2.1 and Theorem 3.1. Suppose that R contains a nontrivial central idempotent ε. Then the map $A \mapsto \varepsilon A + (1-\varepsilon)UA^{\mathrm{tr}}U$ (cf. [4, 14]), where A^{tr} denotes the transpose of A and $U = e_{1r} + e_{2\,r-1} + \ldots + e_{r-1\,2} + e_{r1}$, is a Jordan automorphism of $\mathcal{T}_r(R)$ which is neither an automorphism nor an anti-automorphism. This completes the proof. □

Theorem 3.3. *Every Jordan isomorphism from a nest algebra onto another complex algebra is either an isomorphism or an anti-isomorphism.*

Proof. Let $\mathcal{T}(\mathcal{N})$ be a nest algebra. If $\mathcal{N}$ is trivial, then $\mathcal{T}(\mathcal{N}) = \mathcal{B}(H)$ is a prime ring and so we have the desired conclusion by Herstein's result [9]. If $\mathcal{N}$ is nontrivial, then we apply Theorems 2.7 and 3.1. □

We remark that using Theorem 3.3 and the facts that every isomorphism between nest algebras is spatial [16, Theorem 4.2] (see also [8, Corollary 17.13]) and the composition of an anti-isomorphism and $*$ (adjoint) is an isomorphism, we can get [13, Theorem 15], which states that every Jordan isomorphism between nest algebras is of the form $T \mapsto STS^{-1}$ or $T \mapsto ST^*S^{-1}$ for some invertible operator S.

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