PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 12, Pages 3543–3550
S 0002-9939(05)07918-9
Article electronically published on July 13, 2005

# SHARP VAN DER CORPUT ESTIMATES AND MINIMAL DIVIDED DIFFERENCES

KEITH M. ROGERS

(Communicated by Andreas Seeger)

Abstract. We find the sharp constant in a sublevel set estimate which arises in connection with van der Corput's lemma. In order to do this, we find the nodes that minimise divided differences. We go on to find the sharp constant in the first instance of the van der Corput lemma. With these bounds we improve the constant in the general van der Corput lemma, so that it is asymptotically sharp.

## 1. Sublevel set estimates

The importance of sublevel set estimates in van der Corput lemmas was highlighted by A. Carbery, M. Christ, and J. Wright [3], [4]. We find the sharp constant in the following sublevel set estimate. We will use this in Section 3 to prove an asymptotically sharp van der Corput lemma.

The constants $C_n$ take different values in each lemma.

**Lemma 1.** *Suppose that $f : [a, b] \to \mathbb{R}$ is $n$ times differentiable on $(a, b)$, where $n \geq 1$, and that $|f^{(n)}(x)| \geq \lambda > 0$. Then*

$$|\{x \in [a, b] : |f(x)| \leq \alpha\}| \leq C_n \left(\alpha/\lambda\right)^{1/n},$$

*where $C_n = (n!\, 2^{2n-1})^{1/n}$.*

We note that $C_n \leq 2n$ for all $n \geq 1$, and by Stirling's formula,

$$C_n \approx \left(\sqrt{2\pi n}\frac{n^n}{e^n}2^{2n-1}\right)^{1/n} = \left(\frac{\pi n}{2}\right)^{1/2n}\frac{4n}{e},$$

so that

$$\lim_{n\to\infty}\frac{C_n}{n} = \frac{4}{e}.$$

The Chebyshev polynomials will be key to the proof of Lemma 1, so we recall some facts that we will need. For a more complete introduction see [10]. Consider $T_n$ defined on $[-1, 1]$ by

$$T_n(\cos\theta) = \cos n\theta,$$

where $0 \leq \theta \leq \pi$. If we take the binomial expansion of the right-hand side of

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

Received by the editors July 8, 2004.
2000 *Mathematics Subject Classification.* Primary 42A05; Secondary 65T40, 26D10.
*Key words and phrases.* Van der Corput lemma, sharp constant, divided differences.

and equate real parts, we obtain

$$\cos n\theta = \sum_{k=0}^{\lfloor n/2 \rfloor} \left( (-1)^k \sum_{j=k}^{\lfloor n/2 \rfloor} \binom{n}{2j} \binom{j}{k} \right) (\cos\theta)^{n-2k},$$

where $\lfloor n/2 \rfloor$ denotes the integer part of $n/2$. Thus we can consider $T_n$ to be the polynomial of degree $n$ defined on the real line by

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \left( (-1)^k \sum_{j=k}^{\lfloor n/2 \rfloor} \binom{n}{2j} \binom{j}{k} \right) x^{n-2k}.$$

It is clear that $|T_n| \leq 1$ on $[-1, 1]$, and that the extrema are attained at

$$\eta_j = \cos(j\pi/n)$$

for $j = 0, \ldots, n$. Finally we calculate the leading coefficient,

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} = \frac{1}{2} \left( (1+1)^n + (1-1)^n \right) = 2^{n-1}.$$

We will also require the following generalization of the classical mean value theorem. It can be found in texts on numerical analysis, for example [8, p. 189]. We include a proof for convenience.

**Theorem 1.** *Suppose that $f : [a, b] \to \mathbb{R}$ is $n$ times differentiable, where $n \geq 1$, and suppose that $x_0 < x_1 < \ldots < x_n$ are distinct points in $[a, b]$. Then there exists $\zeta \in (a, b)$ such that*

$$f^{(n)}(\zeta) = \sum_{j=0}^{n} c_j f(x_j),$$

*where $c_j = (-1)^{j+n} n! \prod_{k:k\neq j} |x_k - x_j|^{-1}$.*

*Proof.* Define the Lagrange interpolating polynomial $P$ by

$$P(x) = \sum_{j=0}^{n} f(x_j) L_j(x), \qquad \text{where} \qquad L_j(x) = \prod_{k:k\neq j} \frac{x - x_k}{x_j - x_k}.$$

By construction, $P(x_j) = f(x_j)$ for $j = 0, \ldots, n$, so that $P - f$ has at least $n + 1$ zeros. Thus, by Rolle's Theorem (iterated many times), there exists $\zeta \in (a, b)$ such that

$$f^{(n)}(\zeta) - P^{(n)}(\zeta) = 0.$$

Now

$$P^{(n)}(\zeta) = \sum_{j=0}^{n} f(x_j) n! \prod_{k:k\neq j} \frac{1}{x_j - x_k},$$

so that

$$f^{(n)}(\zeta) = P^{(n)}(\zeta) = \sum_{j=0}^{n} f(x_j) (-1)^{n+j} n! \prod_{k:k\neq j} \frac{1}{|x_k - x_j|},$$

as desired. $\square$

We apply this result to the Chebyshev polynomials. It is easy to see that $T_n^{(n)} = n!\,2^{n-1}$, and that $T_n(\eta_j) = (-1)^{j+n}$ for the Chebyshev extrema $\eta_j = \cos(j\pi/n)$. Thus we obtain

$$\sum_{j=0}^{n} \prod_{k:k\neq j} n!\,|\eta_k - \eta_j|^{-1} = n!\,2^{n-1},$$

so that

$$\sum_{j=0}^{n} \prod_{k:k\neq j} |\eta_k - \eta_j|^{-1} = 2^{n-1}. \tag{1}$$

*Proof of Lemma* 1. Suppose that $E = \{x \in [a,b] : |f(x)| \leq \alpha\}$, and that $|E| > 0$. First we map $E$ to an interval of the same measure without increasing distance. Centering at the origin and scaling by $2/|E|$ if necessary, we map the interval to $[-1,1]$. Now by (1) we have

$$\sum_{j=0}^{n} \prod_{k:k\neq j} |\eta_k - \eta_j|^{-1} = 2^{n-1},$$

where $\eta_j = \cos(j\pi/n)$. Thus, mapping back to $E$, there exist $n+1$ points $x_0, \ldots, x_n \in E$ such that

$$\sum_{j=0}^{n} \prod_{k:k\neq j} |x_k - x_j|^{-1} \leq 2^{n-1}\frac{2^n}{|E|^n} = \frac{2^{2n-1}}{|E|^n}.$$

On the other hand, by Lemma 1 there exists $\zeta \in (a,b)$ such that

$$f^{(n)}(\zeta) = \sum_{j=0}^{n} c_j f(x_j),$$

where $|c_j| = n!\prod_{k:k\neq j} |x_k - x_j|^{-1}$. Putting these together,

$$\lambda \leq \left|\sum_{j=0}^{n} c_j f(x_j)\right| \leq \sum_{j=0}^{n} n! \prod_{k:k\neq j} |x_k - x_j|^{-1}\alpha \leq n!\,\frac{2^{2n-1}}{|E|^n}\alpha,$$

so that

$$|E| \leq \left(n!\,2^{2n-1}\right)^{1/n} (\alpha/\lambda)^{1/n},$$

as desired. □

The sharpness may be observed by considering $f(x) = T_n(x)$ on $[-1,1]$ with $\lambda = n!\,2^{n-1}$ and $\alpha = 1$.

## 2. Divided differences

Divided differences were first considered by Newton and are important in interpolation theory. For a given set of nodes $x_0 < \ldots < x_m \in [-1,1]$, the divided differences are defined recursively by

$$f[x_0, \ldots, x_n] = \frac{f[x_0, \ldots, x_{n-1}] - f[x_1, \ldots, x_n]}{x_0 - x_n},$$

where $f[x_0] = f(x_0)$ and $f[x_1] = f(x_1)$. It is not difficult to calculate that

$$f[x_0, \ldots, x_n] = \sum_{j=0}^{n} f(x_j)(-1)^{j+n} \prod_{k:k\neq j} |x_k - x_j|^{-1}.$$

The following theorem shows that the Chebyshev extrema are optimal for the minimization of divided differences.

**Theorem 2.** *The Chebyshev extrema $\eta_j = \cos(j\pi/n)$ for $j = 0, \dots, n$ are the unique nodes for which*

$$f[\eta_0, \dots, \eta_n] \leq 2^{n-1} ||f||_\infty$$

*for all $f \in C[-1, 1]$.*

*Proof.* That the inequality holds is clear from (1). To show the uniqueness, we suppose there exist $x_0, \dots, x_n \in [-1, 1]$ other than the Chebyshev extrema, such that

$$f[x_0, \dots, x_n] \leq 2^{n-1} ||f||_\infty$$

for all $f \in C[-1, 1]$. If we consider $f \in C[-1, 1]$, so that $||f||_\infty = 1$ and

$$f(x_j) = (-1)^{j+n}$$

for $j = 0, \dots, n$, we see that

$$\sum_{j=0}^{n} \prod_{k:k\neq j} |x_k - x_j|^{-1} \leq 2^{n-1}.$$

It is clear that $|T_n(x_j)| < 1$ for some $j$, as $x_0, \dots, x_n$ are not the Chebyshev extrema. Thus

$$\sum_{j=0}^{n} T(x_j)(-1)^{j+n} n! \prod_{k:k\neq j} |x_k - x_j|^{-1} < n!\, 2^{n-1}.$$

On the other hand,

$$\sum_{j=0}^{n} T(x_j)(-1)^{j+n} n! \prod_{k:k\neq j} |x_k - x_j|^{-1} = n!\, 2^{n-1}$$

by Theorem 1, and we have a contradiction. □

## 3. Van der Corput lemmas

The following lemma is perhaps the cornerstone of the theory of oscillatory integrals, and is proven, using different methods, in J.G. van der Corput [5], A. Zygmund [12], and E.M. Stein [11] with constants $2\sqrt{2}$, 4, and 3, respectively. We find the sharp constant.

**Lemma 3.** *Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable, $f'$ is monotone, and $|f'| \geq \lambda > 0$ on $(a, b)$. Then*

$$\left| \int_a^b e^{if(x)}\, dx \right| \leq \frac{2}{\lambda}.$$

*Proof.* By replacing $f$ by $-f$ if necessary, we may assume $f' \geq \lambda > 0$. We also assume $f'$ is increasing. The proof when $f'$ is decreasing is similar.

As $f'$ is not defined on $a$ or $b$, we initially consider the integral over a slightly smaller interval. Let $\epsilon > 0$, and define $I_\epsilon$ by

$$I_\epsilon = \int_{a+\epsilon}^{b-\epsilon} e^{if(x)}\, dx.$$

As $I_\epsilon = |I_\epsilon| e^{i\theta_\epsilon}$ for some argument $\theta_\epsilon \in [0, 2\pi)$, we see that

$$|I_\epsilon| = e^{-i\theta_\epsilon} \int_{a+\epsilon}^{b-\epsilon} e^{if(x)}\, dx = \int_{a+\epsilon}^{b-\epsilon} \cos\left(f(x) - \theta_\epsilon\right)\, dx.$$

By a change of variables,

$$|I_\epsilon| = \int_{f(a+\epsilon)-\theta_\epsilon}^{f(b-\epsilon)-\theta_\epsilon} \frac{\cos y}{f'\left(f^{-1}(y+\theta_\epsilon)\right)}\, dy.$$

Now as $1/(f' \circ f^{-1})$ is positive and decreasing, there exists a point $c \in [a+\epsilon, b-\epsilon]$ such that

$$|I_\epsilon| = \frac{1}{f'(a+\epsilon)} \int_{f(a+\epsilon)-\theta_\epsilon}^{c} \cos y\, dy = \frac{1}{f'(a+\epsilon)} \left(\sin c - \sin(f(a+\epsilon) - \theta_\epsilon)\right),$$

by the second mean value theorem for integrals (see, for example, [2, p. 304]). Thus

$$|I_\epsilon| \le \frac{1 + \sin\left(\theta_\epsilon - f(a+\epsilon)\right)}{\lambda} \le \frac{2}{\lambda}. \tag{2}$$

Finally, we let $\epsilon$ tend to zero, and we are done. □

The sharp constant for functions with continuous derivative was known, and a proof of this result may be found in [9]. That the constant 2 is sharp is observed by considering $f(x) = x$ on $(0, \pi)$ with $\lambda = 1$.

The following lemma, with constants taken to be

$$C_n = 2^{5/2} \pi^{1/n} (1 - 1/n),$$

is due to G.I. Arhipov, A.A. Karacuba and V.N. Čubarikov [1]. It is a more precise formulation of what is generally known as van der Corput's lemma. We improve their constants, so that the bound becomes sharp as $n$ tends to infinity.

**Lemma 4.** *Suppose that $f : [a, b] \to \mathbb{R}$ is $n$ times differentiable on $(a, b)$, where $n \ge 2$, and $|f^{(n)}(x)| \ge \lambda > 0$. Then*

$$\left| \int_a^b e^{if(x)}\, dx \right| \le C_n \frac{n}{\lambda^{1/n}},$$

*where $C_n \le 2^{5/3}$ for all $n \ge 2$ and $C_n \to 4/e$ as $n \to \infty$.*

*Proof.* We integrate over $E_1$ and $E_2$ separately, where

$$E_1 = \{x \in [a, b] : |f'(x)| \le \alpha\} \quad \text{and} \quad E_2 = \{x \in [a, b] : |f'(x)| > \alpha\}.$$

If we consider $g = f'$, then $|g^{(n-1)}| \ge \lambda$, so that

$$|E_1| \le \left((n-1)!\, 2^{2(n-1)-1}\right)^{1/(n-1)} \left(\frac{\alpha}{\lambda}\right)^{1/(n-1)},$$

by Lemma 1. So trivially

$$\left| \int_{E_1} e^{if(x)}\, dx \right| \le \left((n-1)!\, 2^{2n-3}\right)^{1/(n-1)} \left(\frac{\alpha}{\lambda}\right)^{1/(n-1)}.$$

Now $E_2$ is made up of at most $2(n-1)$ intervals on each of which $|f'| \ge \alpha$ and $f'$ is monotone, so by Corollary 3 we have

$$\left| \int_{E_2} e^{if(x)}\, dx \right| \le 2(n-1) \frac{2}{\alpha}.$$

Finally we optimize with respect to $\alpha$ and deduce that

$$\left|\int_a^b e^{if(x)}\,dx\right| \le \left(\frac{(n-1)!\,2^{2n-1}}{(n-1)^{n-2}}\right)^{1/n} \frac{n}{\lambda^{1/n}} = C_n \frac{n}{\lambda^{1/n}}. \tag{3}$$

Now $C_n$ tends to $4/e$ as $n$ tends to infinity, by Stirling's formula, and we can check the first few terms to see it is always less than or equal $2^{5/3}$. □

To see that the bound becomes sharp as $n$ tends to infinity, we consider $f_n$ defined on $[-1,1]$ by

$$f_n(x) = \frac{T_n(x)}{n}.$$

When $n \ge 2$, we have

$$\left|\int_{-1}^1 e^{if_n(x)}\,dx\right| \ge \left|\int_{-1}^1 \cos(f_n(x))\,dx\right| \ge \left|\int_{-1}^1 1 - \frac{(f_n(x))^2}{2}\,dx\right|,$$

by Taylor's Theorem. Now as $|f_n| \le 1/n$, we see that

$$\left|\int_{-1}^1 e^{if_n(x)}\,dx\right| \ge 2 - \frac{1}{n^2},$$

which tends to 2 as $n$ tends to infinity. On the other hand

$$f_n^{(n)} = (n-1)!\,2^{n-1},$$

so that

$$\left|\int_{-1}^1 e^{if_n(x)}\,dx\right| \le \left(\frac{(n-1)!\,2^{2n-1}}{(n-1)^{n-2}}\right)^{1/n} \frac{n}{((n-1)!\,2^{n-1})^{1/n}},$$

by (3) in Lemma 4. We manipulate this to obtain

$$\left|\int_{-1}^1 e^{if_n(x)}\,dx\right| \le \left(\frac{2^n n^n}{(n-1)^{n-2}}\right)^{1/n}.$$

This bound also tends to 2 as $n$ tends to infinity. Hence Lemma 4 is asymptotically sharp.

We note that as

$$\left|\int_{-2}^2 e^{ix^2/2}\,dx\right| = 3.33346\ldots > \frac{4}{e} \times 2$$

and

$$\left|\int_{-3}^3 e^{i(x^3/6-x)}\,dx\right| = 4.61932\ldots > \frac{4}{e} \times 3,$$

the asymptote could not be approached strictly from below.

**Corollary 5.** *Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a real polynomial of degree $n \ge 1$. Then for all $a, b \in \mathbb{R}$,*

$$\left|\int_a^b e^{if(x)}\,dx\right| < \frac{C_n}{|a_n|^{1/n}},$$

*where $C_n < 11/2$ for all $n \ge 1$ and $C_n \to 4$ as $n \to \infty$.*

*Proof.* When $n = 1$, we obtain the result simply by integrating. For higher degrees we have $f^{(n)} = n!\, a_n$, so that

$$|I| = \left| \int_a^b e^{if(x)}\, dx \right| \leq \left( \frac{(n-1)!\, 2^{2n-1}}{(n-1)^{n-2}} \right)^{1/n} \frac{n}{(n!\, |a_n|)^{1/n}}$$

by (3) in the proof of Lemma 4. We can manipulate to obtain

$$|I| \leq \left( \frac{2^{2n-1} n^{n-1}}{(n-1)^{n-2}} \right)^{1/n} \frac{1}{|a_n|^{1/n}}.$$

This bound tends to $4/|a_n|^{1/n}$ and we can check the first few terms to see it is always less than the stated bound. □

The constant in (3) of Lemma 4 is unfortunately not absolutely sharp. Indeed, R. Kershner [6], [7] has shown that the absolutely sharp constant when $n = 2$ can be given in terms of the Fresnel integrals, and is

$$2\sqrt{2} \max_{\theta \in [0, 2\pi]} \int_0^{\sqrt{\pi/2 - \theta}} \cos(x^2 + \theta)\, dx = 3.3643\ldots.$$

When $n$ is greater than two, the problem appears to be more complex. Some tentative numerical experiments have suggested that the polynomials $a_1 x + a_3 x^3$ with the property $a_3/a_1^3 = -0.3547...$ are optimal for maximizing

$$\max_{a,b} \left| \int_a^b e^{i(a_1 x + a_3 x^3)}\, dx \right| a_3^{1/3}. \tag{4}$$

They share the property of having local maxima that take the values $\pm 0.5935\ldots$. These polynomials appear to be nonstandard, so it may be difficult to find the other absolutely sharp constants. On the other hand, as the maximum in (4) appears to be $2.6396\ldots < 4$, it may be possible to show that the asymptote in Corollary 5 is approached from below.

## Acknowledgment

This is part of the author's Ph.D. thesis, written under the supervision of Professor Michael Cowling at the University of New South Wales. Thanks to Michael for all his invaluable help. Thanks to Steve Finch for bringing to my attention the sharp results when $n = 2$. Thanks also to the referee(s) for constructive criticism.

## References

[1] G.I. Arhipov, A.A. Karacuba and V.N. Čubarikov, 'Trigonometric integrals', *Math. USSR Izvestija* **15** (1980), 211–239.

[2] R. G. Bartle, *The elements of real analysis*, second edition, John Wiley & Sons, New York, 1976. MR0393369 (52:14179)

[3] A. Carbery, M. Christ and J. Wright, 'Multidimensional van der Corput and sublevel set estimates', J. Amer. Math. Soc. **12** (1999), no. 4, 981–1015. MR1683156 (2000h:42010)

[4] A. Carbery and J. Wright, 'What is van der Corput's lemma in higher dimensions?', Publ. Mat. **2002**, Vol. Extra, 13–26. MR1964813 (2004a:42016)

[5] J.G. van der Corput, 'Zahlentheoretische Abschätzungen', *Math. Ann.* **84** (1921), 53–79.

[6] R. Kershner, 'Determination of a van der Corput-Landau absolute constant', *Amer. J. Math.* **57** (1935) 840-846.

[7] R. Kershner, 'Determination of a van der Corput absolute constant', *Amer. J. Math.* **60** (1938) 549-554.

[8] E. Isaacson and H. B. Keller, *Analysis of numerical methods*, Wiley, New York, 1966. MR0201039 (34:924)

[9] H. L. Montgomery, *Ten lectures on the interface between analytic number theory and harmonic analysis*, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1994. MR1297543 (96i:11002)

[10] T. J. Rivlin, *Chebyshev polynomials*, second edition, Wiley, New York, 1990. MR1060735 (92a:41016)

[11] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Univ. Press, Princeton, NJ, 1993. MR1232192 (95c:42002)

[12] A. Zygmund, *Trigonometric series. 2nd ed. Vols. I, II*, Cambridge Univ. Press, New York, 1959. MR0107776 (21:6498)

School of Mathematics, University of New South Wales, Sydney, NSW 2052, Australia

*E-mail address*: K.M.Rogers.99@cantab.net

*Current address*: Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain