MORITA EQUIVALENCE FOR QUANTUM HEISENBERG MANIFOLDS

BEATRIZ ABADIE

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ABSTRACT. We discuss Morita equivalence within the family \{D_{c\mu\nu}^c : c \in \mathbb{Z}, c > 0, \mu, \nu \in \mathbb{R}\} of quantum Heisenberg manifolds. Morita equivalence classes are described in terms of the parameters \mu, \nu and the rank of the free abelian group \(G_{\mu\nu} = 2\mu\mathbb{Z} + 2\nu\mathbb{Z} + \mathbb{Z}\) associated to the \(C^\ast\)-algebra \(D_{c\mu\nu}^c\).

INTRODUCTION

Quantum Heisenberg manifolds \(\{D_{c\mu\nu}^c : c \in \mathbb{Z}, c > 0, \mu, \nu \in \mathbb{R}\}\) were constructed by Rieffel in [Rf4] as a quantization deformation of certain homogeneous spaces \(H/N_c, H\) being the Heisenberg group.

It was shown in [Ab1] 3.4 that \(K_0(D_{c\mu\nu}^c) \cong \mathbb{Z}^3 \oplus \mathbb{Z}_c\), which implies that \(D_{c\mu\nu}^c\) and \(D_{c'\mu'\nu'}^{c'}\) are not isomorphic unless \(c = c'\). Besides, \(D_{c\mu\nu}^c\) and \(D_{c'\mu'\nu'}^{c'}\) are isomorphic when \((2\mu, 2\nu)\) and \((2\mu', 2\nu')\) belong to the same orbit under the usual action of \(GL_2(\mathbb{Z})\) on \(\mathbb{T}^2\) ([AE Theorem 2.2]; see also [Ab2] 3.3). The range of traces on \(D_{c\mu\nu}^c\) was discussed in [Ab2], where it was shown that the range of the homomorphism induced on \(K_0(D_{c\mu\nu}^c)\) by any tracial state on \(D_{c\mu\nu}^c\) has range \(G_{\mu\nu} = \mathbb{Z} + 2\mu\mathbb{Z} + 2\nu\mathbb{Z}\).

As a consequence ([Ab2] 3.17]), the isomorphism condition stated above turns out to be necessary when the rank of \(G_{\mu\nu}\) is either 1 or 3. Rieffel showed in [Rf4] that \(D_{c\mu\nu}^c\), is simple if and only if \(\{1, \mu, \nu\}\) is linearly independent over the field of rational numbers (i.e. rank \(G_{\mu\nu} = 3\)). It might be interesting to know whether in this case the classification can be made by means of the results of Elliott and Gong ([EG]).

The quantum Heisenberg manifold \(D_{c\mu\nu}^c\) was described in [AEE] as a crossed product by a Hilbert \(C^\ast\)-bimodule. In order to discuss Morita equivalence within this family, we adapt to this setting some of the techniques employed in the analogous discussion for non-commutative tori ([Rf3]) and Heisenberg \(C^\ast\)-algebras ([Pa2]). Thus we generalize in Section 1 Green’s result (discussed by Rieffel in [Rf2 Situation 10]) on the Morita equivalence of the crossed products \(C_0(M/K) \rtimes H\) and \(C_0(M/H) \rtimes K\), for free and proper commuting actions on a locally compact space \(M\). This result provides the main tool used to discuss Morita equivalence for quantum Heisenberg manifolds (Section 2).

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1. Morita equivalence of crossed products by certain Hilbert $C^*$-bimodules over commutative $C^*$-algebras

For a Hilbert $C^*$-bimodule $X$ over a $C^*$-algebra $A$, the crossed product $A \rtimes X$ was introduced in [AEF] (see also [Pi]) as the universal $C^*$-algebra for which there exist a *-homomorphism $i_A : A \longrightarrow A \rtimes X$ and a continuous linear map $i_X : X \longrightarrow A \rtimes X$ such that

\[ i_X(ax) = i_A(a)i_X(x), \quad i_A((x,y)L) = i_X(x)i_X(y)^*, \quad i_X(ax) = i_X(x)i_A(a), \quad i_A((x,y)R) = i_X(x)^*i_X(y). \]

The crossed product $A \rtimes X$ carries a dual action $\delta$ of $S^1$, defined by $\delta_z(i_A(a)) = i_A(a)$, $\delta_z(i_X(x)) = z i_X(x)$, for $a \in A$, $x \in X$ and $z \in S^1$. Moreover, if a $C^*$-algebra $B$ carries an action $\delta$ of $S^1$ such that $B$ is generated as a $C^*$-algebra by the fixed point subalgebra $B_0 = \{ b \in B : \delta_z(b) = b \ \forall z \in S^1 \}$ and the first spectral subspace $B_1 = \{ b \in B : \delta_z(b) = zb \ \forall z \in S^1 \}$, then $B$ is isomorphic to $B_0 \rtimes B_1$ (where $B_1$ has the obvious Hilbert $C^*$-bimodule structure over $B_0$), via an isomorphism that takes the action $\delta$ into the dual action.

If $X$ is an $A$-Hilbert $C^*$-bimodule and $\alpha \in \text{Aut}(A)$, we denote by $X_\alpha$ the Hilbert $C^*$-bimodule over $A$ obtained by leaving unchanged the left structure, and by setting

\[ x \cdot x_\alpha a := x\alpha(a), \quad (x,y)^{X_\alpha} := \alpha^{-1}(\langle x,y \rangle_R), \]

where the undecorated notation refers to the original right structure of $X$.

For $\alpha \in \text{Aut}(A)$ and the usual $A$-Hilbert $C^*$-bimodule structure on $A$, the crossed product $A \rtimes A_\alpha$ is easily checked to be the usual crossed product $A \rtimes A Z$.

**Definition 1.1.** Given a proper action $\alpha$ of $\mathbb{Z}$ on a locally compact Hausdorff space $M$ and a unitary $u \in C_0(M)$, let $X^{\alpha,u}$ denote the set of functions $f \in C_0(M)$ satisfying $f = u\alpha(f)$, and such that the map $x \mapsto |f(x)|$, which is constant on $\alpha$-orbits, belongs to $C_0(M/\alpha)$. Then $X^{\alpha,u}$ is a Hilbert $C^*$-bimodule over $C_0(M/\alpha)$ for pointwise multiplication on the left and the right, and inner products given by $\langle f, g \rangle_L = f\overline{g}, \langle f, g \rangle_R = \overline{f}g$.

**Proposition 1.2.** Let $\alpha$ and $\beta$ be free and proper commuting actions of $\mathbb{Z}$ on a locally compact Hausdorff space $M$, and let $u$ be a unitary in $C_0(M)$. Then the $C^*$-algebras $C_0(M/\alpha) \rtimes X^{\alpha,u}$ and $C_0(M/\beta) \rtimes X^{\alpha,u}$ are Morita equivalent.

**Proof.** Let $U : Z \times Z \longrightarrow U(C_0(M))$ be given by

\[ U(n, k) = \begin{cases} 1, & \text{if either } n = 0 \text{ or } k = 0, \\ \prod_{l \in S_k} \alpha^j(b^i(u^*), & \text{for } n, k > 0, \\ \prod_{l \in S_k} \alpha^j(b^i(u^*), & \text{for either } n \text{ or } k < 0, \text{ and } nk \neq 0, \end{cases} \]

where $S_l = \{0, 1, \ldots, l-1\}$ if $l > 0$ and $S_l = \{-1, -2, \ldots, l\}$ if $l < 0$. Straightforward computations show that $U(m + n, k) = U(m, k)\beta(m)(U(n, k))$, and $U(n, k + l) = U(n, k)\alpha(l)(U(n, l))$.

Consider the proper actions $\gamma^\alpha$ and $\gamma^\beta$ of $\mathbb{Z}$ on $C_0(M) \rtimes Z$ and $C_0(M) \rtimes Z$, respectively, given by

\[ \gamma^\alpha_n(\phi)(n) = U(n, k)\alpha^k(\phi(n)) \] and \[ \gamma^\beta_n(\psi)(k) = U^*(n, k)\beta^m(\psi(k)), \]

for $\phi \in C_0(Z, C_0(M)) \subset C_0(M) \rtimes Z$ and $\psi \in C_0(Z, C_0(M)) \subset C_0(M) \rtimes Z$.

These two actions correspond, respectively, to $\gamma^\alpha U$ and $\gamma^\beta U^*$ in [Ab1] Propositions 1.2 and 2.1. By virtue of [Ab1] Theorem 2.12, the generalized fixed-point algebras, in the sense of [RGL] Definition 1.4, $D^\alpha$ and $D^\beta$ of $C_0(M) \rtimes Z$ and
$C_0(M) \rtimes_{\alpha} \mathbb{Z}$ under the actions $\gamma^\alpha$ and $\gamma^\beta$, respectively, are Morita equivalent. The result will then be proved once we show that $D^\alpha \cong C_0(M/\alpha) \times X^\alpha_{\beta}$ and $D^\beta \cong C_0(M/\beta) \times X^\beta_{\alpha}$.

Recall from [Ab1, Proposition 2.1] that $D^\alpha$ is defined to be the closed span in $\mathcal{M}(C_0(M) \rtimes_{\beta} \mathbb{Z})$ of the set $\{P_\alpha(\phi \ast \psi) : \phi, \psi \in C_c(\mathbb{Z} \times M)\}$, where

$$P_\alpha(\phi)(x, n) = \sum_{k \in \mathbb{Z}} [\gamma^\alpha_k(\phi)](x, n),$$

for $\phi \in C_c(\mathbb{Z} \times M) \subset C_0(M) \rtimes_{\beta} \mathbb{Z}$, $x \in M$, and $n \in \mathbb{Z}$.

The $C^*$-algebra $D^\alpha$ can also be described ([Ab1, Proposition 2.8]) as the closure in $\mathcal{M}(C_0(M) \rtimes_{\beta} \mathbb{Z})$ of the $*$-subalgebra $C^\alpha = \{F \in C_c(\mathbb{Z}, C_0(M)) : \gamma^\alpha(F) = F$ and $\pi_\alpha(\text{supp } F(n))$ is precompact for all $n \in \mathbb{Z}\}$, where $\pi_\alpha$ denotes the canonical projection $\pi_\alpha : M \rightarrow M/\alpha$.

Now, since $C^\alpha$ is contained in $C_b(M) \rtimes_{\beta} \mathbb{Z}$, which is closed in $\mathcal{M}(C_0(M) \rtimes_{\beta} \mathbb{Z})$, so is $D^\alpha$. Moreover, the $C^*$-algebra $D^\alpha$ is invariant under the dual action $\hat{\beta}$ of $T$ on $C_b(M) \rtimes_{\beta} \mathbb{Z}$:

$$[\gamma^\alpha(\hat{\beta}_z F)](n, x) = U(n, 1)(x)(\hat{\beta}_z(F))(n, \alpha^{-1}x)$$

$$= U(n, 1)(x)z^n F(n, \alpha^{-1}x)$$

$$= z^n F(n, x)$$

$$= (\hat{\beta}_z F)(n, x),$$

for $F \in C^\alpha$, $x \in M$, $n \in \mathbb{Z}$, and $z \in \mathbb{T}$. Besides, $\text{supp } (\hat{\beta}_z(F)(n)) = \text{supp } F(n)$ for all $n \in \mathbb{Z}$, so $\hat{\beta}_z(F) \in C^\alpha$ for all $z \in \mathbb{T}$.

We next show that the action $\hat{\beta}$ on $D^\alpha$ is semi-saturated. That is, that, as a $C^*$-algebra, $D^\alpha$ is generated by the fixed-point subalgebra $D_0$ and the first spectral subspace $D_1 = \{d \in D^\alpha : \hat{\beta}_z(d) = zd \quad \forall z \in \mathbb{T}\}$ for the restriction of the dual action $\hat{\beta}$.

Since the maps $P_i : D^\alpha \rightarrow D_i$ given by $P_i(a) = \int_{\mathbb{T}} z^{-i} \hat{\beta}_z(a) \, dz$ are surjective contractions, $D_i$ is the closure of $P_i(C^\alpha)$. Now, for $i = 0, 1$, $C_i = C^\alpha \cap F_i$, and $D_i = D^\alpha \cap F_i$, where $F_0$ and $F_1$ are, respectively, the fixed-point subalgebra and the first spectral subspace of $C_b(M) \rtimes_{\beta} \mathbb{Z}$, which are known to be the $\delta_i$-maps; that is, $F_i = \{F \in C_c(\mathbb{Z}, C_b(M)) : \text{supp } F = \{i\}\}$.

Note that

$$C^\alpha \cap F_0 = \{f \delta_0 : f \in C_0(M) : \pi_\alpha(\text{supp } f) \text{ is precompact and } f = \alpha(f)\}$$

can be identified with $C_c(M/\alpha)$ via $f \delta_0 \mapsto \hat{f}$, where $\hat{f} = f \circ \alpha = f$, and that this map extends to a $*$-isomorphism between $D_0$ and $C_0(M/\alpha)$.

Now, $D_1$ is a Hilbert $C^*$-bimodule over $D_0$ for

$$\begin{align*}
(1a) \quad (f \delta_0) \ast (g \delta_2) &= (f g) \delta_1, \\
(1b) \quad (f \delta_1, g \delta_1)_L &= (f \delta_1) \ast (g \delta_1)^* = (f g) \delta_0, \\
(1c) \quad (f \delta_1, g \delta_1)_R &= (f \delta_1)^* \ast (g \delta_1) = (\beta^{-1}(f g)) \delta_0.
\end{align*}$$

Note that $D_1$ is full on the left (and on the right, by a similar argument) as a Hilbert $C^*$-bimodule over $C_0(M/\alpha)$. For $(D_1, D_1)_L$, the closed linear span in $C_0(M/\alpha) \equiv D_0$ of the set $\{f \delta_1, g \delta_1 : f \delta_1, g \delta_1 \in D_1\}$ is a closed ideal of $C_0(M/\alpha)$. Therefore, unless $(D_1, D_1)_L = C_0(M/\alpha)$, there exists $x_0 \in M$ such that $f(x_0) = 0$ for all $f \delta_1 \in D_1$. 


Now, given \( x_0 \in M \), we can choose (\cite[Situation 10]{Rf2}) a neighborhood \( U \) of \( x_0 \) such that \( U \cap \alpha^k(U) = \emptyset \) for \( k \neq 0 \). Let \( g \in C_c(M)^+ \) be such that \( \text{supp } g \subset U \) and \( g(x_0) = 1 \).

Then
\[
[P_{\alpha}(g^{1/2} \delta_0) * (g^{1/2} \delta_1)](x,n) = (P_{\alpha}(g \delta_1))(x,n) = \left( \sum_k U(1,k)(x)g(\alpha^{-k}(x)) \right) \delta_1(n),
\]
so \( P_{\alpha}(g^{1/2} \delta_0) * (g^{1/2} \delta_1) \in D_1 \) and equals 1 at \( (x_0,1) \).

In order to prove that \( C^\alpha \subset C^*(D_0, D_1) \), it suffices to show that \( f \delta_k \in C^*(D_0, D_1) \) for \( f \delta_k \in C^\alpha, k \in \mathbb{Z}. \) Since \( C^\alpha \) is closed under involution, we may assume that \( k \geq 0. \) We show this fact, which clearly holds for \( k = 0 \) and \( k = 1 \), by induction on \( k \).

If \( f \delta_k \in C^\alpha \) and \( \epsilon > 0 \), since \( \pi_{\alpha}(\text{supp } f) \) is precompact in \( M/\alpha \), and \( D_1 \) is full over \( C_0(M/\alpha) \), we can find \( \phi_i, \psi_i \in D_1, i = 1, \ldots, p \), such that
\[
\| \sum_i (\phi_i * \psi_i^*) * f \delta_1 - f \delta_1 \|_{D_1} = \| \sum_i (\phi_i, \psi_i)_{L^1} - f \|_{C_0(M)} < \epsilon.
\]

Now, since \( \phi_i \) and \( \psi_i^* \) belong to \( C^\alpha(D_0, D_1) \) for \( i = 1, \ldots, p \), so does \( f \). This shows that \( D^\alpha = C^*(D_0, D_1) \) and, consequently, by [AEE] Theorem 3.1, that \( D^\alpha \cong D_0 \times D_1 \).

It only remains to note now that \( D_0 \times D_1 \cong C_0(M/\alpha) \rtimes X_{\beta}^{\alpha,u} \). As noted above, \( D_0 \) is isomorphic to \( C_0(M/\alpha) \). On the other hand, the map \( f \delta_1 \mapsto f \) takes \( C^\alpha \cap F_1 \) to \( X_{\beta}^{\alpha,u} \). By keeping track of the formulae in (1a)–(1c), one easily checks that the map is an isometry, so it extends to an isometry from \( D_1 \) to \( X_{\beta}^{\alpha,u} \), which is onto because its image contains the dense set
\[
X_0^{\alpha,u} = \{ f \in X_{\beta}^{\alpha,u} : \text{the map } x \mapsto |f(x)| \text{ is compactly supported on } M/\alpha \}.
\]
(Note that \( X_0^{\alpha,u} \) is dense in \( X_{\beta}^{\alpha,u} \), because, if \( \{ e_\lambda \} \) is an approximate identity for \( C_c(M/\alpha) \), then \( e_\lambda f \) converges to \( f \) for all \( f \in X_{\beta}^{\alpha,u} \).)

This shows that \( D^\alpha \) is isomorphic to \( C_0(M/\alpha) \rtimes X_{\beta}^{\alpha,u} \). Analogously, \( D^\beta \) is isomorphic to \( C_0(M/\beta) \rtimes X_{\beta}^{\alpha,u} \).

\[ \square \]

2. Morita equivalence for quantum Heisenberg manifolds

In [AEE] (see also [AE 2]) the quantum Heisenberg manifold \( D_{\mu \nu}^{c} \) was shown to be the crossed product of \( C(\mathbb{T}^2) \), the \( C^* \)-algebra of continuous functions on the torus, by the Hilbert \( C^* \)-bimodule \( M_{\mu \nu}^{c} \), where \( \alpha_{\mu \nu}(x,y) = (x + 2\mu, y + 2\nu) \), and
\[
M^{c} = \{ f \in C_b(\mathbb{R} \times \mathbb{T}) : f(x+1, y) = e^{-2\pi i c y} f(x, y) \}
\]
is the Hilbert \( C^* \)-bimodule obtained by letting \( C(\mathbb{T}^2) \) act by pointwise product, and by defining the inner products \( (f,g)_L = f\overline{g}, (f,g)_R = \overline{f}g \).

Remark 2.1. The \( C^* \)-algebras \( D_{\mu \nu}^{c} \) and \( D_{\mu' \nu'}^{c} \) are isomorphic when the projections of \( (2\mu, 2\nu) \) and \( (2\mu', 2\nu') \) on the torus are in the same orbit under the usual action of \( GL_2(\mathbb{Z}) \) ([AE Theorem 2.2], see also [Ab2 Remark 3.3]).

Proposition 2.2. Let \( \mu \neq 0. \) Then \( D_{\mu \nu}^{c} \) and \( D_{\frac{1}{\mu}, \frac{1}{\nu}}^{c} \) are Morita equivalent.
Proof. We follow the lines of [KR, 1.1] and apply Proposition 1.2 to the following setting: \( \alpha \) and \( \beta \) consist of translation on \( \mathbb{R} \times T \) by \( (\frac{1}{2\mu}, 0) \) and \( (1, 2\nu) \), respectively, and \( u \in C_{b}(\mathbb{R} \times T) \) is given by \( u(x, y) = e(-cy) \), where \( T \) is viewed as \( \mathbb{R}/\mathbb{Z} \) and, for a real number \( h \), \( e(h) = e^{2\pi ih} \).

Then, by Proposition 1.2, \( C((\mathbb{R} \times T)/\alpha) \rtimes X_{\alpha}^{\alpha,u} \) and \( C((\mathbb{R} \times T)/\beta) \rtimes X_{\beta}^{\beta,u} \) are Morita equivalent, where

\[
X_{\alpha,u} = \{ F \in C_{b}(\mathbb{R} \times T) : F(x - \frac{1}{2\mu}, y) = e(cy)F(x,y) \} \quad \text{and} \quad X_{\beta,\alpha,u} = \{ F \in C_{b}(\mathbb{R} \times T) : F(x - 1, y - 2\nu) = e(-cy)F(x,y) \}
\]

and, for \( (\mu', \nu') = \left( \frac{\mu}{4\mu}, \frac{\nu}{2\nu} \right) \), set

\[
J_{\alpha} : M_{\alpha,\nu} \to X_{\alpha}^{\alpha,u} \quad \text{and} \quad J_{\beta} : M_{\beta,\mu} \to X_{\beta,\alpha,u}.
\]

Note that

\[
(J_{\alpha}f)(x) = f(2\mu x, y), \quad (J_{\beta}f)(x) = e(c(x + 1)\nu)f(x, 2\nu x - y).
\]

so the definitions make sense.

For \( i = \alpha, \beta \), it is easily checked that \( J_{i} \) is a bijection and that, for \( \phi \in C(T^{2}) \), \( f, g \in M_{C}^{c} \):

\[
J_{i}(\phi \cdot f) = H_{i}(\phi) \cdot J_{i}(f), \quad J_{i}(f \cdot \phi) = J_{i}(f) \cdot H_{i}(\phi),
\]

\[
\langle J_{i}f, J_{i}g \rangle_{L} = H_{i}((f, g)_{L}), \quad \langle J_{i}f, J_{i}g \rangle_{R} = H_{i}((f, g)_{R}).
\]

This shows that \( D_{\mu,\nu}^{c} = C(T^{2}) \rtimes M_{\mu,\nu}^{c} \) and \( D_{\mu',\nu'}^{c} = C(T^{2}) \rtimes M_{\mu',\nu'}^{c} \) are isomorphic, respectively, to \( C((\mathbb{R} \times T)/\alpha) \rtimes X_{\alpha}^{\alpha,u} \) and \( C((\mathbb{R} \times T)/\beta) \rtimes X_{\beta,\alpha,u} \), and they are, consequently, Morita equivalent to each other. \hfill \Box

Corollary 2.3. Let \( \mu \notin \mathbb{Q} \), and let \( A = \left( \begin{array}{cc} \frac{a}{c} & \frac{b}{d} \\ c & d \end{array} \right) \in GL_{2}(\mathbb{Z}) \). If

\[
2\mu' = \frac{2a\mu + b}{2c\mu + d} \quad \text{and} \quad 2\nu' = \frac{2\nu}{2c\mu + d},
\]

then the quantum Heisenberg manifolds \( D_{\mu,\nu}^{c} \) and \( D_{\mu',\nu'}^{c} \) are Morita equivalent.

Proof. It suffices to check the statement for \( A_{1} = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \) and \( A_{2} = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \), since \( A_{1} \) and \( A_{2} \) generate \( GL_{2}(\mathbb{Z}) \) ([Ku, Appendix B]), and \( (\mu, \nu) \mapsto (\mu', \nu') \) defines an action of \( GL_{2}(\mathbb{Z}) \) on \( (\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{R} \). For \( A = A_{1} \) we get isomorphic \( C^{*} \)-algebras by Remark 2.1. For \( A = A_{2} \), we get \( (\mu', \nu') = \left( \frac{1}{4\mu}, \frac{1}{2\nu} \right) \), and the result follows from Proposition 2.2. \hfill \Box
Proposition 2.4. Let \( \{1, \mu, \nu\} \) be linearly independent over \( \mathbb{Q} \), and let \( A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \text{GL}_3(\mathbb{Z}) \). If
\[
2\mu' = \frac{2a\mu + 2b\nu + c}{2g\mu + 2h\nu + i} \quad \text{and} \quad 2\nu' = \frac{2d\mu + 2e\nu + f}{2g\mu + 2h\nu + i},
\]
then the quantum Heisenberg manifolds \( D^{c}_{\mu,\nu} \) and \( D^{c}_{\mu',\nu'} \) are Morita equivalent.

Proof. As in the proof of Theorem 1.7 in [Pa2], \( A = A_1A_2A_3 \), where
\[
A_1 = \begin{pmatrix} A & B & C \\ D & E & F \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} G & 0 & H \\ 0 & 1 & 0 \\ I & 0 & J \end{pmatrix}, \quad A_3 = \begin{pmatrix} K & L & 0 \\ M & N & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]
and \( A_i \in \text{GL}_3(\mathbb{Z}) \), for \( i = 1, 2, 3 \).

Since the map \((\mu, \nu) \mapsto (\mu', \nu')\) defines an action of \( \text{GL}_3(\mathbb{Z}) \) on the set \( \{(\mu, \nu) \in \mathbb{R}^2 : \{1, \mu, \nu\} \text{ is linearly independent over } \mathbb{Q}\} \), it suffices to check the statement for \( A_i, i = 1, 2, 3 \).

For \( A = A_1 \) and \( A = A_3 \) the \( C^* \)-algebras \( D^{c}_{\mu,\nu} \) and \( D^{c}_{\mu',\nu'} \) are isomorphic by Remark 2.1. Thus it suffices to show the result for \( A = A_2 \). The map \((G \quad H \quad J) \mapsto A_2\) is a group homomorphism from \( \text{GL}_2(\mathbb{Z}) \) into \( \text{GL}_3(\mathbb{Z}) \), and \( \text{GL}_2(\mathbb{Z}) \) is generated by \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), which implies that we only need to prove the statement for \( A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \) and \( A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \). For \( A_1 \) we get \( 2\mu' = 2\mu + 1 \), \( 2\nu' = 2\nu \), so \( D^{c}_{\mu,\nu} \) and \( D^{c}_{\mu',\nu'} \) are isomorphic by Remark 2.1. Proposition 2.2 takes care of the case \( A = A_2 \), since then we have \((\mu', \nu') = (\frac{1}{4\nu} \frac{\mu}{2\nu}) \). \( \square \)

Notation 2.5. We denote by \( G_{\mu,\nu} \) the subgroup of \( \mathbb{R} \) generated by \( \{1, 2\mu, 2\nu\} \). It was shown in [Ab2] Theorem 3.16] that the homomorphism induced on \( K_0(D^{c}_{\mu,\nu}) \) by any tracial state on \( D^{c}_{\mu,\nu} \) has range \( G_{\mu,\nu} \).

Remark 2.6. If rank \( G_{\mu,\nu} = 2 \), then there exist an irrational number \( \nu' \) and integers \( p, q \in \mathbb{Z}, \ p \neq 0, (p, q) = 1 \), such that \( D^{c}_{\mu,\nu} \) and \( D^{c}_{\mu',\nu'} \) are isomorphic.

Proof. We proceed as in [Pa1] Proposition 1.5. Let \( \mu_0 = 2\mu, \nu_0 = 2\nu \). Since the group generated by \( \{1, \mu_0, \nu_0\} \) has rank 2, either \( \mu_0 \) or \( \nu_0 \) is irrational. We may assume that \( \nu_0 \) is irrational, because, by Remark 2.4, \( D^{c}_{\mu,\nu} \) and \( D^{c}_{\mu',\nu'} \) are isomorphic. Besides, there exist \( M, N, P \in \mathbb{Z}, \) with \( N \neq 0 \) such that \( M + N\mu_0 + P\nu_0 = 0 \), so we have \( \mu_0 = \frac{k}{l} \nu_0 + \frac{m}{n}, \) with \( (k, l) = 1 \). If \( k = 0 \), then \( \mu_0 \in \mathbb{Q} \), and we are done. Otherwise take \( a, b \in \mathbb{Z} \) such that \( ak + bl = 1 \), so that \( (\frac{-l}{a} \frac{k}{b}) \in \text{GL}_2(\mathbb{Z}) \), and set
\[
(\mu_0', \nu_0') = \left( \frac{-l}{a} \frac{k}{b} \right) (\mu_0, \nu_0).
\]
Then
\[
\mu_0' = -l(\frac{k}{l} \nu_0 + \frac{m}{n}) + k\nu_0 = \frac{-lm}{n} \in \mathbb{Q}
\]
and
\[
\nu_0' = a(\frac{k}{l} \nu_0 + \frac{m}{n}) + b\nu_0 = \frac{1}{l} \nu_0 + \frac{am}{n} \notin \mathbb{Q}.
\]
Let $\nu$ and $D$ for Morita equivalent quantum Heisenberg manifolds, and:

Two quantum Heisenberg manifolds particular, for $D$ which, as shown above, is Morita equivalent to $\nu$ and, if $c$ is isomorphic.

Proposition 2.2 to $(\nu, q)$.

Proof. By Remark 2.1 we may assume that $p$ and $q$ are positive. By applying Proposition 2.2 to $(\mu, \nu) = (q/2, \nu)$, we get that $D_{0, \nu}^c \cong D_{q/2, \nu}^c$ is Morita equivalent to $D_{q, \nu}^c$, thus proving the proposition for $p = 1$. For $p > 1$, let $r_0 = q$, $r_1 = p$, and, if $r_{i+1} \neq 1$, define $r_{i+2}$ by $r_i = m_{i+1}r_{i+1} + r_{i+2}$, where $0 \leq r_{i+2} < r_{i+1}$, and $m_{i+1} \in \mathbb{Z}$.

Actually, $r_{i+2} > 0$; otherwise $r_{i+1}$ divides $r_i$, and it follows that $r_{i+1}$ divides $r_j$ for all $j \leq i$. In particular, $r_{i+1}$ divides both $p$ and $q$, which contradicts the fact that $r_{i+1} \neq 1$. Now, since $r_{i+1} < r_i$, there is an index $i_0$ for which $r_{i_0} = 1$.

On the other hand, it follows from Proposition 2.2 that, for any real number $\kappa$, $D_{r_0, \nu}^c \cong D_{r_1, \nu}^c \cong D_{r_2, \nu}^c \cdots$ is Morita equivalent to $D_{r_0, \nu}^c \cong D_{r_1, \nu}^c \cong D_{r_2, \nu}^c \cdots$, which is Morita equivalent to $D_{0, \nu}^c$.

Theorem 2.8. Two quantum Heisenberg manifolds $D_{\mu, \nu}^c, D_{\mu', \nu'}^c$ are Morita equivalent if and only if $c = c'$ and there exists a positive real number $r$ such that

$Z + 2\mu Z + 2\nu Z = r(Z + 2\mu' Z + 2\nu' Z)$.

In particular, the rank of the free abelian group $G_{\mu, \nu}$ is the same for Morita equivalent quantum Heisenberg manifolds, and:

1. If rank $G_{\mu, \nu} = 1$ = rank $G_{\mu', \nu'}$, then $D_{\mu, \nu}^c$ is Morita equivalent to $D_{\mu', \nu'}^c$. In particular, $D_{\mu, \nu}^c$ is Morita equivalent to the commutative Heisenberg manifold $D_{0, \nu}^c$.

2. If rank $G_{\mu, \nu} = 2$ = rank $G_{\mu', \nu'}$, let $\{\alpha, \frac{q}{q'}\}$ and $\{\alpha', \frac{q}{q'}\}$ be bases of $G_{\mu, \nu}$ and $G_{\mu', \nu'}$, respectively, where $\alpha$ and $\alpha'$ are irrational numbers and $p, p', q, q' \in \mathbb{Z}$, $(p, q) = (p', q') = 1$. Then $D_{\mu, \nu}^c$ and $D_{\mu', \nu'}^c$ are Morita equivalent if and only if there exists $\left(\begin{array}{cc}a & b \\c & d \end{array}\right) \in GL_2(\mathbb{Z})$ such that

$q' \alpha' = \frac{aqd + b}{cqd + d}$.

In particular, $D_{\mu, \nu}^c$ is Morita equivalent to $D_{0, \nu}^c$.

3. If rank $G_{\mu, \nu} = 3$ = rank $G_{\mu', \nu'}$, then $D_{\mu, \nu}^c$ and $D_{\mu', \nu'}^c$ are Morita equivalent if and only if there exists $\left(\begin{array}{ccc}a & b & c \\g & h & i \end{array}\right) \in GL_3(\mathbb{Z})$ such that

$2\mu' = \frac{2a\mu + 2b\nu + c}{2g\mu + 2h\nu + i}$ and $2\nu' = \frac{2d\mu + 2e\nu + f}{2g\mu + 2h\nu + i}$.

Proof. It was shown in [Ab1, 3.4] that $K_0(D_{\mu, \nu}^c) = \mathbb{Z}^3 \oplus \mathbb{Z}_c$, which implies that $D_{\mu, \nu}^c$ and $D_{\mu', \nu'}^c$ are not Morita equivalent for $c \neq c'$.
Besides ([Ab2] Theorem 3.16), all tracial states on $D_{\mu\nu}^c$ induce the same homomorphism on $K_0(D_{\mu\nu}^c)$, whose range is the group $G_{\mu\nu} = 2\mu\mathbb{Z} + 2\nu\mathbb{Z} + \mathbb{Z}$. Since ([Re1] 2.2) there is a bijection between finite traces of Morita equivalent unital C*-algebras, we must have $G_{\mu\nu} = rG_{\mu'\nu'}$ for some positive real number $r$ when $D_{\mu\nu}^c$ and $D_{\mu'\nu'}^c$ are Morita equivalent. An immediate consequence of this fact is that the rank of $G_{\mu\nu}$ is invariant under Morita equivalence.

If rank $G_{\mu\nu} = 1$, then, by [Ab2, Remark 3.5], $D_{\mu\nu}^c$ is isomorphic to $D_{0,0}^c$ for some non-zero integer $p$, so $D_{\mu\nu}^c$ is isomorphic to $D_{0,0}^c$ by Remark 2.1. Now, by Proposition 2.7, $D_{0,0}^c$ is Morita equivalent $D_{0,0}^c$.

If rank $G_{\mu\nu} = 2$ = rank $G_{\mu'\nu'}$ and $G_{\mu\nu} = rG_{\mu'\nu'}$ for some positive $r$, let $\{\alpha, \frac{q}{r}\}$ and $\{\alpha', \frac{q'}{r}\}$ be bases of $G_{\mu\nu}$ and $G_{\mu'\nu'}$, respectively, where $\alpha, \alpha'$ are irrational numbers, and $p, p', q, q'$ are integers, with $(p, q) = (p', q') = 1$. Since $\mathbb{Z} \subseteq G_{\mu\nu} (G_{\mu'\nu'})$ we have that $p(p') = \pm 1$ and, by Remark 2.1, we may assume $p = p' = 1$. Then we have that $\alpha Z + 1/qZ = r(\alpha' Z + 1/q' Z)$, which implies that $\alpha qZ + Z = (rq/q')(\alpha' q' Z + Z)$. A standard argument shows that

$$q\alpha = \frac{aq\alpha' + b}{cq\alpha' + d} \text{ for some } \left(\begin{array}{cc}a & b \\c & d \end{array}\right) \in \text{GL}_{2}(\mathbb{Z}).$$

Therefore $D_{0,0}^c$ and $D_{0,0}^c$ are Morita equivalent by Corollary 2.3.

On the other hand, by Remark 2.6, $D_{\mu\nu}^c$ and $D_{\mu'\nu'}^c$ are isomorphic, respectively, to $D_{m,n}^c$ and $D_{m',n'}^c$, for some irrational numbers $\beta$ and $\beta'$ and integers $m, m', n, n'$ such that $(m, n) = (m', n') = 1$. Therefore $\{2\beta, \frac{1}{m}\}$ and $\{2\beta', \frac{1}{m}\}$ are bases of $G_{\mu\nu}$ and $G_{\mu'\nu'}$, respectively, and it follows from the argument above that $D_{n,0}^c$ and $D_{n',0}^c$ are Morita equivalent. It only remains to note now that, by Proposition 2.7 and Remark 2.1, $D_{n,0}^c$ and $D_{n',0}^c$ are Morita equivalent to $D_{\mu\nu}^c$ and $D_{\mu'\nu'}^c$, respectively.

Finally, if rank $G_{\mu\nu} = 3$ = rank $G_{\mu'\nu'}$ and $G_{\mu\nu} = rG_{\mu'\nu'}$ for some positive $r$, then let $A = \left(\begin{array}{ccc}a & b & c \\d & e & f \\g & h & i \end{array}\right) \in \text{GL}_{3}(\mathbb{Z})$ be the transpose of the matrix that changes coordinates between the bases $\{2r\mu', 2r'\nu', r\}$ and $\{2\mu, 2\nu, 1\}$ of $G_{\mu\nu}$. Then

$$2\mu' = \frac{2a\mu + 2b\nu + c}{2g\mu + 2h\nu + i} \text{ and } 2\nu' = \frac{2d\mu + 2e\nu + f}{2g\mu + 2h\nu + i},$$

which implies, by Proposition 2.4, that $D_{\mu\nu}^c$ and $D_{\mu'\nu'}^c$ are Morita equivalent. □

**References**


Centro de Matemáticas, Facultad de Ciencias, Iguá 4225, CP 11 400, Montevideo, Uruguay
E-mail address: abadie@cmat.edu.uy