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COMPARISON THEOREMS OF HILLE–WINTNER TYPE FOR DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. We obtain an analogue of the Hille–Wintner comparison theorem for the nonoscillation of second-order linear dynamic equations on time scales. Several examples are given including applications to difference equations.

1. Introduction and preliminary results

We will be concerned with proving the analogue of the Hille–Wintner comparison theorem concerning the oscillation and nonoscillation of the two second-order linear dynamic equations

$$L_1 x = (r_1(t)x^\Delta)^\Delta + q_1(t)x^\sigma = 0, \tag{1.1}$$

$$L_2 x = (r_2(t)x^\Delta)^\Delta + q_2(t)x^\sigma = 0, \tag{1.2}$$

where r_i and q_i, $i = 1, 2$, are real-valued, right-dense continuous functions on a time scale $\mathbb{T} \subset \mathbb{R}$, with $\sup \mathbb{T} = \infty$. We also assume throughout that

$$0 < r_1(t) \le r_2(t), \quad t \in \mathbb{T}, \quad \int_a^\infty \frac{1}{r_1(t)} \Delta t = \infty, \quad \int_a^\infty q_i(t)\Delta t \tag{1.3}$$

exist for $i = 1, 2$, and that either

$$0 \le \int_t^\infty q_2(s)\Delta s \le \int_t^\infty q_1(s)\Delta s \tag{1.4}$$

for all large t, or

$$\left| \int_t^\infty q_2(s)\Delta s \right| \le \int_t^\infty q_1(s)\Delta s \tag{1.5}$$

for all large t.

If $r_i > 0$, q_i, $i = 1, 2$, are real-valued continuous functions satisfying

$$0 < r_1(t) \le r_2(t), \quad q_2(t) \le q_1(t)$$

for t in the real interval $[a, \infty)$, then the Sturm comparison theorem says that if

$$(r_1(t)x')' + q_1(t)x = 0$$

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is nonoscillatory on $[a,\infty)$ (i.e. all nontrivial solutions have finitely many zeros in $[a,\infty)$), then

$$(r_2(t)x')' + q_2(t)x = 0$$

is also nonoscillatory on $[a,\infty)$. The Hille–Wintner theorem replaces the point-wise comparisons on the coefficient functions by integral comparisons. (See Remark 2.2 for additional details.) There are many recent additional results for oscillation of dynamic equations on time scales for both the linear and nonlinear case (see, e.g., [1], [4]-[13], [18], [20]).

For completeness, we recall the following concepts related to the notion of time scales. A **time scale** $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ and, since oscillation of solutions is our primary concern, we make the blanket assumption that $\sup \mathbb{T} = \infty$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The **forward jump operator** and the **backward jump operator** are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : \ s < t\},$$

where $\sup \emptyset = \inf \mathbb{T}$. A point $t \in \mathbb{T}$ is said to be **left-dense** if $\rho(t) = t$ and $t > \inf \mathbb{T}$, **right-dense** if $\sigma(t) = t$, **left-scattered** if $\rho(t) < t$ and **right-scattered** if $\sigma(t) > t$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be **right-dense continuous** (rd-continuous) provided g is continuous at right-dense points and at left-dense points in $\mathbb{T}$, left-hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$. The **graininess function** μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$.

Definition 1.1. Fix $t \in \mathbb{T}$ and let $x : \mathbb{T} \to \mathbb{R}$. Define $x^\Delta(t)$ to be the number (if it exists) with the property that given any $\epsilon > 0$ there is a neighbourhood U of t with

$$|[x(\sigma(t)) - x(s)] - x^\Delta(t)[\sigma(t) - s]| \le \epsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

In this case, we say $x^\Delta(t)$ is the **(delta) derivative** of x at t and that x is **(delta) differentiable** at t.

The following theorem is due to Hilger [15].

Theorem 1.2. *Assume that $g : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}$.*

(i) *If g is differentiable at t, then g is continuous at t.*

(ii) *If g is continuous at t and t is right-scattered, then g is differentiable at t with*

$$g^\Delta(t) = \frac{g(\sigma(t)) - g(t)}{\sigma(t) - t}.$$

(iii) *If g is differentiable and t is right-dense, then*

$$g^\Delta(t) = \lim_{s \to t} \frac{g(t) - g(s)}{t - s}.$$

(iv) *If g is differentiable at t, then $g(\sigma(t)) = g(t) + \mu(t)g^\Delta(t)$.*

Definition 1.3. If $G^\Delta(t) = g(t)$, then the **Cauchy (delta) integral** of g is defined by

$$\int_a^t g(s)\Delta s := G(t) - G(a).$$

For a more general definition of the delta integral see [2], [3].

We assume throughout that $t_0 \geq 0$ and $t_0 \in \mathbb{T}$. By the interval $[t_0, \infty)$ we mean the set $[t_0, \infty) \cap \mathbb{T}$. The theory of time scales dates back to Hilger [15]. The monographs [2], [3] and [17] also provide an excellent introduction.

The following lemma [12, Lemma 13] will be useful in the proof of Theorem 2.1.

Lemma 1.4. *Assume*

$$\liminf_{t\to\infty} \int_T^t q_1(s)\Delta s \geq 0 \quad \textit{and not identically } 0 \tag{1.6}$$

for all large T, and $\int_a^\infty \frac{1}{r_1(s)}\Delta s = \infty$. If (1.1) *has a positive solution $x = x(t)$ for all sufficiently large t, then $x^\Delta(t) > 0$ for all sufficiently large t.*

2. Main results

We recall that a solution of equation (1.1) is said to be **oscillatory** on $[a, \infty)$ in case it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be **nonoscillatory** on $[a, \infty)$. Equation (1.1) is said to be oscillatory on $[a, \infty)$ in case all of its solutions are oscillatory on $[a, \infty)$.

We now state and prove our main result.

Theorem 2.1 (Hille–Wintner Theorem)**.** *Assume that* (1.3) *and* (1.4) *hold. Let*

$$\hat{\mathbb{T}} := \{t \in \mathbb{T} : \mu(t) > 0\}$$

and let χ denote the characteristic function of $\hat{\mathbb{T}}$. Assume further that there is an $M > 0$ such that

$$r_1(t)\chi(t) \leq M\mu(t), \quad t \in \mathbb{T}. \tag{2.1}$$

Then if $L_1x = 0$ is nonoscillatory on $[a, \infty)$, it follows that $L_2x = 0$ is nonoscillatory on $[a, \infty)$.

Proof. We can assume without loss of generality that $\sup \hat{\mathbb{T}} = \infty$, since otherwise $\mathbb{T}$ is eventually a real interval, (2.1) holds trivially, and so the result reduces to the usual Hille–Wintner theorem. Since $L_1x = 0$ is nonoscillatory on $[a, \infty)$, there is a $T \in [a, \infty)$ and a solution x of (1.1) with $x(t) > 0$ on $[T, \infty)$. If we make the Riccati substitution

$$z(t) := \frac{r_1(t)x^\Delta(t)}{x(t)}, \quad t \geq T,$$

then (cf. [2, Theorem 4.36]) z is a solution of the Riccati equation

$$R_1 z = z^\Delta + q_1(t) + \frac{z^2}{r_1(t) + \mu(t)z} = 0 \tag{2.2}$$

on $[T, \infty)$ and satisfies

$$r_1(t) + \mu(t)z(t) > 0 \tag{2.3}$$

on $[T,\infty)$. Now let us denote

$$F(t) := \frac{z^2(t)}{r_1(t)+\mu(t)z(t)} \geq 0.$$

Then integrating (2.2) gives

$$z(t) + \int_T^t q_1(s)\Delta s + \int_T^t F(s)\Delta s = z(T). \tag{2.4}$$

If $q_1(t) \equiv 0$ for large t, then (1.4) implies $q_2(t) \equiv 0$ for large t, and the conclusion of the theorem is immediate. Hence we can assume that $q_1(t) \not\equiv 0$ for large t and in this case (1.4) implies (1.6). Using (1.3) it follows without loss of generality from Lemma 1.4 that $z(t) > 0$ for $t \geq T$ and by (1.4), $\int_T^t q_1(s)\Delta s \geq 0$ for $t \geq T$, where T is sufficiently large. Then an easy application of (1.3) gives that $\int_T^\infty F(s)\Delta s < \infty$. Again using (1.3) we get from (2.4) that $\lim_{t\to\infty} z(t)$ exists. We claim that $\lim_{t\to\infty} z(t) = 0$. Let $\{t_k\} \subset \hat{\mathbb{T}}$ with $\lim_{k\to\infty} t_k = \infty$. By (2.4) we have that

$$0 \leq \int_T^{t_k} q_1(s)\Delta s + \int_T^{t_k} F(s)\Delta s \leq z(T).$$

We recall (cf. [2, Theorem 1.75]) that $\int_t^{\sigma(t)} g(s)\Delta s = \mu(t)g(t)$ and so we have, for n_0 sufficiently large,

$$\sum_{k=n_0}^{\infty} \mu(t_k)F(t_k) = \sum_{k=n_0}^{\infty} \int_{t_k}^{\sigma(t_k)} F(t)\Delta t \leq \int_T^\infty F(s)\Delta s < \infty.$$

Therefore it follows that

$$0 = \lim_{k\to\infty} \mu(t_k)F(t_k) = \lim_{k\to\infty} \frac{\mu(t_k)z^2(t_k)}{r_1(t)+\mu(t_k)z(t_k)} = \lim_{k\to\infty} \frac{z^2(t_k)}{\frac{r_1(t)}{\mu(t_k)} + z(t_k)}. \tag{2.5}$$

Hence given an $\varepsilon > 0$ there is a positive integer k_0 such that

$$0 < \frac{z^2(t_k)}{\frac{r_1(t_k)}{\mu(t_k)} + z(t_k)} < \varepsilon$$

for $k \geq k_0$. This implies that

$$z^2(t_k) < \varepsilon\left(\frac{r_1(t_k)}{\mu(t_k)} + z(t_k)\right),$$

which implies that

$$\left(z(t_k) - \frac{\varepsilon}{2}\right)^2 < \frac{\varepsilon^2}{4} + \varepsilon\frac{r_1(t_k)}{\mu(t_k)} \leq \frac{\varepsilon^2}{4} + \varepsilon M.$$

Therefore

$$\left|z(t_k) - \frac{\varepsilon}{2}\right| < \frac{\varepsilon}{2} + \sqrt{\varepsilon M}$$

and consequently

$$|z(t_k)| < \varepsilon + \sqrt{\varepsilon M}.$$

Since $\varepsilon > 0$ is arbitrary, we get that

$$\lim_{k\to\infty} z(t_k) = 0.$$

But from (2.4) we know that $\lim_{t\to\infty} z(t) = z_0 \geq 0$ exists, and hence we get the desired result

$$\lim_{t\to\infty} z(t) = 0.$$

Now letting $t \to \infty$ in (2.4) we get

$$\int_T^\infty q_1(s)\Delta s + \int_T^\infty F(s)\Delta s = z(T).$$

Define, for large t,

$$v(t) := \int_t^\infty q_2(s)\Delta s + \int_t^\infty F(s)\Delta s.$$

Then using (1.4),

$$0 \leq v(t) \leq \int_t^\infty q_1(s)\Delta s + \int_t^\infty F(s)\Delta s = z(t).$$

Note that

$$v^\Delta(t) = -q_2(t) - F(t) = -q_2(t) - \frac{z^2(t)}{r_1(t) + \mu(t)z(t)}.$$

We now claim that

$$\frac{z^2(t)}{r_1(t) + \mu(t)z(t)} \geq \frac{v^2(t)}{r_1(t) + \mu(t)v(t)}, \quad t \geq T.$$

This follows from the fact that for each fixed t, $H(w) := \frac{w^2}{r_1(t)+\mu(t)w}$ is strictly increasing for $w \geq 0$ and the fact that $v(t) \geq 0$. But this implies that the Riccati dynamic inequality

$$v^\Delta + q_2(t) + \frac{v^2}{r_1(t) + \mu(t)v} \leq 0$$

has a solution on $[T, \infty)$ with $r_1(t) + \mu(t)v(t) > 0$, and this means (cf. [12, Lemma 5]) that

$$(r_1(t)x^\Delta)^\Delta + q_2(t)x^\sigma = 0$$

nonoscillatory on $[a, \infty)$. Since $0 < r_1(t) \leq r_2(t)$ we have by the Sturm comparison theorem [2, Theorem 5.60] that $L_2x = 0$ is nonoscillatory on $[a, \infty)$. □

Remark 2.2. If $\mathbb{T} = \mathbb{R}$, $r_i(t) \equiv 1$, $i = 1, 2$, and (1.4) holds, then the above result was first obtained by Hille [16] with the additional assumption that the $q_i(t)$, $i = 1, 2$, are positive. Wintner in [21] showed that (1.4) is sufficient for the conclusion to hold without the assumption of positivity. Taam [20] showed later that the conclusion of the theorem holds with (1.5) replacing (1.4). We shall see, however, that for the general case of time scales, additional assumptions on $q_1(t)$ and on the set of right-scattered points are needed to obtain the analogous result when (1.5) replaces (1.4).

Theorem 2.3. *Suppose that* (1.3) *and* (1.5) *hold and, in addition, assume there exists* $0 < m < M$ *such that*

$$m\mu(t) \leq r_1(t)\chi(t) \leq M\mu(t), \quad t \in \mathbb{T}. \tag{2.6}$$

Suppose also that $q_1(t)$ *is positive near* ∞ *for* $t \in \hat{\mathbb{T}}$ *and*

$$\liminf_{t\to\infty,\, t\in\hat{\mathbb{T}}} \frac{q_2(t)}{q_1(t)} > -1. \tag{2.7}$$

Then $L_1x = 0$ *nonoscillatory on* $[a, \infty)$ *implies* $L_2x = 0$ *is nonoscillatory on* $[a, \infty)$.

Proof. Let $x(t)$, $z(t)$, and $F(t)$ be as in the proof of Theorem 2.1. Then

$$z(t) = \int_t^\infty q_1(s)\Delta s + \int_t^\infty F(s)\Delta s$$

and $z(t)$ satisfies (2.3) for $t \in [T, \infty)$. Define

$$v(t) := \int_t^\infty q_2(s)\Delta s + \int_t^\infty F(s)\Delta s.$$

Note that, for $t \in [T, \infty)$,

$$\begin{aligned} |v(t)| &\leq \left| \int_t^\infty q_2(s)\Delta s \right| + \int_t^\infty F(s)\Delta s \\ &\leq \int_t^\infty q_1(s)\Delta s + \int_t^\infty F(s)\Delta s \\ &= z(t). \end{aligned}$$

We would like to prove that

$$\frac{z^2(t)}{r_1(t) + \mu(t)z(t)} \geq \frac{v^2(t)}{r_1(t) + \mu(t)v(t)}, \tag{2.8}$$

for large t. Note that for any right-dense point and for any point where $z(t) = v(t)$, (2.8) is true. Hence it remains to show that (2.8) is true for all sufficiently large right-scattered points where $z(t) > v(t)$. For such points note that (2.8) is equivalent to

$$r_1(t)(z(t) + v(t)) \geq -\mu(t)v(t)z(t). \tag{2.9}$$

Evidently, $z(t) + v(t) > 0$, and so (2.9) is equivalent to

$$\frac{r_1(t)}{\mu(t)} \geq -\frac{v(t)z(t)}{z(t) + v(t)} = -\frac{v(t)}{1 + \frac{v(t)}{z(t)}}. \tag{2.10}$$

If there is no sequence $\{t_n\}$ of points in $\hat{\mathbb{T}}$ such that $\lim_{n\to\infty} t_n = \infty$ and

$$\liminf_{n\to\infty} \frac{v(t_n)}{z(t_n)} = -1, \tag{2.11}$$

then, since $\lim_{t\to\infty} v(t) = 0$, it would follow that

$$\lim_{t\to\infty,\, t\in\hat{\mathbb{T}}} \frac{v(t)}{1 + \frac{v(t)}{z(t)}} = 0.$$

Since for $t \in \hat{\mathbb{T}}$, we have

$$\frac{r_1(t)}{\mu(t)} \geq m > 0,$$

it would follow that (2.10) and hence (2.8) hold for all sufficiently large t. It remains to prove that (2.11) does not hold for any sequence in $\hat{\mathbb{T}}$ tending to ∞. We do this by contradiction. So assume there is such a sequence $\{t_n\}$ with

$$\lim_{n\to\infty} \frac{v(t_n)}{z(t_n)} = -1. \tag{2.12}$$

Since we are assuming that $q_1(t) > 0$ for large t we get that

$$z^\Delta(t) = -q_1(t) - \frac{z^2(t)}{r_1(t) + \mu(t)z(t)} < 0$$

for large t. Since $z(t) > 0$ and $z^{\Delta}(t) < 0$ for large t we can apply L'Hôpital's rule for the time scale case [2, Theorem 1.120] to get

$$\begin{aligned} -1 &= \lim_{n\to\infty} \frac{v(t_n)}{z(t_n)} = \lim_{n\to\infty} \frac{v^{\Delta}(t_n)}{z^{\Delta}(t_n)} \\ &= \lim_{n\to\infty} \frac{-q_2(t_n) - \frac{z^2(t_n)}{r_1(t_n)+\mu(t_n)z(t_n)}}{-q_1(t_n) - \frac{z^2(t_n)}{r_1(t_n)+\mu(t_n)z(t_n)}} \\ &= \lim_{n\to\infty} \frac{(r_1(t_n)+\mu(t_n)z(t_n))q_2(t_n) + z^2(t_n)}{(r_1(t_n)+\mu(t_n)z(t_n))q_1(t_n) + z^2(t_n)}. \end{aligned}$$

That is,

$$\lim_{n\to\infty} \frac{a_n \frac{q_2(t_n)}{q_1(t_n)} + b_n}{a_n + b_n} = -1, \tag{2.13}$$

where

$$a_n := r_1(t_n) + \mu(t_n)z(t_n) > 0, \quad b_n := \frac{z^2(t_n)}{q_1(t_n)} > 0.$$

By (2.7), there is an $\varepsilon > 0$ such that

$$\frac{q_2(t_n)}{q_1(t_n)} \geq -1 + \varepsilon$$

for large n, where we suppose also that $\varepsilon < 2$. Let $0 < \delta < \varepsilon$ be given. Then by (2.13) for large n,

$$a_n \frac{q_2(t_n)}{q_1(t_n)} + b_n < (-1+\delta)(a_n + b_n).$$

Hence we have for large n that

$$(-1+\varepsilon)a_n + b_n \leq a_n \frac{q_2(t_n)}{q_1(t_n)} + b_n < (-1+\delta)(a_n + b_n).$$

But this implies that

$$(\varepsilon - \delta)a_n < (-2+\delta)b_n < 0$$

for large n, which is a contradiction, since the left-hand side is positive. Therefore it follows that (2.8) holds and this implies that

$$v^{\Delta}(t) \leq -q_2(t) - \frac{v^2(t)}{r_1(t) + \mu(t)v(t)}.$$

Consequently, it follows that $(r_1(t)x^{\Delta})^{\Delta} + q_2(t)x^{\sigma}$ is nonoscillatory on $[a, \infty)$. Then by the Sturm comparison theorem $L_2 x = 0$ is nonoscillatory on $[a, \infty)$. □

3. Examples

Example 3.1. Consider the Euler–Cauchy-like equation

$$x^{\Delta\Delta} + \frac{\gamma}{t\sigma(t)} x^{\sigma} = 0. \tag{3.1}$$

(This dynamic equation in general is not an Euler–Cauchy equation according to the definition given in [2]; however if $\mathbb{T} = \mathbb{R}$, then this is an Euler–Cauchy differential equation.) For $\gamma < \frac{1}{4}$ it is known [18] that (3.1) is nonoscillatory near infinity provided

$$\mu(t) = o(t) \quad \text{as } t \to \infty, \quad \text{i.e.} \quad \frac{\mu(t)}{t} \to 0 \quad \text{as } t \to \infty. \tag{3.2}$$

Since $\int_t^\infty \frac{\gamma}{s\sigma(s)}\Delta s = \frac{\gamma}{t}$, we get from Theorem 2.1 that if (3.2) holds, if $\mu(t)$ is bounded below by a positive number for $t \in \hat{\mathbb{T}}$, and if

$$0 \le \limsup_{t\to\infty} \; t \int_t^\infty q(s)\Delta s < \frac{1}{4},$$

then $x^{\Delta\Delta} + q(t)x^\sigma = 0$ is nonoscillatory on $[a, \infty)$.

To see that the above result is sharp we note the following example.

Example 3.2. If

$$\liminf_{t\to\infty} \; t \int_t^\infty q(s)\Delta s > \frac{1}{4},$$

then $x^{\Delta\Delta} + q(t)x^\sigma = 0$ is oscillatory on $[a, \infty)$. We show that this follows from the following result, which is the linear version of [13, Corollary 3.3].

Theorem 3.3. *Assume that $q(t)$ is positive, $\int_a^\infty \frac{1}{r(t)}\Delta t = \infty$ holds and for any $t_0 \ge a$ there is a $t_1 > t_0$ such that*

$$\limsup_{t\to\infty} \int_{t_1}^t \left[\sigma(s)\left[q(s) - \left(\frac{1}{2s\sigma(s)}\right) + \frac{1}{4s^2C_1(s)}\right] - \frac{A_1^2(s)C_1(s)}{4B_1(s)}\right]\Delta s = \infty, \tag{3.3}$$

where

$$A_1(s) = \frac{-1}{sC_1(s)}\left(1 + \frac{1}{s}\mu(s) - C_1(s)\right),$$
$$B_1(s) = \frac{s + \mu(s)}{s^2}, \quad C_1(s) = 1 + \frac{\mu(s)}{(s - t_0)}.$$

Then $(r(t)x^\Delta)^\Delta + q(t)x^\sigma = 0$ is oscillatory on $[a, \infty)$.

An easy calculation shows that for the dynamic equation (3.1) we have

$$C_1(s) = \frac{\sigma(s) - t_0}{s - t_0}, \quad A_1(s) = -\frac{\sigma(s)}{s^2C_1(s)} + \frac{1}{s}, \quad B_1(s) = \frac{\sigma(s)}{s^2}.$$

One can then show that

$$\frac{\sigma(s)}{4s^2C_1(s)} - \frac{A_1^2(s)C_1(s)}{4B_1(s)} = \frac{1}{4s}\left(2 - \frac{s(\sigma(s) - t_0)}{\sigma(s)(s - t_0)}\right).$$

Since this last expression is asymptotic to $\frac{1}{4s}$ as $s \to \infty$ and since $\int_{t_1}^{\infty} \frac{1}{s}\Delta s = \infty$, for any time scale which is unbounded above, we get that for $t_1 > t_0$,

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(\frac{\gamma}{s} - \frac{1}{2s} + \frac{\sigma(s)}{4s^2 C_1(s)} - \frac{A_1^2(s)C_1(s)}{4b_1(s)} \right) \Delta s = \infty$$

if $\gamma > \frac{1}{4}$. Hence by Theorem 3.3 we have that if $\gamma > \frac{1}{4}$, then (3.1) is oscillatory on $[a, \infty)$.

In Example 3.1 we assumed that $\mathbb{T}$ was a time scale satisfying (3.2). A time scale that is important in the theory of orthogonal polynomials and quantum theory (cf., e.g., [14], [19]) is $\mathbb{T} = q^{\mathbb{N}_0} := \{1, q, q^2, q^3, \cdots\}$. For this time scale, $\mu(t) = (q-1)t$, so (3.2) is not satisfied. We now give an application of Theorem 2.1 for this time scale.

Example 3.4. It is not difficult to show that $x(t) = t^\alpha$ is a solution of the dynamic equation

$$x^{\Delta\Delta} + \frac{C_\alpha}{t\sigma(t)} x^\sigma = 0, \tag{3.4}$$

where $C_\alpha := \frac{(q^\alpha - 1)(q^{1-\alpha} - 1)}{(q-1)^2}$, for $t \in \mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$. If we let $0 < \alpha < 1$, then $C_\alpha > 0$. We have with $t = q^n$,

$$\int_t^\infty \frac{C_\alpha}{s\sigma(s)} \Delta s = \frac{C_\alpha}{t} = \frac{(q^\alpha - 1)(q^{1-\alpha} - 1)}{(q-1)^2 q^n}$$

and so if $Q(t)$ is defined on $\mathbb{T}$, then

$$\int_t^\infty Q(s)\Delta s = (q-1) \sum_{k=n}^{\infty} q^k Q(q^k).$$

Hence, if

$$0 \le \sum_{k=n}^{\infty} q^k Q(q^k) \le \frac{(q^\alpha - 1)(q^{1-\alpha} - 1)}{(q-1)^3 q^n},$$

for all large n, then by Theorem 2.1, $x^{\Delta\Delta} + Q(t)x^\sigma = 0$ is nonoscillatory on $\mathbb{T} = q^{\mathbb{N}_0}$.

Remark 3.5. It was shown in [9, Example 16] (see also [2, Example 4.48]) that $x^{\Delta\Delta} + \frac{c}{(q-1)t\sigma(t)} x^\sigma = 0$ is oscillatory on $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, if $c > 1$. We can use the result of Example 3.4 to show that this is sharp. To see this, notice that for fixed $0 < \alpha < 1$, $h(x) := \frac{(x^\alpha - 1)(x^{1-\alpha} - 1)}{x-1}$ satisfies $h'(x) > 0$, $x > 1$ and $\lim_{x\to\infty} h(x) = 1$. Hence given any $c_0 < 1$, we can choose q sufficiently large so that $h(q) = (q-1)C_\alpha > c_0$ and so since (3.4) has the nonoscillatory solution $x(t) = t^\alpha$ on $\mathbb{T} = q^{\mathbb{N}_0}$, it follows by the Sturm comparison theorem that $x^{\Delta\Delta} + \frac{c_0}{(q-1)t\sigma(t)} x^\sigma = 0$ is nonoscillatory on $\mathbb{T} = q^{\mathbb{N}_0}$.

Example 3.6. We note that Examples 3.1 and 3.4 involve an application of Theorem 2.1 and that Theorem 2.3 does not apply since (2.6), (2.7) need not hold (i.e., $\mu(t)$ is unbounded in Example 3.4). As a simple example of Theorem 2.3, let

$$r_i(t) \equiv 1,\ i = 1, 2, \quad q_1(t) := \frac{\gamma}{t\sigma(t)}, \quad q_2(t) := \frac{\lambda}{t^2}(-1)^t$$

for $t \in \mathbb{T} = \mathbb{N} := \{1, 2, 3, \cdots\}$, where $\lambda > 0$. We have that

$$\int_t^\infty q_1(s)\Delta s = \frac{\gamma}{t} \quad \text{and} \quad \left|\int_t^\infty q_2(s)\Delta s\right| \le \frac{\lambda}{t^2}$$

for $t \in \mathbb{N}$. Notice that $\int_t^\infty q_2(s)\Delta s$ assumes positive and negative values for arbitrarily large t, so that Theorem 2.1 does not apply. Clearly, for any $\lambda, \gamma > 0$, (1.5) holds for all large t. Moreover

$$\frac{q_2(t)}{q_1(t)} = \frac{\lambda}{\gamma}(-1)^t\left(1 + \frac{1}{t}\right)$$

so that (2.7) holds if $\lambda < \gamma$. Hence, if $0 < \lambda < \gamma < \frac{1}{4}$, then $L_2 x = 0$ is nonoscillatory on $\mathbb{N}$ by Theorem 2.3. We note that the results in the above examples may not be established by any other criteria known to the authors. More sophisticated examples are left to the reader.

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