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ON THE PREDICTABILITY OF DISCRETE DYNAMICAL SYSTEMS II

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Abstract. Given a metrizable compact topological n-manifold X with boundary and a metric d compatible with the topology of X, we prove that "most" continuous functions $f : X \to X$ are non-sensitive at "most" points of X but are sensitive at every point of an infinite set which is dense in the set of all periodic points of f. We also establish some results concerning sets of periodic points and non-wandering points.

1. Introduction

Throughout the present paper we fix an integer $n \geq 1$, a metrizable compact topological n-manifold X with (or without) boundary [6], and a metric d which is compatible with the topology of X. Moreover, $C(X)$ (resp. $H(X)$) denotes the set of all continuous functions from X into X (resp. the set of all homeomorphisms from X onto X) endowed with the supremum metric: $\tilde{d}(f, g) = \sup_{x \in X} d(f(x), g(x))$.

If M is a Baire space, we say that "most elements of M" satisfy a certain property P if the set of all $x \in M$ that do not satisfy property P is of the first category in M. The terms "typical" and "generic" are often used instead of "most".

Let us recall that a function $f : X \to X$ is said to be non-sensitive at a point $a \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that for any choice of points $a_0 \in B(a; \delta)$, $a_1 \in B(f(a_0); \delta)$, $a_2 \in B(f(a_1); \delta), \ldots$, we have that

$$d(a_m, f^m(a)) < \epsilon \quad \text{for every } m \geq 0.$$

The notion of non-sensitivity was first defined in [2] and was redefined in [4]. It is a way to describe mathematically the idea of predictability: if f is non-sensitive at a, then we can predict the future evolution of a in the discrete dynamical system (X, f) forever as accurately as we want, provided we can compute the initial condition a and the values of f precisely enough.

Although the notion of non-sensitivity is very strong, it occurs very often for functions in the space $H(X)$. Indeed, the following results were proved in [4]:

(A) Most functions in $H(X)$ are non-sensitive at most points of X.

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(B) If $n \geq 2$ and B^n is the closed unit ball of $\mathbf{R}^n$, then most functions in $H(B^n)$ are non-sensitive at every point of a full (Lebesgue) measure subset of B^n.

The main goal of the present paper is to study the notion of non-sensitivity for functions in the space $C(X)$. Our main result is Theorem 1, which implies the following versions of (A) and (B) for the space $C(X)$:

(C) Most functions in $C(X)$ are non-sensitive at most points of X.
(D) If μ is a finite positive Borel measure on X, then most functions in $C(X)$ are non-sensitive μ-almost everywhere on X.

We remark that (A) and (B) were proved in [4] by means of independent arguments; in the present context we establish both (C) and (D) from a single more general result. Moreover, (D) can be applied not only to the Lebesgue measure on B^n but to any finite positive Borel measure on an arbitrary X.

We also consider the opposite direction, that is, the problem of existence of points where the functions *are* sensitive. We prove that for most functions $f \in C(X)$, the set of all points where f is sensitive is infinite and is dense in the set of all periodic points of f (Theorem 16 and Corollary 17). The proof of this fact is based on a certain result concerning periodic points (Theorem 5), which also has some other interesting consequences (Corollaries 7, 8 and 9). Corollary 8 generalizes to an arbitrary X results already established by Agronsky, Bruckner and Laczkovich [1] and Simon [7] in the case $X = [0, 1]$ (Remark 10).

We also prove that if μ is a finite positive Borel measure on X, then for most functions $f \in C(X)$, the set Ω_f of all non-wandering points of f has μ-measure zero (Theorem 11). This was established in [1] in the case $X = [0, 1]$ with $\mu =$ Lebesgue measure (Remark 14).

2. On non-sensitivity

For each $x \in X$ and each $r > 0$, let

$$B(x; r) = \{y \in X; d(y, x) < r\} \quad \text{and} \quad B^*(x; r) = \{y \in X; 0 < d(y, x) < r\}.$$

Given $A \subset X$, we denote by $\overline{A}$, $\operatorname{Int} A$, $\operatorname{Bd} A$ and $\operatorname{diam} A$ the closure, the interior, the boundary and the diameter of A in X, respectively. Moreover, for each $\delta > 0$, $N_\delta(A) = \bigcup_{a \in A} B(a; \delta)$. Finally, we denote by $\mathcal{G}_X$ the set of all closed subsets A of X such that $\operatorname{Int} A \neq \emptyset$ and A has a fundamental system of neighborhoods which are homeomorphic to B^n (the closed unit ball of $\mathbf{R}^n$). Note that each point $a \in X$ has a fundamental system of neighborhoods that belong to $\mathcal{G}_X$.

The following extension theorem will be used many times in the sequel:

> *If A is a closed subspace of a compact space Y and Z is a topological space which is homeomorphic to B^n, then every continuous function $\phi : A \to Z$ has a continuous extension $\tilde{\phi} : Y \to Z$.*

Since Z is homeomorphic to the product space $[0, 1]^n$, this result follows by applying Tietze extension theorem to each component function of ϕ (regarded as a function from A into $[0, 1]^n$).

Theorem 1. *Suppose that for each integer $r \geq 1$, it is given a finite collection $\mathcal{C}_r$ of pairwise disjoint sets of $\mathcal{G}_X$ with diameters $< 1/r$. Put $Q_r = \bigcup_{A \in \mathcal{C}_r} A$ $(r \geq 1)$. Then, for most functions $f \in C(X)$, there is a sequence $(t_k)_{k \geq 1}$ of positive integers*

such that $t_k \to \infty$ as $k \to \infty$ and f is non-sensitive at every point of the set

$$\bigcap_{r=1}^{\infty} \bigcup_{k=r}^{\infty} Q_{t_k}.$$

Proof. Given a collection $\mathcal{C}$ of sets, we shall mean by a $\mathcal{C}$-tree a pair (T, φ), where T is a finite rooted tree [5] and φ is a bijective correspondence between the set $V(T)$ of all vertices of T and a collection of pairwise disjoint sets in $\mathcal{C}$. If (T, φ) is a $\mathcal{C}$-tree, we usually omit the correspondence φ and speak just of the $\mathcal{C}$-tree T; moreover, we make no distinction between a vertex of T and its corresponding set of $\mathcal{C}$. If T is a $\mathcal{C}$-tree and $V_1, V_2 \in V(T)$, we write "$V_1 > V_2$" or "$V_2 < V_1$" to mean that V_1 and V_2 are adjacent and that the unique path connecting V_2 to the root of T passes through V_1. Two $\mathcal{C}$-trees T_1 and T_2 are said to be disjoint if $A \cap B = \emptyset$ whenever $A \in V(T_1)$ and $B \in V(T_2)$.

For each integer $k \geq 1$, let $\mathcal{O}_k$ be the set of all $f \in C(X)$ such that, for some integer $t \geq k$, there are finitely many pairwise disjoint $\mathcal{G}_X$-trees $T_1, \ldots, T_s$ so that:

(i) $\operatorname{diam} A < 1/k$ for all $A \in V(T_1) \cup \ldots \cup V(T_s)$.
(ii) $\mathcal{C}_t \subset V(T_1) \cup \ldots \cup V(T_s)$.
(iii) If $A, B \in V(T_i)$ and $A > B$, then $f(B) \subset \operatorname{Int} A$.
(iv) If R_i is the root of T_i, then there is an $S_i \in V(T_i)$ such that $f(R_i) \subset \operatorname{Int} S_i$.

Clearly, each $\mathcal{O}_k$ is open in $C(X)$. Let $f \in \bigcap_{k=1}^{\infty} \mathcal{O}_k$. For each $k \geq 1$, there are an integer $t_k \geq k$ and pairwise disjoint $\mathcal{G}_X$-trees $T_{k,1}, \ldots, T_{k,s_k}$ so that (i) through (iv) hold with t_k in place of t and $T_{k,1}, \ldots, T_{k,s_k}$ in place of $T_1, \ldots, T_s$. For each $k \geq 1$, let $0 < \delta_k < 1/k$ be such that

$$f(N_{\delta_k}(B)) \subset \operatorname{Int} A$$

whenever $A, B \in V(T_{k,i})$ and $A > B$ $(1 \leq i \leq s_k)$, and such that

$$f(N_{\delta_k}(R_{k,i})) \subset \operatorname{Int} S_{k,i},$$

where $R_{k,i}$ and $S_{k,i}$ are related to $T_{k,i}$ $(1 \leq i \leq s_k)$ as R_i and S_i are related to T_i in property (iv). So, if $a \in \bigcup\{A; A \in V(T_{k,i})$ for some $1 \leq i \leq s_k\}$ and if we choose $a_0 \in B(a; \delta_k)$, $a_1 \in B(f(a_0); \delta_k)$, $a_2 \in B(f(a_1); \delta_k), \ldots$, then

$$d(a_m, f^m(a)) < \delta_k + 1/k < 2/k \quad \text{for every } m \geq 0.$$

This implies that f is non-sensitive at every point of $\bigcap_{r=1}^{\infty} \bigcup_{k=r}^{\infty} Q_{t_k}$.

It remains to show that each $\mathcal{O}_k$ is dense in $C(X)$. Fix $k \geq 1$, $f \in C(X)$ and $\epsilon > 0$. Let $0 < \delta < \min\{1/k, \epsilon/2\}$ be such that

$$d(f(x), f(y)) < \epsilon/2 \quad \text{whenever } d(x, y) < \delta \quad (x, y \in X).$$

Choose a positive integer t such that $1/t < \delta$. We shall construct pairwise disjoint $\mathcal{G}_X$-trees $T_1, \ldots, T_s$ satisfying the following properties:

(a) $\operatorname{diam} A < \delta$ for every $A \in V(T_1) \cup \ldots \cup V(T_s)$.
(b) $\mathcal{C}_t \subset V(T_1) \cup \ldots \cup V(T_s)$.
(c) If $A, B \in V(T_i)$ and $A > B$, then $f(B) \cap A \neq \emptyset$.
(d) If R_i is the root of T_i, then there is an $S_i \in V(T_i)$ such that at least one of the following two properties hold:
  (1) $f(R_i) \cap S_i \neq \emptyset$.
  (2) There is a $C_i \subset X$ homeomorphic to B^n with $\operatorname{diam} C_i < \epsilon/2$, $f(R_i) \subset \operatorname{Int} C_i$ and $S_i \subset \operatorname{Int} C_i$.

In order to explain how to construct the $\mathcal{G}_X$-trees $T_1, \dots, T_s$, we shall need a variable $\mathcal{B}$, which will denote the set of all elements of $\mathcal{G}_X$ that have already been used in the construction up to the current step (at the beginning we have $\mathcal{B} = \emptyset$). We begin by choosing an $A_1 \in \mathcal{C}_t$ and by putting it as a vertex of T_1 (note that now $\mathcal{B} = \{A_1\}$). Suppose that in a certain moment T_1 consists of the vertices $A_1 < A_2 < \cdots < A_j$. We look at the set $f(A_j)$. There are three possibilities:

CASE 1: $f(A_j) \cap A \neq \emptyset$ for some $A \in \mathcal{B}$.

In this case we stop the construction of T_1 (for the time being). So, A_j is the root R_1 of T_1 and S_1 may be any set of $\mathcal{B}$ such that $f(R_1) \cap S_1 \neq \emptyset$.

CASE 2: $f(A_j) \cap A = \emptyset$ for every $A \in \mathcal{B}$ and $f(A_j) \cap A \neq \emptyset$ for some $A \in \mathcal{C}_t - \mathcal{B}$.

Let $A_{j+1} \in \mathcal{C}_t - \mathcal{B}$ be such that $f(A_j) \cap A_{j+1} \neq \emptyset$. We put A_{j+1} as a vertex of T_1 adjacent to A_j and satisfying $A_j < A_{j+1}$.

CASE 3: $f(A_j) \cap A = \emptyset$ for every $A \in \mathcal{C}_t \cup \mathcal{B}$.

Choose an $A_{j+1} \in \mathcal{G}_X$ disjoint from each element of $\mathcal{C}_t \cup \mathcal{B}$ such that $\operatorname{diam} A_{j+1} < \delta/j$ and $f(A_j) \cap A_{j+1} \neq \emptyset$. We put A_{j+1} as a vertex of T_1 adjacent to A_j and satisfying $A_j < A_{j+1}$.

If Case 1 never happens, the construction can go on forever. In this case we will stop the construction of T_1 as soon as we obtain an A_m for which there is a $C \subset X$ homeomorphic to B^n with $\operatorname{diam} C < \epsilon/2$, $f(A_m) \subset \operatorname{Int} C$ and $A_k \subset \operatorname{Int} C$ for some $1 \leq k \leq m$ (so A_m will be the root R_1 of T_1 and S_1 may be A_k). We claim that we will obtain such an A_m in a finite number of steps. Indeed, suppose that this is not the case and consider the infinite chain $A_1 < A_2 < A_3 < \cdots$. For each $j \geq 1$, choose an $x_j \in A_j$. Let $a \in X$ be a cluster point of the sequence $(x_j)_{j \geq 1}$ and let C be a neighborhood of a which is homeomorphic to B^n and has diameter $< \epsilon/2$. Since $\mathcal{C}_t$ is finite, there must exist a $p \geq 1$ such that $A_j \notin \mathcal{C}_t$ for all $j \geq p$, and so

$$\operatorname{diam} A_j < \frac{\delta}{j-1} \quad \text{for all } j \geq p.$$

Hence, we can choose a $k \geq 1$ large enough so that $A_k \subset \operatorname{Int} C$. Since $f(A_j) \cap A_{j+1} \neq \emptyset$ for every j, we can also choose an $m \geq k$ large enough so that $f(A_m) \subset \operatorname{Int} C$. This contradicts our assumption and proves our claim.

Now, suppose that we have already constructed $T_1, \dots, T_{i-1}$. If $\mathcal{B} \supset \mathcal{C}_t$, we are done. If this is not the case, we choose an $A'_1 \in \mathcal{C}_t - \mathcal{B}$ and put it as a vertex of T_i. If in a certain moment T_i consists of the vertices $A'_1 < A'_2 < \cdots < A'_j$, we then look at $f(A'_j)$. Cases 2 and 3 are treated as before. However, Case 1 should be divided in two possibilities:

CASE 1a: $f(A'_j) \cap A \neq \emptyset$ for some $A \in V(T_i)$.

We stop the construction of T_i (for the time being). So, A'_j is the root R_i of T_i and S_i may be any set of $V(T_i)$ such that $f(R_i) \cap S_i \neq \emptyset$.

CASE 1b: $f(A'_j) \cap A = \emptyset$ for every $A \in V(T_i)$ and $f(A'_j) \cap A \neq \emptyset$ for some $A \in \mathcal{B} - V(T_i)$.

Let $\tilde{A} \in \mathcal{B} - V(T_i)$ be such that $f(A'_j) \cap \tilde{A} \neq \emptyset$. Then $\tilde{A}$ is a vertex of a previous tree; say $\tilde{A} \in V(T_{i_0})$, where $1 \leq i_0 < i$. In this case we will have no tree T_i for the time being. We will just enlarge T_{i_0} by putting the chain $A'_1 < A'_2 < \cdots < A'_j$ as a new branch of it, satisfying the relation $A'_j < \tilde{A}$.

If Cases 1a and 1b never happen, we will stop the construction of T_i when we obtain an A'_m for which there is a $C' \subset X$ homeomorphic to B^n with $\operatorname{diam} C' < \epsilon/2$, $f(A'_m) \subset \operatorname{Int} C'$ and $A'_k \subset \operatorname{Int} C'$ for some $1 \le k \le m$ (so A'_m will be the root R_i of T_i and S_i may be A'_k).

It is immediate to check that the $\mathcal{G}_X$-trees $T_1, \dots, T_s$ so constructed have all the desired properties.

Let $I = \{i \in \{1, \dots, s\}; f(R_i) \cap S_i \neq \emptyset\}$ and $J = \{1, \dots, s\} - I$. For each $A \in V(T_1) \cup \dots \cup V(T_s)$, choose a neighborhood V_A of A which is homeomorphic to B^n and has diameter $< \delta$. By choosing the V_A's small enough, we may also assume that the family $\{V_A\}_{A \in V(T_1) \cup \dots \cup V(T_s)}$ is pairwise disjoint and that

$$f(V_{R_i}) \subset \operatorname{Int} C_i \quad \text{for every } i \in J.$$

For each $B \in V(T_i) - \{R_i\}$ $(1 \le i \le s)$, let A_B be the unique element of $V(T_i)$ such that $B < A_B$, choose an $a_B \in f(B) \cap A_B$ and let $b_B \in B$ be such that $f(b_B) = a_B$. Let $\varphi_B : V_B \to V_B$ be a continuous function such that $\varphi_B(B) = \{b_B\}$ and $\varphi_B(x) = x$ for all $x \in \operatorname{Bd} V_B$, and define

$$g(x) = f(\varphi_B(x)) \quad \text{for all } x \in V_B.$$

For each $i \in I$, choose an $a_i \in f(R_i) \cap S_i$ and let $b_i \in R_i$ be such that $f(b_i) = a_i$. Let $\varphi_i : V_{R_i} \to V_{R_i}$ be a continuous function such that $\varphi_i(R_i) = \{b_i\}$ and $\varphi_i(x) = x$ for all $x \in \operatorname{Bd} V_{R_i}$, and define

$$g(x) = f(\varphi_i(x)) \quad \text{for all } x \in V_{R_i}.$$

For each $i \in J$, choose an $a_i \in S_i$, define

$$g(x) = a_i \text{ for } x \in R_i \quad \text{and} \quad g(x) = f(x) \text{ for } x \in \operatorname{Bd} V_{R_i},$$

and extend g to map V_{R_i} continuously into C_i. Finally, put

$$g(x) = f(x) \quad \text{for all } x \in X - \bigcup \{V_A; A \in V(T_1) \cup \dots \cup V(T_s)\}.$$

Then $g \in C(X)$, $\tilde{d}(g, f) < \epsilon/2$, $g(B) = \{a_B\} \subset A_B$ for every $B \in V(T_i) - \{R_i\}$ and $g(R_i) = \{a_i\} \subset S_i$ $(1 \le i \le s)$. Now, let $\psi \in C(X)$ be such that $\tilde{d}(\psi, \mathrm{id}_X) < \epsilon/2$ (where id_X is the identity mapping of X), $\psi(a_B) \in \operatorname{Int} A_B$ for every $B \in V(T_i) - \{R_i\}$ and $\psi(a_i) \in \operatorname{Int} S_i$ $(1 \le i \le s)$. Put $h = \psi \circ g \in C(X)$. Then $\tilde{d}(h, g) < \epsilon/2$ (hence $\tilde{d}(h, f) < \epsilon$) and properties (iii) and (iv) hold with h in place of f (and so $h \in \mathcal{O}_k$). This completes the proof. □

Corollary 2. *Most functions in $C(X)$ are non-sensitive at most points of X.*

Proof. Let $z_1, z_2, z_3, \dots$ be a sequence of distinct elements of X which is dense in X. For each $r \ge 1$, let $A_{r,1}, \dots, A_{r,r}$ be pairwise disjoint elements of $\mathcal{G}_X$ with $\operatorname{diam} A_{r,i} < 1/r$ and $z_i \in \operatorname{Int} A_{r,i}$ for each $1 \le i \le r$. Consider the sets $\mathcal{C}_r = \{A_{r,1}, \dots, A_{r,r}\}$ for $r \ge 1$, and apply Theorem 1.

Corollary 3. *If μ is a finite positive Borel measure on X, then most functions in $C(X)$ are non-sensitive μ-almost everywhere on X.*

Proof. We shall see in Lemma 4 below that for each integer $r \ge 1$ there is a finite collection $\mathcal{C}_r$ of pairwise disjoint sets of $\mathcal{G}_X$ such that $\mu(X - \bigcup_{A \in \mathcal{C}_r} A) < 1/r$ and $\operatorname{diam} A < 1/r$ for every $A \in \mathcal{C}_r$. Now, it is enough to apply Theorem 1.

Lemma 4. *Suppose μ is a finite positive Borel measure on X. For every $\delta > 0$, there are pairwise disjoint sets $B_1, \ldots, B_s \in \mathcal{G}_X$ such that $\mu(X - (B_1 \cup \ldots \cup B_s)) < \delta$ and* $\operatorname{diam} B_i < \delta$ *for every $1 \leq i \leq s$.*

Proof. Put $D = \{x \in \mathbf{R}^n; \|x\| < 1\}$ and $H = \{(x_1, \ldots, x_n) \in \mathbf{R}^n; x_n \geq 0\}$. By a box in $\mathbf{R}^n$ we mean a set of the form

$$\{(x_1, \ldots, x_n) \in \mathbf{R}^n; a_i \leq x_i < a_i + \epsilon \text{ for } 1 \leq i \leq n\},$$

where $(a_1, \ldots, a_n) \in \mathbf{R}^n$ and $\epsilon > 0$. A closed box in $\mathbf{R}^n$ is the closure of a box in $\mathbf{R}^n$.

Since X is compact, there are sets $V_1, \ldots, V_\ell, \ldots, V_t \subset X$ such that:

(1) For $1 \leq j \leq \ell$, there is a homeomorphism $\psi_j : V_j \to B^n$ with $\psi_j(\operatorname{Int} V_j) = D$.
(2) For $\ell < j \leq t$, there is a homeomorphism $\psi_j : V_j \to B^n \cap H$ with $\psi_j(\operatorname{Int} V_j) = D \cap H$.
(3) The sets $\psi_1^{-1}(\frac{1}{2}B^n), \ldots, \psi_\ell^{-1}(\frac{1}{2}B^n), \psi_{\ell+1}^{-1}(\frac{1}{2}B^n \cap H), \ldots, \psi_t^{-1}(\frac{1}{2}B^n \cap H)$ cover X.

For each $1 \leq j \leq \ell$ (resp. $\ell < j \leq t$), choose $r_j \in [1/2, 1]$ such that the boundary of the set $A_j = \psi_j^{-1}(r_j B^n)$ (resp. $A_j = \psi_j^{-1}(r_j B^n \cap H)$) has μ-measure zero.

Fix $\eta > 0$. Write $\psi_1(\operatorname{Int} A_1)$ as a countable union of disjoint boxes $C_{1,i}$ ($i \in \mathbf{N}^*$) of diameter $< \eta$. Put $B_{1,i} = \psi_1^{-1}(C_{1,i})$ ($i \in \mathbf{N}^*$); then $\operatorname{Int} A_1 = \bigcup_{i=1}^\infty B_{1,i}$. Now, write $\psi_2((\operatorname{Int} A_2) - A_1)$ as a countable union of disjoint boxes $C_{2,i}$ ($i \in \mathbf{N}^*$) of diameter $< \eta$. Put $B_{2,i} = \psi_2^{-1}(C_{2,i})$ ($i \in \mathbf{N}^*$); then $(\operatorname{Int} A_2) - A_1 = \bigcup_{i=1}^\infty B_{2,i}$. Now, consider $\psi_3((\operatorname{Int} A_3) - (A_1 \cup A_2))$ and continue this process. Since each ψ_j is a uniform homeomorphism, by choosing η small enough we have that each $B_{j,i}$ has diameter $< \delta$. Since $X - \bigcup_{j=1}^t \bigcup_{i=1}^\infty B_{j,i} \subset \operatorname{Bd} A_1 \cup \ldots \cup \operatorname{Bd} A_t$, we may choose a finite number $B_{j_1,i_1}, \ldots, B_{j_s,i_s}$ of the $B_{j,i}$ so that

$$\mu(X - (B_{j_1,i_1} \cup \ldots \cup B_{j_s,i_s})) < \delta.$$

Since each $C_{j,i}$ is the union of an increasing sequence of closed boxes, we may choose for each $1 \leq k \leq s$ a closed box $C_k \subset C_{j_k,i_k}$ so that

$$\mu(X - (\psi_{j_1}^{-1}(C_1) \cup \ldots \cup \psi_{j_s}^{-1}(C_s))) < \delta.$$

Hence, the sets $B_1 = \psi_{j_1}^{-1}(C_1), \ldots, B_s = \psi_{j_s}^{-1}(C_s)$ have all the desired properties.

3. Some results on periodic points

In the sequel, $i(X)$ denotes the interior of the manifold X and $\mathcal{B}_X$ denotes the set of all closed subsets of $i(X)$ which are homeomorphic to B^n. Note that each point $x \in i(X)$ has a fundamental system of neighborhoods that belong to $\mathcal{B}_X$. Given $f \in C(X)$ and $x \in X$, $\operatorname{Orb}(f, x) = \{x, f(x), f^2(x), \ldots\}$ is the orbit of x under f and $F_f = \{y \in X; f(y) = y\}$ is the set of all fixed points of f.

Theorem 5. *For most functions $f \in C(X)$, the set of all periodic points of f is non-empty and the following property holds:*

(P) *For each integer $m \geq 1$, each $x \in F_{f^m}$ and each neighborhood N_x of x in X, there is a $V \in \mathcal{B}_X$ such that $V \subset N_x - \{x\}$, the sets $V, f(V), \ldots, f^{m-1}(V)$ are pairwise disjoint and $f^m(V) \subset \operatorname{Int} V$.*

In order to prove this theorem we shall need the following

Lemma 6. *Suppose $f \in C(X)$, $p \in X$ is a periodic point of f with period k, and m is a multiple of k. Then, for every $\alpha > 0$, there is a function $g \in C(X)$ such that:*

(i) $\tilde{d}(g, f) < \alpha$.
(ii) $g(x) = f(x)$ *if* $x \in X - [B^*(p;\alpha) \cup B^*(f(p);\alpha) \cup \ldots \cup B^*(f^{k-1}(p);\alpha)]$.
(iii) g *has a periodic point with period m in* $i(X) \cap B^*(p;\alpha)$.

Proof. Choose neighborhoods $V_0, V_1, \ldots, V_k$ of $p, f(p), \ldots, f^k(p)$, respectively, which are homeomorphic to B^n, so that $V_0 \subset V_k$, $V_1, \ldots, V_k$ are pairwise disjoint, $V_i \subset B(f^i(p);\alpha/2)$ for $1 \le i \le k$, and $f(V_i) \subset V_{i+1}$ for $0 \le i \le k-1$. Choose distinct points $y_0, \ldots, y_{m-1}, z_1, \ldots, z_m \in i(X)$ so that

$$y_i, y_{i+k}, y_{i+2k}, \ldots, y_{i+(m-k)} \in \operatorname{Int}(V_i) - \{f^i(p)\} \quad (0 \le i \le k-1),$$
$$z_i, z_{i+k}, z_{i+2k}, \ldots, z_{i+(m-k)} \in \operatorname{Int}(V_i) - \{f^i(p)\} \quad (1 \le i \le k-1) \text{ and}$$
$$z_k, z_{2k}, \ldots, z_m \in \operatorname{Int}(V_0) - \{p\}.$$

For each $0 \le i \le k-1$, define

$$h(y_i) = z_{i+1}, h(y_{i+k}) = z_{i+k+1}, \ldots, h(y_{i+(m-k)}) = z_{i+(m-k)+1}$$

and $h(x) = f(x)$ if $x \in (\operatorname{Bd} V_i) \cup \{f^i(p)\}$, and then extend h continuously to map V_i into V_{i+1}. Put $h(x) = f(x)$ for $x \in X - (V_0 \cup \ldots \cup V_{k-1})$. Then $h \in C(X)$, $h(x) = f(x)$ if $x \in [X - (V_0 \cup \ldots \cup V_{k-1})] \cup \operatorname{Orb}(f,p)$, $h(V_i) \subset V_{i+1}$ for $0 \le i \le k-1$, and $h(y_j) = z_{j+1}$ for $0 \le j \le m-1$. Now, let $\psi \in C(X)$ be such that $\psi(z_m) = y_0$, $\psi(z_j) = y_j$ for $1 \le j \le m-1$, $\psi(V_i) \subset V_i$ for $0 \le i \le k-1$, and $\psi(x) = x$ if $x \in [X - (V_0 \cup \ldots \cup V_{k-1})] \cup \operatorname{Orb}(f,p)$. Let $g = h \circ \psi \in C(X)$. Then $g(V_i) \subset V_{i+1}$ $(0 \le i \le k-1)$ and $g(x) = f(x)$ if $x \in [X - (V_0 \cup \ldots \cup V_{k-1})] \cup \operatorname{Orb}(f,p)$, and so (i) and (ii) hold. Moreover, z_m is a periodic point of g with period m and z_m lies in $i(X) \cap B^*(p;\alpha)$. This completes the proof.

Proof of Theorem 5. Let $\mathcal{O}$ be the set of all $f \in C(X)$ for which there are a $V \in \mathcal{B}_X$ and an integer $s \ge 1$ such that $f^s(V) \subset \operatorname{Int} V$. For each integer $m \ge 1$ and each integer $k \ge 1$, let $\mathcal{O}_{m,k}$ be the set of all $f \in C(X)$ such that either $F_{f^m} = \emptyset$ or the following property is satisfied: there are finitely many pairwise disjoint sets $V_1, \ldots, V_r \in \mathcal{B}_X$ such that $V_j, f(V_j), \ldots, f^{m-1}(V_j)$ are pairwise disjoint $(1 \le j \le r)$, $f^m(V_j) \subset \operatorname{Int} V_j$ $(1 \le j \le r)$ and each $x \in F_{f^m}$ has a neighborhood with diameter $< 1/k$ which contains at least two distinct V_j's.

Clearly, the sets $\mathcal{O}$, $\mathcal{O}_{m,k}$ are open, and for each $f \in \mathcal{O} \cap (\bigcap_{m,k} \mathcal{O}_{m,k})$ the set of all periodic points of f is non-empty (by Brouwer's fixed point theorem) and property (P) holds. We have to show that the sets $\mathcal{O}$ and $\mathcal{O}_{m,k}$ are dense in $C(X)$. Fix $f \in C(X)$ and $\epsilon > 0$.

Choose $a \in X$. Let $b \in X$ be a cluster point of the sequence $(f^\ell(a))_{\ell \ge 1}$ and let W be a neighborhood of b in X which is homeomorphic to B^n. Let $\ell_1 \ge 1$ be an integer such that $f^{\ell_1}(a) \in \operatorname{Int} W$ and let $\ell_2 > \ell_1$ be the smallest integer such that $f^{\ell_2}(a) \in \operatorname{Int} W$. Choose a point $y \in i(X) \cap \operatorname{Int} W$ and let $\phi, \psi \in C(X)$ be such that

$$\phi(x) = x \text{ and } \psi(x) = x \text{ if } x \in X - \operatorname{Int} W,$$
$$\phi(W) \subset W,\ \psi(W) \subset W,\ \phi(y) = f^{\ell_1}(a) \text{ and } \psi(f^{\ell_2}(a)) = y.$$

Put $g = \psi \circ f \circ \phi \in C(X)$. Then, y is a periodic point of g with period $s = \ell_2 - \ell_1$. Moreover, by choosing W small enough, we have $\tilde{d}(g, f) < \epsilon/2$. Now, let $U \in \mathcal{B}_X$ be

a neighborhood of y such that the sets $U, g(U), \ldots, g^{s-1}(U)$ are pairwise disjoint. Let $V \in \mathcal{B}_X$ be a neighborhood of y contained in $\operatorname{Int} U$ and let $\varphi \in C(X)$ be such that $\varphi(x) = x$ if $x \in X - U$, $\varphi(U) \subset U$ and $\varphi(V) = \{y\}$. Let $h = g \circ \varphi \in C(X)$. Then, $h^s(V) \subset \operatorname{Int} V$. Moreover, by choosing U small enough, we also have $\tilde{d}(h, g) < \epsilon/2$. This proves that $\mathcal{O}$ is dense in $C(X)$.

We shall now show that $\mathcal{O}_{m,k}$ is dense in $C(X)$. If $F_{f^m} = \emptyset$, then f lies in $\mathcal{O}_{m,k}$. So, assume $F_{f^m} \neq \emptyset$. Choose finitely many open balls $U_1, \ldots, U_t$ in X of radii $< \frac{1}{2k}$ that meet F_{f^m} so that $F_{f^m} \subset U_1 \cup \ldots \cup U_t$. For each $1 \leq j \leq t$, choose an $x_j \in F_{f^m} \cap U_j$. Without loss of generality we may assume that $x_1, \ldots, x_t$ are distinct. Let k_j be the period of x_j $(1 \leq j \leq t)$. Fix $\eta > 0$. Let $0 < \alpha_1 < \eta$ be such that $B(x_1; \alpha_1) \subset U_1$ and the balls

$$B(x_1; \alpha_1), B(f(x_1); \alpha_1), \ldots, B(f^{k_1-1}(x_1); \alpha_1)$$

are pairwise disjoint and do not meet the set $\operatorname{Orb}(f, x_j)$ for every j such that $\operatorname{Orb}(f, x_j) \neq \operatorname{Orb}(f, x_1)$. By applying Lemma 6, we see that there is a $g_1 \in C(X)$ such that $\tilde{d}(g_1, f) < \alpha_1 < \eta$,

$$g_1(x) = f(x) \text{ for every } x \in \operatorname{Orb}(f, x_1) \cup \ldots \cup \operatorname{Orb}(f, x_t),$$

and g_1 has a periodic point $a_1 \in i(X) \cap B^*(x_1; \alpha_1) \subset i(X) \cap U_1$ with period m. Note that $a_1 \notin \operatorname{Orb}(f, x_1) \cup \ldots \cup \operatorname{Orb}(f, x_t)$. Choose $0 < \alpha_2 < \eta$ such that $B(x_1; \alpha_2) \subset U_1$ and the balls

$$B(x_1; \alpha_2), B(g_1(x_1); \alpha_2), \ldots, B(g_1^{k_1-1}(x_1); \alpha_2)$$

are pairwise disjoint and do not meet $\operatorname{Orb}(g_1, a_1)$ or $\operatorname{Orb}(g_1, x_j)$ for every j such that $\operatorname{Orb}(g_1, x_j) \neq \operatorname{Orb}(g_1, x_1)$. It follows from Lemma 6 that there is a $g_2 \in C(X)$ such that $\tilde{d}(g_2, g_1) < \alpha_2 < \eta$,

$$g_2(x) = g_1(x) \text{ for every } x \in \operatorname{Orb}(g_1, a_1) \cup \operatorname{Orb}(g_1, x_1) \cup \ldots \cup \operatorname{Orb}(g_1, x_t),$$

and g_2 has a periodic point $b_1 \in i(X) \cap B^*(x_1; \alpha_2) \subset i(X) \cap U_1$ with period m. Note that $b_1 \notin \operatorname{Orb}(g_1, a_1) \cup \operatorname{Orb}(g_1, x_1) \cup \ldots \cup \operatorname{Orb}(g_1, x_t)$. Now, consider x_2 instead of x_1 and continue this process. We will finally obtain a function g $(= g_{2t})$ which has periodic points $a_j, b_j \in i(X) \cap U_j$ with period m $(1 \leq j \leq t)$ so that the sets

$$\operatorname{Orb}(g, a_1), \operatorname{Orb}(g, b_1), \ldots, \operatorname{Orb}(g, a_t), \operatorname{Orb}(g, b_t)$$

are pairwise disjoint. Moreover, by choosing $\eta > 0$ small enough, we can also guarantee that $\tilde{d}(g, f) < \epsilon/2$ (since $\tilde{d}(g, f) < 2t\eta$) and $F_{g^m} \subset U_1 \cup \ldots \cup U_t$.

Now, choose neighborhoods $Z_1, \ldots, Z_t, W_1, \ldots, W_t \in \mathcal{B}_X$ of $a_1, \ldots, a_t, b_1, \ldots, b_t$, respectively, so that $Z_j \subset U_j$, $W_j \subset U_j$ $(1 \leq j \leq t)$ and the sets

$$g^i(Z_j),\ g^i(W_j) \quad (1 \leq j \leq t, 0 \leq i \leq m-1)$$

are pairwise disjoint. Let $Z'_1, \ldots, Z'_t, W'_1, \ldots, W'_t \in \mathcal{B}_X$ be neighborhoods of $a_1, \ldots, a_t$, $b_1, \ldots, b_t$, respectively, such that $Z'_j \subset \operatorname{Int} Z_j$ and $W'_j \subset \operatorname{Int} W_j$ $(1 \leq j \leq t)$. Let $\Psi \in C(X)$ be such that

$$\Psi(x) = x \text{ if } x \in X - (Z_1 \cup \ldots \cup Z_t \cup W_1 \cup \ldots \cup W_t),$$
$$\Psi(Z_j) \subset Z_j,\ \Psi(W_j) \subset W_j,\ \Psi(Z'_j) = \{a_j\} \text{ and } \Psi(W'_j) = \{b_j\} \quad (1 \leq j \leq t).$$

Put $h = g \circ \Psi \in C(X)$. Then, $Z'_j, h(Z'_j), \ldots, h^{m-1}(Z'_j)$ are pairwise disjoint and $h^m(Z'_j) \subset \operatorname{Int} Z'_j$ $(1 \leq j \leq t)$. The same properties also hold with W'_j in place of Z'_j. Moreover, by choosing $Z_1, \ldots, Z_t, W_1, \ldots, W_t$ small enough we can also guarantee

that $\tilde{d}(h,g) < \epsilon/2$ and $F_{h^m} \subset U_1 \cup \ldots \cup U_t$. Hence, $h \in \mathcal{O}_{m,k}$ and $\tilde{d}(h,f) < \epsilon$, which completes the proof of Theorem 5.

By combining Theorem 5 with Brouwer's fixed point theorem we obtain the following results.

Corollary 7. *For most functions $f \in C(X)$, the set of all periodic points of f is non-empty and has no isolated points.*

Corollary 8. *For most functions $f \in C(X)$, the set of all periodic points of f with period m is dense in the set of all periodic points of f with period q whenever m is a multiple of q.*

Corollary 9. *Suppose X has the fixed point property. Then, for most functions $f \in C(X)$, F_{f^m} is a perfect set, and the set of all periodic points of f with period m is dense in F_{f^m} $(m \geq 1)$. In particular, most functions in $C(X)$ have uncountably many periodic points with period m, for each $m \geq 1$.*

Remark 10. a) Agronsky, Bruckner and Laczkovich [1] proved that for most functions $f \in C([0,1])$, any neighborhood of a periodic point of f contains periodic points of f of arbitrarily large periods. Soon after Simon [7] strengthened this result by obtaining Corollary 8 in the case $X = [0,1]$. In [1] it was also proved that F_{f^m} is a perfect set for most functions $f \in C([0,1])$.

b) For $X = B^n$ with $n \geq 2$, Corollary 9 was obtained by the author in [3].

Recall that a point $x \in X$ is said to be a non-wandering point of a function $f \in C(X)$ if, for every neighborhood U of x, $f^k(U) \cap U \neq \emptyset$ for infinitely many integers $k \geq 1$. We denote by Ω_f the set of all non-wandering points of f.

Theorem 11. *If μ is a finite positive Borel measure on X, then for most functions $f \in C(X)$, $\mu(\Omega_f) = 0$.*

Proof. Let $Z = \{x \in X; \mu(\{x\}) > 0\}$. Since μ is a finite measure, the set Z is countable, and so $X - Z$ is dense in X.

For each $k \geq 1$, let $\mathcal{O}_k$ be the set of all functions $f \in C(X)$ such that there are finitely many pairwise disjoint $\mathcal{G}_X$-trees $T_1, \ldots, T_s$ with the following properties:

(i) If $A, B \in V(T_i)$ and $A > B$, then $f(B) \subset \operatorname{Int} A$.

(ii) If R_i is the root of T_i, then there is an $S_i \in V(T_i)$ such that $f(R_i) \subset \operatorname{Int} S_i$.

(iii) Let $R_i = A_{i,1} > A_{i,2} > \cdots > A_{i,t_i} = S_i$ be the chain of successive elements of $V(T_i)$ connecting R_i to S_i. If

$$Y_1 = \bigcup \{A; A \in V(T_i) - \{A_{i,1}, \ldots, A_{i,t_i}\} \text{ for some } 1 \leq i \leq s\} \quad \text{and}$$

$$Y_2 = \bigcup_{i=1}^{s} [(A_{i,t_i} - f(A_{i,1})) \cup (A_{i,t_i-1} - f^2(A_{i,1})) \cup \ldots \cup (A_{i,1} - f^{t_i}(A_{i,1}))],$$

then $\mu(X - (Y_1 \cup Y_2)) < 1/k$.

Since μ is regular, each $\mathcal{O}_k$ is open in $C(X)$. It is also clear that $\mu(\Omega_f) = 0$ for every $f \in \bigcap_{k=1}^{\infty} \mathcal{O}_k$. Let us show that each $\mathcal{O}_k$ is dense in $C(X)$. Fix $k \geq 1$, $f \in C(X)$ and $\epsilon > 0$. Let $0 < \delta < \min\{1/k, \epsilon/2\}$ be such that $d(f(x), f(y)) < \epsilon/2$ whenever $d(x,y) < \delta$ $(x, y \in X)$. By Lemma 4, there are pairwise disjoint sets $B_1, \ldots, B_r \in \mathcal{G}_X$ such that

$$\mu(X - (B_1 \cup \ldots \cup B_r)) < 1/k \quad \text{and} \quad \operatorname{diam} B_i < \delta \text{ for every } 1 \leq i \leq r.$$

By arguing as in the proof of Theorem 1 (with $\mathcal{C}_t = \{B_1, \ldots, B_r\}$), we can construct pairwise disjoint $\mathcal{G}_X$-trees $T_1, \ldots, T_s$ and a function $h \in C(X)$ so that $\tilde{d}(h, f) < \epsilon$, $\{B_1, \ldots, B_r\} \subset V(T_1) \cup \ldots \cup V(T_s)$ and (i) and (ii) hold with h in place of f. Moreover, by the way the function h is constructed in the proof of Theorem 1, we also have that for each $A \in V(T_1) \cup \ldots \cup V(T_s)$, the set $h(A)$ consists of exactly one point, which we may suppose in $X - Z$ (since $X - Z$ is dense in X), and so $\mu(h(A)) = 0$. Thus, (iii) also holds with h in place of f, which completes the proof.

Let us denote by m_n the n-dimensional Lebesgue measure.

Corollary 12. *For most functions $f \in C(X)$, Ω_f is nowhere dense in X.*

Proof. Let $(O_k)_{k\geq 1}$ be a countable basis for the topology of X consisting of sets O_k for which there is a homeomorphism $h_k : \overline{O_k} \to B^n$ with

$$h_k(i(X) \cap O_k) = \{x \in \mathbf{R}^n; \|x\| < 1\}.$$

For each $k \geq 1$ and each Borel set S of X, define $\mu_k(S) = m_n(h_k(S \cap \overline{O_k}))$. By Theorem 11, for most functions $f \in C(X)$, $\mu_k(\Omega_f) = 0$, and so, $\Omega_f \not\supset O_k$. This proves that $\operatorname{Int} \Omega_f = \emptyset$ for most functions $f \in C(X)$, as desired.

Corollary 13. *For most functions $f \in C(B^n)$, $m_n(\Omega_f) = 0$.*

Remark 14. The case $n = 1$ of Corollary 13 was obtained in [1].

Remark 15. Recall that a point $x \in X$ is said to be a chain recurrent point of a function $f \in C(X)$ if, for every $\epsilon > 0$, there is a finite sequence $x_0, x_1, \ldots, x_k$ with $k \geq 1$, $x_0 = x_k = x$, and $d(f(x_i), x_{i+1}) < \epsilon$ for $0 \leq i < k$. We denote by $CR(f)$ the set of all chain recurrent points of f. Clearly, $CR(f) \supset \Omega_f$. The proof of Theorem 11 actually establishes the following result: *If μ is a finite positive Borel measure on X, then for most functions $f \in C(X)$, $\mu(CR(f)) = 0$.*

4. A further result on non-sensitivity

Theorem 16. *For most functions $f \in C(X)$, the set of all points where f is sensitive is dense in the set of all periodic points of f.*

Proof. Let $f \in C(X)$ be a function such that property (P) of Theorem 5 holds.

Suppose x is a periodic point of f and let N be a neighborhood of x in X. Let m be the period of x and put $g = f^m$. By property (P), there is a $V \in \mathcal{B}_X$ such that $V \subset N$ and $g(V) \subset \operatorname{Int} V$. Consider the two sets $A = V \cap \Omega_g$ and $B = \bigcap_{k=0}^{\infty} g^k(V)$. Clearly, A and B are non-empty closed subsets of $\operatorname{Int} V$. Moreover, B is connected. We claim that A is not connected, and so $A \neq B$. Indeed, let $z \in \operatorname{Int}(V) \cap F_g$. By property (P), there is a $W \in \mathcal{B}_X$ such that $W \subset (\operatorname{Int} V) - \{z\}$ and $g(W) \subset \operatorname{Int} W$. Hence, the sets $(X - W) \cap A$ and $(\operatorname{Int} W) \cap A$ form a separation of A, which proves our claim. Let us prove that $A \subset B$. Suppose that this is not the case and let $y \in A - B$. Then $y \notin g^t(V)$ for a certain $t \geq 1$. Let $\gamma > 0$ be such that $B(y; \gamma) \subset V$ and the distance between y and $g^t(V)$ is greater than γ. Since $y \in \Omega_g$, there is an $s \geq t$ such that $B(y; \gamma) \cap g^s(B(y; \gamma)) \neq \emptyset$. Since $g^s(B(y; \gamma)) \subset g^t(V)$, we have a contradiction. Now, choose $b \in B - A$ and define

$$F = V \cap \bigcap_{r=1}^{\infty} \overline{\bigcup_{k \geq r} (g^k)^{-1}(\{b\})}.$$

Then $g(F) \subset F$, $F \neq \emptyset$ (since $b \in B$) and $b \notin F$ (since $b \notin \Omega_g$). Let $\epsilon > 0$ be the distance between b and F. Choose an $a \in F$ and a $\delta > 0$. There is a $k \geq 1$ such that the intersection $B(a;\delta) \cap (g^k)^{-1}(\{b\})$ is non-empty. If a_0 belongs to this intersection, then $d(a_0, a) < \delta$ and $d(g^k(a_0), g^k(a)) = d(b, g^k(a)) \geq \epsilon$. This shows that the family $\{g^s; s \geq 1\}$ is not equicontinuous at a. Therefore, $\{f^s; s \geq 1\}$ is not equicontinuous at a and, in particular, f is sensitive at a.

By combining Corollary 7 with Theorem 16, we obtain the following

Corollary 17. *Most functions in $C(X)$ are sensitive at infinitely many points of X.*

Remark 18. All the results presented in the present paper remain valid if instead of $C(X)$ we consider the space $CO(X)$ of all continuous functions from X *onto* X endowed with the supremum metric. The proofs are essentially the same. We have just to observe that all small pertubations needed in the proofs can be made by means of continuous onto maps (which follows easily from the extension theorem stated at the beginning of Section 2).

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