BOUNDS FOR THE INDEX OF THE CENTRE
IN CAPABLE GROUPS

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Abstract. A group $H$ is called capable if it is isomorphic to $G/Z(G)$ for some group $G$. Let $H$ be a capable group. I. M. Isaacs (2001) showed that if $H$ is finite, then the index of the centre is bounded above by some function of $|H'|$. We show that if $|H'| < \infty$, then $|H : Z(H)| \leq |H'|^{c \log_2 |H'|}$ with some constant $c$ and this bound is essentially best possible. We complete a result of Isaacs, showing that if $H'$ is a cyclic group, then $|H : Z(H)| \leq |H'|^2$.

1. Introduction

Let $G$ be an arbitrary group. According to a classical theorem of Schur, if $|G : Z(G)| < \infty$, then $|G'| < \infty$. An easy argument based on the ultra product method shows that there is a bound for the order of the derived subgroup in terms of the index of the centre. The best bound was given by Wiegold [7] showing that if $|G : Z(G)| = n$, then $|G'| \leq n^{\log_2 n}$. Infinite extraspecial groups show that the converse of the theorem of Schur does not hold in general. However, P. Hall (see [6], p. 423) observed that if $|G'| < \infty$, then $|G : Z_2(G)|$ is bounded above in terms of $|G'|$ (where $Z_2(G)$ denotes the second member of the upper central series of $G$). The first explicit bound was given by I. D. Macdonald [3]. Improving this bound we proved in [5] that

$$|G : Z_2(G)| \leq |G'|^{c \log_2 |G'|}$$

and our examples show that this estimate is sharp apart from the value of the constant $c$.

A group $H$ is said to be capable if there exists some group $G$ such that $G/Z(G)$ is isomorphic to $H$. I. M. Isaacs [2] proved that if $H$ is a capable group and $|H'| = n$, then $|H : Z(H)|$ is bounded above by some function $f$ of $n$, or equivalently, if $G$ is an arbitrary group and $|G' : G' \cap Z(G)| = n$, then $|G : Z_2(G)| \leq f(n)$. However, he has not given an explicit function $f(n)$. In our present paper we give the essentially best possible bound.

Theorem 1. If $G$ is a group (not necessarily finite) and $|G' : G' \cap Z(G)| = n$, then $|G : Z_2(G)| \leq n^{c \log_2 n}$ with $c = 2$.

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Using this result for $H = G/Z(G)$ we obtain the following.

**Corollary 2.** If $H$ is a capable group and $|H'| = n$, then

$$|H : Z(H)| \leq n^{c \log_2 n}$$

with $c = 2$.

Actually, the preceding result can be regarded as a converse of Wiegold’s theorem. The sequence of groups $G_n$ we constructed in [5] shows that these estimates are sharp apart from the value of the constant $c$. The proof of Theorem 1 shows that the value of the constant $c$ is at most 2. We also mention that H. Heineken [1] constructed capable groups $H$ for all odd prime numbers $p$ and for all natural numbers $n$ such that $|H'| = |Z(H)| = p^n$ and $|H : Z(H)| = p^{2n+\frac{1}{2}}$. Since these are the best known examples, we think that the constant $c$ can be further improved.

**Question 3.** Is it true that if $H$ is a capable group and $|H'| = n$, then $|H : Z(H)| \leq n^{\frac{1}{2} \log_2 n + c_2}$ for some constant $c_2$?

For groups with infinite derived subgroup a similar argument yields:

**Theorem 4.** If $G$ is a group and $|G' : G' \cap Z(G)| = \kappa$ is an infinite cardinal, then $|G : Z_2(G)| \leq 2^\kappa$.

**Corollary 5.** If $H$ is a capable group and $|H'| = \kappa$ is an infinite cardinal, then $|H : Z(H)| \leq 2^\kappa$.

**Remark 6.** Related to infinite groups, similar results are included in [4] and [5]. For each infinite cardinal $\kappa$ we constructed a group $G$ such that $|G'| = \kappa$, $Z(G) = 1$ and $|G| = 2^\kappa$ (see [5]). It follows that the previous estimates are sharp.

The second part of our paper deals with groups with cyclic derived subgroups.

For a capable group $H$, I. M. Isaacs [2] proved that if $H$ is finite, $H'$ is cyclic and all elements of order 4 in $H'$ are central in $H$, then $|H : Z(H)| \leq |H'|^2$. In the present paper we prove that the assumption about elements of order 4 can be omitted.

**Theorem 7.** If $H$ is a finite capable group and $H'$ is cyclic, then $|H : Z(H)| \leq |H'|^2$.

For an arbitrary group $G$, we prove the following estimate.

**Theorem 8.** If $G$ is a finite group with $G'$ cyclic of order $n$, then $|G : Z_2(G)| \leq n\varphi(n)$, where $\varphi$ is Euler’s totient function.

The previous estimate is sharp for the holomorph of a cyclic group.

2. **Groups with arbitrary derived subgroups**

In this section we prove Theorem 1 and Theorem 4.

**Lemma 9.** Let $H$ be a subgroup of $G$ generated by $k$ elements and $|G'| = n$. Then $|G : C_G(H)| \leq n^k$. 

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Remark. It is enough to generate the commutators \([c, g] \mid c \in C; g \in G\). Let \(x\) be an arbitrary element of \(G - C\). Then
\[ [c, g] = [x, c^{-1}gc]^{-1}[cx, g] \in [G - C, G]. \]

Lemma 10. Let \(G\) be an arbitrary group and \(C < G\) be a proper subgroup. Then \(G' = [G - C, G]\).

Proof. It is enough to generate the commutators \([c, g] \mid c \in C; g \in G\). Let \(x\) be an arbitrary element of \(G - C\). Then
\[ [c, g] = [x, c^{-1}gc]^{-1}[cx, g] \in [G - C, G]. \]

Lemma 11. Let \(Z = G' \cap Z(G)\), and let \(U, V\) be subgroups of \(G\) such that \(Z \leq U, V \leq G'\). Then there exist elements \(y, z\) of \(G\) with the following properties.

1. If \(Z \subseteq U\), then \(U \cap C_G(y) \subseteq U\).
2. If \(V \subseteq G'\), then \(V \subseteq \langle V, [y, z] \rangle\).

Proof. Set \(C = C_G(U)\). Suppose that \(Z \subseteq U\). Now, \(C \supseteq G\); thus \(U \cap C_G(y) \subseteq U\) for all \(y \in G - C\). Lemma 10 yields that \(G' = [G - C, G]\). Consequently, if \(V \subseteq G'\), then we can choose \(y \in G - C\) and \(z \in G\) such that \(V \subseteq \langle V, [y, z] \rangle\). In the case of \(Z = U\) and \(V \subseteq G'\), then we can choose arbitrary \([y, z] \notin V\).

Lemma 12. Let \(Z = G' \cap Z(G)\), and suppose that \(|G' : Z| = n\). Let \(T\) be a subgroup with \(G' \leq T \leq G\) having the following properties.

1. \(G' = T'Z\).
2. \(G' \cap Z(T) = Z\).
3. \(T/Z\) can be generated by \(k\) elements.

Then there exists \(M \leq G\) such that \([M, G, G] = 1\) and \(|G : M| \leq n^k\).


Remark 13. The statement of Lemma 12 is also true if \(n\) and \(k\) are infinite cardinals.

Lemma 14. Let \(G\) be a finite group and \(|G' : Z| = n\). Then there exists \(T\) as in Lemma 12 with \(k \leq 2\log_2 n\).

Proof. We define the elements \(y_{i+1}, z_{i+1} \mid 0 \leq i \leq l - 1\) recursively by applying Lemma 11 for \(V_i = \langle Z, [y_1, z_1], [y_2, z_2], \ldots, [y_i, z_i] \rangle \) and \(U_i = C_{G'}(V_i)\). Now we have that
\[ Z = V_0 \leq V_1 \leq V_2 \leq \cdots \leq V_l = G' \]
and
\[ G' = U_0 \geq U_1 \geq U_2 \geq \cdots \geq U_l = Z, \]
where \(l\) is the smallest integer such that \(V_l = G'\) and \(U_l = Z\). It is clear that \(l \leq \log_2 n\). Now \(T = \langle Z, y_1, z_1, y_2, z_2, \ldots, y_l, z_l \rangle\) has the required properties.
Proof of Theorem 1. It follows immediately from Lemma 12 and Lemma 14 that there exists a subgroup $M$ of $G$ such that $|G : M| \leq n^2 \log n$ and $M \leq Z_2(G)$. □

Proof of Theorem 4. First, we choose a subgroup $T_1$ such that $T_1^2 = G'$ and $|T_1| \leq \kappa$. Let $Q$ be a coset representative system for $Z$ in $G' \setminus Z(G)$. We choose elements $y_q$ for all $q \in Q$ such that $y_q \notin C_G(q)$. The set $T_2 = \{y_q \mid q \in Q\}$ has cardinality $\kappa$ and clearly $C_{G'}(Y) = Z$. Let $T = \langle T_1, T_2 \rangle$. Then $|T| = \kappa$, and the same argument as in Lemma 12 completes the proof. □

3. Groups with cyclic derived subgroups

In this section we focus our attention on groups with cyclic derived subgroups.

Lemma 15. Let $G$ be a group, and write $Z = G' \cap Z(G)$. Assume that $G'$ is a $p$-group and $G'/Z$ is cyclic of order $n$. Then there exists a subgroup $M \leq G$ such that $[M, G, G] = 1$ and $|G : M| \leq n^2$.

Proof. Let $x \in G' - Z$ such that $x^p \in Z(G)$. Set $C = C_G(x)$. It follows that $C_G(y) \cap G' = Z(G) \cap G'$ for all $y \in G - C$. Using Lemma 10 we can find $a \in G - C$ and $b \in G$ such that $\langle Z, [a, b] \rangle = G'$. Let $T = \langle Z, a, b \rangle$, and note that $T$ satisfies the three conditions of Lemma 12 with $k = 2$. □

Proof of Theorem 7. We reduce to the case where $G'$ is a $p$-group. For each prime divisor $p$ of $|G'|$ let $N_p$ be the normal $p$-complement of $G'$ and work in the factor group $G/N_p$. This factor group satisfies the hypotheses with $n$ replaced by a divisor of $n_p$, the $p$-part of $n$. Using the preceding lemma, we know that there exists a subgroup $M_p \leq G$ such that $[M_p, G, G] \leq N_p$ and $|G : M_p| \leq (n_p)^2$. Let $M = \bigcap M_p$. Then $[M, G, G] \leq \bigcap N_p = 1$ and $|G : M| \leq \prod (n_p)^2 = n^2$. □

Proof of Theorem 8. Using the multiplicativity of Euler’s $\varphi$ function, as in the previous proof, we can reduce to the case where $G'$ is a $p$-group. If $G' \cap Z(G) > 1$, then by Theorem 7, the index of the second center is at most $(n/p)^2 < n \varphi(n)$. In the case of $p = 2$ the unique element of order 2 in $G'$ is central in $G$, thus $G' \cap Z(G) = 1$. We can assume therefore that $G' \cap Z(G) = 1$ and in particular $p > 2$. Now let $D = C_G(G')$, and note that $[G, D, D] = 1$. Therefore $D' \leq Z(G)$ by the Three Subgroup Lemma. Then $D' \leq G' \cap Z(G) = 1$; so $D$ is abelian. It is obvious that $G/D \leq \text{Aut}(G')$. Since $G'$ is a cyclic $p$-group and $p > 2$, we have that $G/D$ is cyclic of order dividing $\varphi(n)$. If $x$ generates $G$ modulo $D$, let $C = C_D(x)$. Then $C$ centralizes $D(x) = G$, and hence $C \leq Z(G)$. Consequently $|D : C| = ||D, x|| \leq n$, and we deduce that $|G : Z(G)| \leq |G : D||D : C| \leq n \varphi(n)$. □

Remark 16. I. M. Isaacs [2] proved that if $H$ is a capable nilpotent group with cyclic derived subgroup and all elements of order 4 are central in $H$, then $|H : Z(H)| = |H'|^2$. In this result the assumption about elements of order 4 cannot be omitted as the example of the dihedral group $D$ of order $2^n$ ($n \geq 3$) shows. It is a capable group, $D'$ is a cyclic group of order $2^{n-2}$ and $|D : Z(D)| = 2^{n-1}$.

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References


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