Cyclic Vectors in the Fock Space Over the Complex Plane

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Abstract. In this paper, we characterize the cyclic vectors in the Fock space over the complex plane. We prove that a function $f(z)$ is cyclic in the Fock space if and only if $f(z)$ is a nonvanishing function in $L^2_\mu(\mathbb{C})$.

1. Introduction

Let $\mathbb{D}$ be the open unit disk of the complex plane $\mathbb{C}$. We denote the polynomial ring on $\mathbb{C}$ by $\mathbb{C}$, and the space of all entire functions by $\text{Hol}(\mathbb{C})$. Let $X$ be a Banach space of holomorphic functions on a domain $\Omega$ in $\mathbb{C}$. For a subset $E$ of $X$, let $\overline{E}$ be the norm closure of $E$ in $X$. In this paper, a function $f$ in $X$ is said to be cyclic if $f \in \overline{E}$ and $f \in X$. In the classical Hardy space $H^2(\mathbb{D})$, it is well known that a function in $H^2(\mathbb{D})$ is cyclic if and only if it is $H^2(\mathbb{D})$-outer; see [5]. Also in the Bergman space $L^2_a(\mathbb{D})$, it is known that $f$ is cyclic if and only if $f$ is $L^2_a(\mathbb{D})$-outer; see [7].

The Fock space, or the so-called Segal-Bargmann space, is the space of all $\mu$-square-integrable entire functions on $\mathbb{C}$, where

$$d\mu(z) = \exp\left(-\frac{|z|^2}{2}\right) dA(z)/2\pi$$

is the Gaussian measure on $\mathbb{C}$ and $dA$ is the ordinary Lebesgue measure. We denote this space by $L^2_\mu(\mathbb{C})$ or $L^2_\mu(\mathbb{C},\mu)$. Then $L^2_\mu(\mathbb{C})$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{\mathbb{C}} f(z) \overline{g(z)} e^{-|z|^2/2} dA(z)$$

for $f, g \in L^2_\mu(\mathbb{C})$, the reproducing kernel functions are given by $K_\lambda(z) = e^{\lambda \overline{z}}$, $\lambda \in \mathbb{C}$, and the polynomial ring $\mathbb{C}$ is dense in $L^2_\mu(\mathbb{C})$; see [2]. We denote by $k_\lambda$ the normalized reproducing kernel at $\lambda \in \mathbb{C}$, that is, $k_\lambda = \frac{K_\lambda}{\|K_\lambda\|}$. It is significant to know which functions in $L^2_\mu(\mathbb{C})$ are cyclic. Obviously a cyclic vector $f$ is an entire function without zeros in $\mathbb{C}$, so that we can write $f = e^h$ for some $h \in \text{Hol}(\mathbb{C})$.

In this paper, we study for which $h \in \text{Hol}(\mathbb{C})$, $e^h$ is cyclic in $L^2_\mu(\mathbb{C})$. In the Fock space $L^2_\mu(\mathbb{C})$, there is no multiplier of $L^2_\mu(\mathbb{C})$ except constant functions; see [6]. So the usual multiplication operator $M_z$ is not defined on $L^2_\mu(\mathbb{C})$. But $M_z$ is defined...
on the dense subspace $C$ of $L^2_\alpha(\mathbb{C})$. In this meaning, we may say that the definition of the cyclicity in $L^2_\alpha(\mathbb{C})$ can be considered as cyclic vectors for the densely defined operator $M_z$. The following is our main theorem.

**Theorem 1.1.** Let $h \in Hol(\mathbb{C})$. Then the following are equivalent:

(i) $e^h \in L^2_\alpha(\mathbb{C})$.
(ii) $h = \alpha z^2 + \beta z + \gamma$ for $\alpha, \beta, \gamma \in \mathbb{C}$ with $|\alpha| < \frac{1}{4}$.
(iii) $e^h$ is cyclic in $L^2_\alpha(\mathbb{C})$.

It is known that there are nonvanishing functions in $H^2(\mathbb{D})$ and $L^2_\alpha(\mathbb{D})$ which are not cyclic in the respective spaces. But the situation is quite different in the Fock space.

**Corollary 1.2.** Every nonvanishing function in $L^2_\alpha(\mathbb{C})$ is cyclic.

2. PROOF OF THE MAIN THEOREM

The proof of Theorem 1.1 consists of two steps.

**Step 1.** Let $h \in Hol(\mathbb{C})$. Then $e^h \in L^2_\alpha(\mathbb{C})$ if and only if $h(z) = \alpha z^2 + \beta z + \gamma$ with $|\alpha| < \frac{1}{4}$. Moreover if $e^h \in L^2_\alpha(\mathbb{C})$, then $e^h C \subset L^2_\alpha(\mathbb{C})$.

**Proof.** Suppose that $e^h \in L^2_\alpha(\mathbb{C})$. First, we shall prove that $h \in C$ and $\text{deg} h \leq 2$, where $\text{deg} h$ denotes the polynomial degree of $h$. In [1], Chen, Guo, and Hou proved that $\lim_{|\lambda| \to \infty} (f, k_\lambda) = 0$ for every $f \in L^2_\alpha(\mathbb{C})$, so that $\Re h(\lambda) < \frac{|\lambda|^2}{4}$ for every $\lambda \in \mathbb{C}$ with sufficiently large modulus. Then obviously $e^h$ is an entire function of finite class $p = 1$, and the order $\rho$ of $e^h$ is

$$
\rho = \limsup_{r \to \infty} \frac{\log \left[ \max_{|\lambda| = r} \Re h(z) \right]}{\log r} 
\leq \limsup_{r \to \infty} \frac{\log \frac{r^2}{4}}{\log r} = 2.
$$

By [1 Corollary 4.5.11], we get that $h \in C$ and $\text{deg} h \leq 2$.

Now, we show that $e^{\alpha z^2 + \beta z + \gamma} \notin L^2_\alpha(\mathbb{C})$ for $\alpha, \beta, \gamma \in \mathbb{C}$ with $|\alpha| \geq \frac{1}{4}$. It is enough to prove that $e^{(az-b)^2} \notin L^2_\alpha(\mathbb{C})$ for $a, b \in \mathbb{C}$ with $|a| \geq \frac{1}{4}$. Without loss of generality, we may assume that $a \geq \frac{1}{2}$. Write $b = r (\cos \theta + i \sin \theta)$, $r \geq 0$. For $t \in \mathbb{R}$, we have

$$
\left| \left< e^{(az-b)^2}, k_t \right> \right| = \frac{1}{||K_t||} \left| \left< (az-b)^2, K_t \right> \right| = \left| e^{(a^2 - \frac{1}{4})t^2 - 2at \cos \theta} \right|.
$$

If $a > \frac{1}{2}$, then we have $\lim_{t \to -\infty} \left| \left< (az-b)^2, k_t \right> \right| = \infty$. If $a = \frac{1}{2}$, then we have $\lim_{t \to -\infty} \left| \left< (az-b)^2, k_t \right> \right| = \infty$ provided $\cos \theta < 0$, $\lim_{t \to -\infty} \left| \left< (az-b)^2, k_t \right> \right| = \infty$ provided $\cos \theta > 0$, and $\left| \left< (az-b)^2, k_t \right> \right| = 1$ for every $t \in \mathbb{R}$ provided $\cos \theta = 0$.

Since $\lim_{|\lambda| \to \infty} (f, k_\lambda) = 0$ for every $f \in L^2_\alpha(\mathbb{C})$, we get that $e^{(az-b)^2} \notin L^2_\alpha(\mathbb{C})$.

Finally, fix $\alpha, \beta, \gamma$ with $|\alpha| < \frac{1}{4}$ and put $h(z) = \alpha z^2 + \beta z + \gamma$. Let $\epsilon$ be a number satisfying

$$(1) \quad 0 < \epsilon < \frac{1}{2} \left( \frac{1}{4} - |\alpha| \right).$$
For large $|z|$, $|p(z)| \leq e^{c|z|^2}$ and $|\beta z + \gamma| \leq e|z|^2$. Thus

$$\left| p(z)e^{h(z)} \right| \leq e^{c|z|^2}2^{2\Re h(z)} \leq e^{2c|z|^2}e^{2(|\alpha| + r)|z|^2} = e^{2(2e + |\alpha|)|z|^2}.$$ 

Then

$$\left| p(z)e^{h(z)} \right|^2 e^{-|z|^2} \leq e^{-\delta|z|^2}$$

for large $|z|$, where $\delta = \frac{1}{2} - 2(2e + |\alpha|)$. By (1), we have that $\delta > 0$. Therefore there exists a positive constant $C$ such that

$$\int_C \left| p(z)e^{h(z)} \right|^2 e^{-\delta|z|^2} dA(z)/2\pi \leq C \int_C e^{-\delta|z|^2} dA(z)/2\pi.$$

Since the last integral is finite, we get that $e^{h(z)}C \subset L^2_\alpha(\mathbb{C})$. This completes the proof. $\square$

If $f$ is cyclic, then obviously $f$ is a nonvanishing function. So it remains only to prove that $e^{h(z)}$ is cyclic in $L^2_\alpha(\mathbb{C})$ for $h(z) = \alpha z^2 + \beta z + \gamma$ with $|\alpha| < \frac{1}{4}$.

**Step 2.** If $h(z) = \alpha z^2 + \beta z + \gamma$ with $|\alpha| < \frac{1}{4}$, then $e^{h(z)}$ is cyclic in $L^2_\alpha(\mathbb{C})$.

**Proof.** By Step 1, $e^{h(z)}C \subset L^2_\alpha(\mathbb{C})$. Let $N$ be a positive integer satisfying

$$(1 + \frac{1}{N})|\alpha| < \frac{1}{4}.$$

Put $p_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k(\alpha^2 + \beta z + \gamma)^k}{k!}$. Let $l$ be a nonnegative integer. Since

$$|p_n(z)| \leq \sum_{k=0}^{\infty} \frac{\{\frac{1}{N}(|\alpha||z|^2 + |\beta||z| + |\gamma|)\}^k}{k!} = e^{\frac{1}{N}(|\alpha||z|^2 + |\beta||z| + |\gamma|)},$$

we have

$$\left| z^l p_n(z)e^{\alpha z^2 + \beta z + \gamma} - z^l e^{-\frac{1}{N}(\alpha z^2 + \beta z + \gamma)}e^{\alpha z^2 + \beta z + \gamma} \right| \leq |z| \left( e^{(1 + \frac{1}{N})(|\alpha||z|^2 + |\beta||z| + |\gamma|)} + e^{(1 - \frac{1}{N})(|\alpha||z|^2 + |\beta||z| + |\gamma|)} \right) \leq 2|z| e^{(1 + \frac{1}{N})(|\alpha||z|^2 + |\beta||z| + |\gamma|)}.$$

Hence

$$\left| z^l p_n(z)e^{\alpha z^2 + \beta z + \gamma} - z^l e^{-\frac{1}{N}(\alpha z^2 + \beta z + \gamma)}e^{\alpha z^2 + \beta z + \gamma} \right|^2 e^{-\frac{1}{N}2|z|^2} \leq 4|z|^2 e^{2(1 + \frac{1}{N})(|\alpha||z|^2 + 2(1 + \frac{1}{N})|\beta||z| + 2(1 + \frac{1}{N})|\gamma|)}.$$

By (2), we have that $2(1 + \frac{1}{N})|\alpha| - \frac{1}{2} < 0$, so that the last function is integrable with respect to $dA$. Since $p_n(z) \to e^{-\frac{1}{N}(\alpha z^2 + \beta z + \gamma)}$ pointwise on $\mathbb{C}$ as $n \to \infty$, by the Lebesgue dominated convergence theorem,

$$\int_C \left| z^l p_n(z)e^{\alpha z^2 + \beta z + \gamma} - z^l e^{-\frac{1}{N}(\alpha z^2 + \beta z + \gamma)}e^{\alpha z^2 + \beta z + \gamma} \right|^2 e^{-\frac{1}{N}2|z|^2} dA(z)/2\pi \to 0 \text{ as } n \to \infty.$$

Therefore we get

$$e^{(1 + \frac{1}{N})(\alpha z^2 + \beta z + \gamma)}C \subset e^{\alpha z^2 + \beta z + \gamma}C.$$
Applying this method again for \( e^{(1-\frac{1}{N})(\alpha z^2 + \beta z + \gamma)} \), we have

\[
\int_{\mathbb{C}} \left| z^r p_\lambda(z) e^{(1-\frac{1}{N})(\alpha z^2 + \beta z + \gamma)} - z^r e^{-\frac{1}{N}(\alpha z^2 + \beta z + \gamma)} e^{(1-\frac{1}{N})(\alpha z^2 + \beta z + \gamma)} \right|^2 e^{-\frac{|z|^2}{2}} dA(z) \to 0 \text{ as } n \to \infty,
\]

so that

\[
e^{(1-\frac{1}{N})(\alpha z^2 + \beta z + \gamma)} \mathcal{C} \subset e^{(1-\frac{1}{N})(\alpha z^2 + \beta z + \gamma)} \mathcal{C}.
\]

Continuing this way, we get

\[
\mathcal{C} \subset e^{(1-\frac{1}{N})(\alpha z^2 + \beta z + \gamma)} \mathcal{C} \subset \cdots \subset e^{(1-\frac{1}{N})(\alpha z^2 + \beta z + \gamma)} \mathcal{C} \subset e^{\alpha z^2 + \beta z + \gamma} \mathcal{C}.
\]

Thus we get \( e^{\alpha z^2 + \beta z + \gamma} \mathcal{C} = L^2_\alpha(\mathbb{C}) \).

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\section*{References}


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