A GENERAL MULTIPLICITY THEOREM
FOR CERTAIN NONLINEAR EQUATIONS
IN HILBERT SPACES

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Abstract. In this paper, we prove the following general result. Let $X$ be a real Hilbert space and $J : X \to \mathbb{R}$ a continuously Gâteaux differentiable, nonconstant functional, with compact derivative, such that

$$\limsup_{\|x\| \to +\infty} \frac{J(x)}{\|x\|^2} \leq 0.$$ 

Then, for each $r \in ]\inf X J, \sup X J[$ for which the set $J^{-1}(]-\infty,r[)$ is not convex and for each convex set $S \subseteq X$ dense in $X$, there exist $x_0 \in S \cap J^{-1}[r, +\infty]$ and $\lambda > 0$ such that the equation

$$x = \lambda J'(x) + x_0$$

has at least three solutions.

The aim of the present paper is to prove the following result.

**Theorem 1.** Let $X$ be a real Hilbert space and let $J : X \to \mathbb{R}$ be a continuously Gâteaux differentiable, nonconstant functional, with compact derivative, such that

$$(1) \quad \limsup_{\|x\| \to +\infty} \frac{J(x)}{\|x\|^2} \leq 0.$$ 

Then, for each $r \in ]\inf X J, \sup X J[$ and each $x_0 \in J^{-1}[r, +\infty]$, at least one of the following assertions holds:

(a) There exists $\lambda > 0$ such that the equation

$$x = \lambda J'(x) + x_0$$

has at least three solutions.

(b) There exists a unique $y \in J^{-1}(r, +\infty)$ such that

$$\|x_0 - y\| = \text{dist}(x_0, J^{-1}(]r, +\infty[)) = \text{dist}(x_0, J^{-1}(r)) .$$

Among the most significant consequences of Theorem 1, there is the general multiplicity theorem announced in the title of the paper. It reads as follows.

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Theorem 2. Let $X$ be a real Hilbert space and let $J : X \to \mathbb{R}$ be a continuously Gâteaux differentiable, nonconstant functional, with compact derivative, such that

$$\limsup_{\|x\| \to +\infty} \frac{J(x)}{\|x\|^2} \leq 0.$$  

Then, for each $r \in ] \inf_X J, \sup_X J[$ for which the set $J^{-1}([r, +\infty[)$ is not convex and for each convex set $S \subseteq X$ dense in $X$, there exist $x_0 \in S \cap J^{-1}([-\infty, r[)$ and $\lambda > 0$ such that the equation

$$x = \lambda J'(x) + x_0$$

has at least three solutions.

To derive Theorem 2 from Theorem 1, we use a very recent result by I. G. Tsar’kov ([5]). We state it below in a form which is enough for our purposes.

**Theorem A ([5], Corollary 2]).** Let $X$ be a real Hilbert space and $C \subset X$ a sequentially weakly closed and nonconvex set.

Then, for each convex set $S \subseteq X$ dense in $X$, there exists $x_0 \in S \setminus C$ such that the set $\{y \in C : \|x_0 - y\| = \text{dist}(x_0, C)\}$ has at least two points.

In practice, when $\dim(X) = \infty$, Theorem A is a more precise version of the celebrated, classical result of Efimov and Stechkin on Chebyshev sets ([1]) (see also [6] for a proof based on convex analysis methods).

Now, the way of drawing Theorem 2 from Theorem 1 is transparent. Let us formalize it.

**Proof of Theorem 2.** Let $r \in ] \inf_X J, \sup_X J[$ be such that the set $J^{-1}([r, +\infty[)$ is not convex and let $S \subseteq X$ be a convex set dense in $X$. Since $J'$ is compact, the functional $J$ turns out to be sequentially weakly continuous ([7], Corollary 41.9). Hence, the set $J^{-1}([r, +\infty[)$ is sequentially weakly closed (possibly not weakly closed). Consequently, by Theorem A, there exists $x_0 \in S \cap J^{-1}([-\infty, r[)$ such that (b) of Theorem 1 does not hold. Hence, (a) of the same theorem holds, and that is the conclusion.

We are going to prove Theorem 1. For the reader’s convenience, we now recall the statements of two well-known results which are the main tools used later. We recall that a real function $\varphi$ on a convex subset $C$ of a vector space is said to be quasi-concave if, for every $r \in \mathbb{R}$, the set $\{x \in C : \varphi(x) > r\}$ is convex. We also recall that a Gâteaux differentiable functional $J$ on a real Banach space $X$ is said to satisfy the Palais-Smale condition if each sequence $\{x_n\}$ in $X$ such that $\sup_{n \in \mathbb{N}} |J(x_n)| < +\infty$ and $\lim_{n \to +\infty} \|J'(x_n)\|_{X^*} = 0$ admits a strongly converging subsequence.

**Theorem B ([3], Theorem 1 and Remark 1]).** Let $X$ be a topological space, $I$ a real interval, and $f : X \times I \to \mathbb{R}$ a function satisfying the following conditions:

1. for every $x \in X$, the function $f(x, \cdot)$ is quasi-concave and continuous;
2. for every $\lambda \in I$, the function $f(\cdot, \lambda)$ is lower semicontinuous and each of its local minima is a global minimum;
3. there exist $\rho > \sup_{\lambda \in I} \inf_{x \in X} f(x, \lambda)$ and $\lambda_0 \in I$ such that the set

$$\{x \in X : f(x, \lambda_0) \leq \rho\}$$

is compact.
Then
\[
\sup_{\lambda \in I} \inf_{x \in X} f(x, \lambda) = \inf_{x \in X} \sup_{\lambda \in I} f(x, \lambda).
\]

**Theorem C** ([2, Corollary 1]). Let \( X \) be a real Banach space and let \( J : X \to \mathbb{R} \) be a continuously Gâteaux differentiable functional satisfying the Palais-Smale condition and having at least two local minima.

Then \( J \) has at least three critical points.

We are now in a position to prove Theorem 1.

**Proof of Theorem 1.** Let \( r \in [\inf_X J, \sup_X J] \) and \( x_0 \in J^{-1}([-\infty, r]) \). Assume that assertion (a) does not hold. So, let us suppose that, for each \( \lambda > 0 \), the equation
\[
(E_{\lambda}) \quad x = \lambda J'(x) + x_0
\]
has at most two solutions. Now, define the function \( f : X \times [0, +\infty[ \to \mathbb{R} \) by setting
\[
f(x, \lambda) = \frac{1}{2} \|x - x_0\|^2 + \lambda(r - J(x))
\]
for all \((x, \lambda) \in X \times [0, +\infty[\). Let us check that \( f \) satisfies the hypotheses of Theorem B, the space \( X \) being endowed with the weak topology. It is clear that (c1) and (c3) (with \( \lambda_0 = 0 \)) are satisfied. So, fix \( \lambda \in [0, +\infty[ \). As we have already observed, the functional \( J \) is sequentially weakly continuous. Hence, the functional \( f(\cdot, \lambda) \) is sequentially weakly lower semicontinuous. Fix \( \epsilon > 0 \) so that \( \frac{1}{2} - \epsilon \lambda > 0 \). By (1), there is \( \delta > 0 \) such that
\[
\sup_{\|x\| > \delta} \frac{J(x)}{\|x\|^2} < \epsilon.
\]
Thus, we have
\[
f(x, \lambda) > \left( \frac{1}{2} - \epsilon \lambda \right) \|x\|^2 - \|x_0\| \|x\| + \frac{1}{2} \|x_0\|^2 + \lambda r
\]
for all \( x \in X \), with \( \|x\| > \delta \). Hence, we get
\[
\lim_{\|x\| \to +\infty} f(x, \lambda) = +\infty.
\]
From this, by the reflexivity of \( X \), by the Eberlein-Šmulian theorem and by a classical result ([7], Example 38.25) we infer that \( f(\cdot, \lambda) \) is weakly lower semicontinuous, has a global minimum and satisfies the Palais-Smale condition. On the other hand, the critical points of \( f(\cdot, \lambda) \) are exactly the solutions of \((E_{\lambda})\). Hence, by assumption, \( f(\cdot, \lambda) \) has at most two critical points. Then, thanks to Theorem C, \( f(\cdot, \lambda) \) has exactly one global minimum and no other local minimum in the strong topology, and so, a fortiori, in the weak topology. Hence, condition (c2) is satisfied. Therefore, Theorem B ensures that
\[
\sup_{\lambda \geq 0} \inf_{x \in X} f(x, \lambda) = \inf_{x \in X} \sup_{\lambda \geq 0} f(x, \lambda).
\]
Clearly, one has
\[
\inf_{x \in X} \sup_{\lambda \geq 0} f(x, \lambda) = \frac{1}{2} \inf_{x \in J^{-1}([r, +\infty[)} \|x - x_0\|^2.
\]
Theorem 3. Let $J^{-1}([r, +\infty[)$ be sequentially weakly closed, there exists $y \in J^{-1}([r, +\infty[)$ such that

$$
\|x_0 - y\| = \text{dist}(x_0, J^{-1}([r, +\infty[))
$$

We claim that $y \in J^{-1}(r)$. Indeed, if $J(y) > r$, since $J$ is continuous and $J(x_0) < r$, there would exist a point $z$ in the line segment joining $x_0$ and $y$ such that $J(z) = r$. So, we would have $\|x_0 - z\| < \text{dist}(x_0, J^{-1}([r, +\infty[))$, an absurdity. In particular, this implies that

$$
\text{dist}(x_0, J^{-1}([r, +\infty[)) = \text{dist}(x_0, J^{-1}(r))
$$

Now, observe that the function $\inf_{x \in X} f(x, \cdot)$ is upper semicontinuous in $[0, +\infty[$ and that $\lim_{\lambda \to +\infty} \inf_{x \in X} f(x, \lambda) = -\infty$, since $r < \sup_{X} J$. Hence, there is $\lambda^{*} \geq 0$ such that

$$
\inf_{x \in X} f(x, \lambda^{*}) = \sup_{\lambda \geq 0} \inf_{x \in X} f(x, \lambda)
$$

So, from (2) and (3), we get

$$
\inf_{x \in X} \left( \frac{1}{2}\|x - x_0\|^2 - \lambda^{*} J(x) \right) = \inf_{x \in J^{-1}(r)} \left( \frac{1}{2}\|x - x_0\|^2 - \lambda^{*} J(x) \right).
$$

From this, we infer that $\lambda^{*} > 0$, since $J(x_0) < r$, and that each global minimum of the restriction of the functional $x \to \frac{1}{2}\|x - x_0\|^2 - \lambda^{*} J(x)$ to $J^{-1}(r)$ is a global minimum of the same functional on $X$. But, as we have seen above, for each $\lambda > 0$, the functional $x \to \frac{1}{2}\|x - x_0\|^2 - \lambda J(x)$ has exactly one global minimum in $X$. On the other hand, a point $y \in J^{-1}(r)$ is a global minimum for the restriction of the functional $x \to \frac{1}{2}\|x - x_0\|^2 - \lambda^{*} J(x)$ to $J^{-1}(r)$ if and only if $\|y - x_0\| = \text{dist}(x_0, J^{-1}(r))$, and so (b) follows.

Remark 1. The conclusion of Theorem 1 can be false when (1) is not satisfied. To see this, take, for instance, $X = \mathbb{R}$, $J(x) = x^3 - x$, $r = 0$ and $x_0 = \frac{1}{2}$. We also believe that some more sophisticated examples should show that the assumption about the compactness of $J$ cannot be omitted.

Further, observe that, applying Theorem 1 to both $J$ and $-J$, we get the following result.

**Theorem 3.** Let $X$ be a real Hilbert space and $J : X \to \mathbb{R}$ a continuously Gâteaux differentiable, nonconstant functional, with compact derivative, such that

$$
\lim_{\|x\| \to +\infty} \frac{J(x)}{\|x\|^2} = 0
$$

Then, for each $r \in ]\inf_{X} J, \sup_{X} J[$ and each $x_0 \in X \setminus J^{-1}(r)$, at least one of the following assertions holds:

(i) There exists $\lambda \in \mathbb{R}$ such that the equation

$$
x = \lambda J'(x) + x_0
$$

has at least three solutions.

(ii) There exists a unique $y \in J^{-1}(r)$ such that

$$
\|x_0 - y\| = \text{dist}(x_0, J^{-1}(r)).
$$

Reasoning as in the proof of Theorem 2, we obtain the following consequence of Theorem 3.
Theorem 4. Let $X$ be a real Hilbert space and $J : X \to \mathbb{R}$ a continuously Gâteaux differentiable, nonconstant functional, with compact derivative, such that

$$\lim_{\|x\| \to +\infty} \frac{J(x)}{\|x\|^2} = 0.$$ 

Then, for each $r \in \left( \inf_X J, \sup_X J \right]$, for which the set $J^{-1}(r)$ is not convex and for each convex set $S \subseteq X$ dense in $X$, there exist $x_0 \in S \setminus J^{-1}(r)$ and $\lambda \in \mathbb{R}$ such that the equation

$$x = \lambda J'(x) + x_0$$

has at least three solutions.

We conclude by presenting an application of Theorem 2 to a two-point boundary value problem.

As usual, let $W^{1,2}_0(0,1)$ denote the closure of $C_0^\infty(0,1]$ in $W^{1,2}(0,1]$, the space of all $u \in L^2(0,1]$ having distributional derivative $u' \in L^2(0,1]$. We consider $W^{1,2}_0(0,1]$ endowed with the inner product

$$\langle u, v \rangle = \int_0^1 u'(t)v'(t)dt.$$ 

For each continuous function $f : \mathbb{R} \to \mathbb{R}$, define the functional $J_f : W^{1,2}_0(0,1] \to \mathbb{R}$ by putting

$$J_f(u) = \int_0^1 F(u(t))dt$$

for all $u \in W^{1,2}_0(0,1]$, where

$$F(\xi) = \int_0^\xi f(s)ds.$$ 

We then have

Theorem 5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous nonconstant and nondecreasing function satisfying

$$(4) \quad \lim_{|\xi| \to +\infty} \frac{F(\xi)}{\xi^2} = 0.$$ 

Then, for each $r \in \left( \inf_{\mathbb{R}} F, \sup_{\mathbb{R}} F \right]$ and for each convex set $S \subseteq C_0^\infty(0,1]$ dense in $W^{1,2}_0(0,1]$, there exist $w \in S \cap J_f^{-1}(-\infty, r]$ and $\lambda > 0$ such that the problem

$$\begin{cases}
-u'' = \lambda f(u) - w''(t) & \text{in } [0,1], \\
u(0) = u(1) = 0
\end{cases}$$

has at least three (classical) solutions.

Proof. By classical results (3), $J_f$ is a continuously Gâteaux differentiable functional on $W^{1,2}_0(0,1]$ with compact derivative, and one has

$$\langle J_f'(u), v \rangle = \int_0^1 f(u(t))v(t)dt$$

for all $u, v \in W^{1,2}_0(0,1]$. Further, from (4), it readily follows that

$$\lim_{\|u\| \to +\infty} \frac{J_f(u)}{\|u\|^2} = 0.$$
So, $J_f$ satisfies the assumption of Theorem 2. Fix $r \in [\inf_R F, \sup_R F]$ (note that $r$ is in the interior of the range of $J_f$ (see the argument below)). Now, let us show that the set $J_f^{-1}(r)$ is not convex. First, we note that, for each $a \in \mathbb{R}$, there exists $u \in W^{1,2}_0([0,1])$ such that $u \left( \frac{1}{2} \right) = a$ and $\int_0^1 F(u(t))dt = r$. Indeed, set

$$A = \left\{ u \in W^{1,2}_0([0,1]) : u \left( \frac{1}{2} \right) = a \right\}.$$ 

Fix $r_1, r_2$ satisfying $\inf_R F < r_1 < r < r_2 < \sup_R F$, and pick $\xi_1, \xi_2$ so that $F(\xi_1) = r_1$, $F(\xi_2) = r_2$. Next, choose $\epsilon > 0$ such that

$$r_1(1 - 4\epsilon) + 4\epsilon \sup_{[-\rho,\rho]} |F| < r < r_2(1 - 4\epsilon) - 4\epsilon \sup_{[-\rho,\rho]} |F|,$$

where $\rho = \max\{ |\xi_1|, |\xi_2|, |a| \}$. Finally, fix two functions $u_1, u_2 \in A$ so that

$$\max \left\{ \sup_{[0,1]} |u_1|, \sup_{[0,1]} |u_2| \right\} \leq \rho, \ u_1(t) = \xi_1, \ u_2(t) = \xi_2$$

for all $t \in [\epsilon, \frac{1}{2} - \epsilon] \cup \left[ \frac{1}{2} + \epsilon, 1 - \epsilon \right]$. Then, we have

$$\int_0^1 F(u_1(t))dt \leq r_1(1 - 4\epsilon) + 4\epsilon \sup_{[-\rho,\rho]} |F| < r$$

as well as

$$\int_0^1 F(u_2(t))dt \geq r_2(1 - 4\epsilon) - 4\epsilon \sup_{[-\rho,\rho]} |F| > r.$$

Since $A$ is connected (being convex) and the functional $u \to \int_0^1 F(u(t))dt$ is continuous, there is $u \in A$ such that $\int_0^1 F(u(t))dt = r$, as claimed. Now, since $f$ is not constant, we can fix $a, b \in \mathbb{R}$ so that $f(a) \neq f(b)$. According to the previous claim, there are $u, v \in W^{1,2}_0([0,1])$ such that $u \left( \frac{1}{2} \right) = a$, $v \left( \frac{1}{2} \right) = b$ and $\int_0^1 F(u(t))dt = \int_0^1 F(v(t))dt = r$. Finally, we claim that, for some $\mu \in [0,1]$, we have

$$\int_0^1 F(u(t) + \mu(v(t) - u(t)))dt \neq r.$$

Arguing by contradiction, assume the contrary. Hence, the derivative of the function $\mu \to \int_0^1 F(u(t) + \mu(v(t) - u(t)))dt$ is zero in $[0,1]$. That is,

$$\int_0^1 f(u(t) + \mu(v(t) - u(t)))(v(t) - u(t))dt = 0$$

for all $\mu \in [0,1]$. From this, it clearly follows that

$$\int_0^1 (f(v(t)) - f(u(t)))(v(t) - u(t))dt = 0.$$

Then, since $f$ is nondecreasing, we infer that

$$(f(v(t)) - f(u(t)))(v(t) - u(t)) = 0$$

for all $t \in [0,1]$. So, since $u \left( \frac{1}{2} \right) \neq v \left( \frac{1}{2} \right)$, we get

$$f \left( u \left( \frac{1}{2} \right) \right) = f \left( v \left( \frac{1}{2} \right) \right),$$

a contradiction. Now, observe that $J_f$ is convex, since $f$ is nondecreasing. Consequently, $J^{-1}([\infty, r])$ is convex. Then, since $J^{-1}(r)$ is not convex, $J^{-1}([r, \infty])$
is not convex either. Now, let $S \subseteq C^\infty_0(]0,1[)$ be a convex set dense in $W^{1,2}_0(]0,1[)$. Theorem 2 ensures the existence of $w \in S \cap J^{-1}_f(]-\infty,r[)$ and $\lambda > 0$ such that the equation
\[ v = \lambda J'_f(v) + w \]
has at least three solutions in $W^{1,2}_0(]0,1[)$. Note that $v$ is one of them if and only if
\[ \int_0^1 v'(t)\omega'(t)dt = \lambda \int_0^1 f(v(t))\omega(t)dt + \int_0^1 w'(t)\omega'(t)dt = \int_0^1 (\lambda f(v(t)) - w''(t))\omega(t)dt \]
for all $\omega \in W^{1,2}_0(]0,1[)$. This clearly implies that $v \in C^2([0,1])$, with
\[-v''(t) = \lambda f(v(t)) - w''(t)\]
for all $t \in [0,1]$. Consequently, the function $v$ is a classical solution of the problem
\[ \begin{cases} -u'' = \lambda f(u) - w''(t) & \text{in } [0,1], \\ u(0) = u(1) = 0. \end{cases} \]
So, this problem has at least three solutions, and the proof is complete. \hfill \Box

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