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$(\mathbb{Z}_2)^k$-ACTIONS WITH $w(F) = 1$

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(Communicated by Paul Goerss)

Dedicated to Professor Zhende Wu on his seventieth birthday

Abstract. Suppose that (Φ, M^n) is a smooth $(\mathbb{Z}_2)^k$-action on a closed smooth n-dimensional manifold such that all Stiefel-Whitney classes of the tangent bundle to each connected component of the fixed point set F vanish in positive dimension. This paper shows that if $\dim M^n > 2^k \dim F$ and each p-dimensional part F^p possesses the linear independence property, then (Φ, M^n) bounds equivariantly, and in particular, $2^k \dim F$ is the best possible upper bound of $\dim M^n$ if (Φ, M^n) is nonbounding.

1. Introduction

Let k be a positive integer. Suppose that $\Phi : (\mathbb{Z}_2)^k \times M^n \longrightarrow M^n$ is a smooth $(\mathbb{Z}_2)^k$-action on a closed smooth n-dimensional manifold. The fixed point set F of (Φ, M^n) consists of a union of closed submanifolds of different dimensions. By $\dim F$ we mean the dimension of the component of F of largest dimension.

In this paper, we are mainly concerned with the case in which all Stiefel-Whitney classes of the tangent bundle to each connected component of the fixed point set F vanish in positive dimension, i.e., $w(F) = 1$, where w denotes the total Stiefel-Whitney class. For the case $k = 1$, it was proved in [L1] that if $\dim M^n > 2 \dim F$, then (Φ, M^n) with $w(F) = 1$ bounds equivariantly. (Related results with $k = 1$ can be found in [CF], [C], and [KS1]). Any involution is equivariantly cobordant to an involution with the property that the p-dimensional part of the fixed set is connected. However, for the case $k > 1$, different components of the p-dimensional part of the fixed set may have different normal representations. This is just the key difficulty for the case $k > 1$. In [L2], a linear independence condition for the fixed point set was introduced. With the help of the condition, the argument can be carried out without the connectedness restriction for the fixed point set, so that we may obtain the result in the general case. Our main result is stated as follows.

Theorem 1.1. *Suppose that (Φ, M^n) is a smooth $(\mathbb{Z}_2)^k$-action on a closed smooth n-dimensional manifold such that each part F^p of the fixed point set F possesses the linear independence property, and $w(F) = 1$. If $\dim M^n > 2^k \dim F$, then (Φ, M^n) bounds equivariantly.*

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Remark. (1) When $k = 1$, as shown in [L1], $2 \dim F$ is the best possible upper bound of $\dim M$ if the involution (Φ, M) with $w(F) = 1$ does not bound. For the general case, Example 1 in Section 3 will show that $2^k \dim F$ is still the best possible upper bound of $\dim M$ if (Φ, M) does not bound.

(2) In Theorem 1.1, the condition that each part F^p of the fixed point set F possesses the linear independence property is necessary. This can be seen from Example 2 in Section 3.

The method used here is the formula given by Kosniowski and Stong [KS2], which we will review in Section 2. The proof of Theorem 1.1 will be finished in Section 3. The case $\dim M = 2^k \dim F$ will be discussed in Section 4.

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2. Kosniowski-Stong formula

Let $G = (\mathbb{Z}_2)^k$, and let $\mathrm{Hom}(G, \mathbb{Z}_2)$ be the set of all homomorphisms $\rho : G \longrightarrow \mathbb{Z}_2 = \{\pm 1\}$, which consists of 2^k distinct homomorphisms. One agrees to let ρ_0 denote the trivial element in $\mathrm{Hom}(G, \mathbb{Z}_2)$, i.e., $\rho_0(g) = 1$ for all $g \in G$. Let $EG \longrightarrow BG$ be the universal principal G-bundle, where $BG = EG/G = (\mathbb{R}P^\infty)^k$ is the classifying space of G. It is well known that

$$H^*(BG; \mathbb{Z}_2) = \mathbb{Z}_2[a_1, ..., a_k]$$

with the a_i one-dimensional generators. In particular, all nonzero elements of $H^1(BG; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^k$ consist of $2^k - 1$ polynomials of degree one in $\mathbb{Z}_2[a_1, ..., a_k]$, i.e.,

$$\begin{gathered} a_1, ..., a_k, \\ a_1 + a_2, ..., a_1 + a_k, a_2 + a_3, ..., a_2 + a_k, ..., \\ \cdots \\ a_1 + \cdots + a_{k-1}, ..., a_2 + a_3 + \cdots + a_k, \\ a_1 + a_2 + \cdots + a_k. \end{gathered}$$

These polynomials of degree one correspond to all nontrivial elements of $\mathrm{Hom}(G, \mathbb{Z}_2)$ (note that actually $H^1(BG; \mathbb{Z}_2) \cong \mathrm{Hom}(G, \mathbb{Z}_2)$), and so for convenience, they are denoted by α_ρ for $\rho \in \mathrm{Hom}(G, \mathbb{Z}_2)$ with $\rho \neq \rho_0$. Also, one agrees to let $\alpha_{\rho_0} = 0$, the zero element of $H^1(BG; \mathbb{Z}_2)$.

Let X be a G-space. Then $X_G := EG \times_G X$—the orbit space of the diagonal action on the product $EG \times X$—is the total space of the bundle $X \longrightarrow X_G \longrightarrow BG$ associated to the universal principal bundle $G \longrightarrow EG \longrightarrow BG$. The space $X_G = EG \times_G X$ is called the *Borel construction* on the G-space X. Applying cohomology with coefficients $\mathbb{Z}_2$ to the Borel construction X_G on the G-space X gives the equivariant cohomology

$$H^*_G(X; \mathbb{Z}_2) := H^*(X_G; \mathbb{Z}_2).$$

Now let (Φ, M^n) be a smooth G-action on a smooth closed manifold with nonempty fixed point set F, and let $\eta_i^{n_i}, i = 1, ..., s$, be vector bundles with G-actions covering the action Φ on M. Then we have equivariant cohomologies $H^*_G(M; \mathbb{Z}_2)$ and $H^*_G(F; \mathbb{Z}_2)$. They are all $H^*(BG; \mathbb{Z}_2)$-modules; in particular, $H^*_G(F; \mathbb{Z}_2)$ is a free $H^*(BG; \mathbb{Z}_2)$-module (see, for example, [AP]).

Let

$$f(\alpha_\rho; x_1, ..., x_n; x_1^1, \ldots, x_{n_1}^1; \ldots; x_1^s, ..., x_{n_s}^s)$$

be a polynomial over $\mathbb{Z}_2$ which is symmetric in each of the sets of variables $\{x_1, ..., x_n\}$ and $\{x_1^i, ..., x_{n_i}^i\}, i = 1, ..., s$. In this polynomial, if we let the j-th Stiefel-Whitney class of M replace the j-th elementary symmetric function in $x_1, ..., x_n, \sigma_j(x)$, and the j_i-th Stiefel-Whitney class of $\eta_i^{n_i}$ replace the j_i-th elementary symmetric function in $\{x_1^i, ..., x_{n_i}^i\}$, then we obtain a class in $H^*(M; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(BG; \mathbb{Z}_2)$, which may be evaluated on the fundamental homology class of M, giving an element

$$f(\alpha_\rho; x_1, ..., x_n; x_1^1, ..., x_{n_1}^1; \ldots; x_1^s, \ldots, x_{n_s}^s)[M]$$

in $H^*(BG; \mathbb{Z}_2) = \mathbb{Z}_2[a_1, ..., a_k]$. On the other hand, let C be a connected component of F with $\dim C = p$. The irreducible real G-representations are all one dimensional and correspond to all elements in $\mathrm{Hom}(G, \mathbb{Z}_2)$. Actually, every irreducible real G-representation has the form $\lambda_\rho : G \times \mathbb{R} \longrightarrow \mathbb{R}$ by $\lambda_\rho(g, x) = \rho(g)x$. λ_{ρ_0} is the trivial representation determined by the trivial element ρ_0. Then, the normal bundle ν_C to C in M decomposes under the action of G into the Whitney sum $\bigoplus_{\rho \neq \rho_0} \nu_{C,\rho}$ of the subbundles $\nu_{C,\rho}$, where each $\nu_{C,\rho}$ is the summand of ν_C on which G acts in the fibers via λ_ρ, and let $\dim \nu_{C,\rho} = q_{C,\rho}$. Here we call $\{q_{C,\rho}\}$ the *normal-dimensional sequence of* C. Similarly, each vector bundle $\eta_i^{n_i}$ restricted to C decomposes under the action of G into the Whitney sum $\bigoplus_\rho \eta_\rho^{n_i}$ of the subbundles $\eta_\rho^{n_i}$, where $\eta_\rho^{n_i}$ is the subbundle on which G acts via ρ, and let $q_\rho^i = \dim \eta_\rho^{n_i}$ so that $n_i = \sum_\rho q_\rho^i$. Now, in the polynomial $f(\alpha_\rho; x_1, \ldots, x_n; x_1^1, ..., x_{n_1}^1; \ldots; x_1^s, ..., x_{n_s}^s)$, we replace $x_1, ..., x_n$ by $z_1, ..., z_p$ and for all $\rho \neq \rho_0$, variables $\alpha_\rho + y_\rho^i, 1 \leq i \leq q_{C,\rho}$. We also replace $x_1^i, ..., x_{n_i}^i$ by the collection, for all ρ, of $\alpha_\rho + v_\rho^{i,j}, 1 \leq j \leq q_\rho^i$. Next, if we replace the j-th elementary symmetric function in

$$\{z_1, ..., z_p\} \text{ by } w_j(C) = w_j(\tau_C),$$
$$\{y_\rho^1, ..., y_\rho^{q_{C,\rho}}\} \text{ by } w_j(\nu_{C,\rho}),$$
$$\{v_\rho^{i,1}, ..., v_\rho^{i,q_\rho^i}\} \text{ by } w_j(\eta_\rho^{n_i}),$$

respectively, then the expression

$$\frac{f(\alpha_\rho; z_1, ..., z_p, \alpha_\rho + y_\rho^1, ..., \alpha_\rho + y_\rho^{q_{C,\rho}}; \alpha_\rho + v_\rho^{i,1}, ..., \alpha_\rho + v_\rho^{i,q_\rho^i})}{\prod_{\rho \neq \rho_0} \prod_{i=1}^{q_{C,\rho}} (\alpha_\rho + y_\rho^i)}$$

is a class in the localization $S^{-1}H_G^*(C; \mathbb{Z}_2)$ of the equivariant cohomology of C (here S is the subset of $H^*(BG; \mathbb{Z}_2)$ generated multiplicatively by nonzero elements in $H^1(BG; \mathbb{Z}_2)$), which may be evaluated on the fundamental homology class of C. This gives an element

$$\frac{f(\alpha_\rho; z_1, ..., z_p, \alpha_\rho + y_\rho^1, ..., \alpha_\rho + y_\rho^{q_{C,\rho}}; \alpha_\rho + v_\rho^{i,1}, ..., \alpha_\rho + v_\rho^{i,q_\rho^i})}{\prod_{\rho \neq \rho_0} \prod_{i=1}^{q_{C,\rho}} (\alpha_\rho + y_\rho^i)}[C]$$

in the quotient field K of $H^*(BG; \mathbb{Z}_2) = \mathbb{Z}_2[a_1, ..., a_k]$.

Kosniowski and Stong [KS2] gave the following formula.

Theorem 2.1 (Kosniowski-Stong)**.** *If $f(\alpha; x; x^i)$ is of degree less than or equal to n, then*

$$f(\alpha; x; x^i)[M] = \sum_C \frac{f(\alpha; z, \alpha + y; \alpha + v)}{\prod(\alpha + y)}[C]$$

in K.

Our interest in this paper is on the $(\mathbb{Z}_2)^k$-actions (Φ, M^n) with $w(C) = 1$ for all connected components C of F. In this case, we may choose $z = 0$ in Theorem 2.1 from the splitting principle. Thus we have

Corollary 2.2. *Suppose that (Φ, M^n) is a $(\mathbb{Z}_2)^k$-action on a smooth closed manifold with $w(C) = 1$ for all fixed connected components C. Then for all $f(\alpha; x)$ of degree less than or equal to n,*

$$f(\alpha; x)[M] = \sum_C \frac{f(\alpha; 0, \alpha + y)}{\prod(\alpha + y)}[C]$$

in K.

Finally, we give the definition of the linear independence property for the p-dimensional part F^p of the fixed point set of a $(\mathbb{Z}_2)^k$-action (Φ, M^n). Generally speaking, all normal-dimensional sequences of components of F^p may not be distinct if F^p is disconnected. However, (Φ, M^n) must be cobordant to a $(\mathbb{Z}_2)^k$-action such that all elements of the normal-dimensional sequence set of its p-dimensional part F'^p are distinct and F'^0 is possibly empty if $p = 0$. In fact, one may form a connected sum for those components in F^p with the same normal-dimensional sequence when $p > 0$, and one may cancel pairs of components with the same normal-dimensional sequence when $p = 0$. This does not change the $(\mathbb{Z}_2)^k$-action (Φ, M^n) up to equivariant cobordism. We say that F^p possesses the *linear independence property* if the following conditions are satisfied:

1) When $p > 0$, the normal-dimensional sequence set

$$\{\{q_{C,\rho}\} | C \text{ is a connected component of } F'^p\}$$

of F'^p possesses the property $(*)$: all monomials $\frac{1}{\prod_{\rho \neq \rho_0} \alpha_\rho^{q_{C,\rho}}}$ are linearly independent in the quotient field of $H^*(BG; \mathbb{Z}_2) = \mathbb{Z}_2[a_1, ..., a_k]$.

2) When $p = 0$, F'^0 is either empty or nonempty, and for the nonempty case, the normal-dimensional sequence set of F'^0 satisfies property $(*)$.

Note: The definition for the linear independence property given here seems to be weaker than that stated in [L2] or [L3]. However, up to equivariant cobordism for any $(\mathbb{Z}_2)^k$-action without bundles of covering action, essentially there is no genuine difference.

3. The proof of Theorem 1.1

Suppose that (Φ, M^n) is a smooth $(\mathbb{Z}_2)^k$-action on a smooth closed manifold such that each p-dimensional part F^p of the fixed point set F possesses the linear independence property, and $w(F) = 1$ and $\dim M > 2^k \dim F$. Since the equivariant cobordism class of any $(\mathbb{Z}_2)^k$-action is determined by its fixed data (see [S]), it suffices to show that each connected component of F with its normal bundle bounds.

First, let us consider all components of dimension greater than zero. Given a p-dimensional part $F^p = \bigsqcup_{j=1}^{\ell} C_j$ with $p > 0$ and each C_j connected, let

$$\bigsqcup_{j=1}^{\ell} \bigoplus_{\rho \neq \rho_0} \nu_{C_j,\rho} \longrightarrow C_j$$

be the normal bundle to F^p in M, and the normal dimensional sequence of each C_j is

$$\{q_{C_j,\rho} | \rho (\neq \rho_0) \in \mathrm{Hom}(G, \mathbb{Z}_2)\}.$$

Without loss of generality, one assumes that all normal-dimensional sequences of F^p are distinct. Now suppose inductively that each connected component of each h-dimensional part F^h with its normal bundle bounds if $h > p$. Given partitions

$$\omega_\rho = (i_1^\rho, ..., i_{r_\rho}^\rho), \quad \rho \neq \rho_0,$$

with $\sum_{\rho \neq \rho_0} |\omega_\rho| = p$, choose

$$f(\alpha; x) = \prod_{\rho \neq \rho_0} \{\sum [\prod_\rho (\alpha_\rho + x_1)]^{i_1^\rho} \cdots [\prod_\rho (\alpha_\rho + x_{r_\rho})]^{i_{r_\rho}^\rho}\}.$$

Since $w(F) = 1$, by Corollary 2.2 one has that

$$f(\alpha; 0, \alpha + y) \quad = \quad \prod_{\rho \neq \rho_0} s_{\omega_\rho}(y) \cdot (\prod_{\rho \neq \rho_0} \alpha_\rho)^s + \text{ terms of higher degree in the } y\text{'s},$$

where $s = \sum_{\rho \neq \rho_0} |\omega_\rho| = p$. Then, by induction and since $\deg f(\alpha; 0, \alpha + y)$ is more than or equal to p in the y's, $\sum_h \frac{f(\alpha,0,\alpha+y)}{\prod(\alpha+y)}[F^h]$ has no contribution for the component F^h with $h \neq p$, i.e.,

$$\sum_{h \neq p} \frac{f(\alpha; 0, \alpha + y)}{\prod(\alpha + y)}[F^h] = 0.$$

On the other hand, one has that the degree of $f(\alpha, x)$ in the x's is

$$\deg f(\alpha; x) = \sum_{\rho \neq \rho_0} |\omega_\rho| + (2^k - 1) \sum_{\rho \neq \rho_0} |\omega_\rho| = 2^k \sum_{\rho \neq \rho_0} |\omega_\rho| = 2^k p \leq 2^k \dim F < n,$$

so by Corollary 2.2,

$$\begin{aligned} 0 &= f(\alpha; x)[M] \\ &= \sum_{j=1}^{\ell} \frac{\prod_{\rho \neq \rho_0} s_{\omega_\rho}(y) \cdot (\prod_{\rho \neq \rho_0} \alpha_\rho)^s + \text{ terms of higher degree in the } y\text{'s}}{\prod_{\rho \neq \rho_0} \prod_{i=1}^{q_{C_j,\rho}} (\alpha_\rho + y_\rho^i)}[C_j] \\ &= (\prod_{\rho \neq \rho_0} \alpha_\rho)^s \sum_{j=1}^{\ell} \frac{\prod_{\rho \neq \rho_0} s_{\omega_\rho}(y)}{\prod_{\rho \neq \rho_0} \alpha_\rho^{q_{C_j,\rho}}}[C_j] \end{aligned}$$

and thus

$$\sum_{j=1}^{\ell} \frac{\prod_{\rho \neq \rho_0} s_{\omega_\rho}(y)[C_j]}{\prod_{\rho \neq \rho_0} \alpha_\rho^{q_{C_j,\rho}}} = 0.$$

Since the F^p possesses the linear independence property, one obtains that for each j,

$$\prod_{\rho \neq \rho_0} s_{\omega_\rho}(y)[C_j] = 0,$$

which means that $\bigoplus_{\rho\neq\rho_0}\nu_{C_j,\rho}\longrightarrow C_j$ bounds. This completes the induction, and shows that all components of dimension greater than zero with their normal bundles bound. As for the 0-dimensional part F^0 of the fixed point set, it is easy to see that either F^0 is empty or the normal-dimensional sequences of all isolated points in F^0 must appear in pairs if F^0 is nonempty. This means that F^0 with its normal bundle cobords away. Thus, (Φ, M^n) bounds equivariantly. This completes the proof of Theorem 1.1. $\square$

Now let us explain the Remarks (1) and (2) in Section 1 by the following two examples.

Example 1. Begin with the involution $(T,\mathbb{R}P^2)$ given by

$$[x_0,x_1,x_2]\longmapsto[-x_0,x_1,x_2],$$

which fixes the disjoint union of a point and a real projective 1-space $\mathbb{R}P^1$. Then the product

$$(T\underbrace{\times\cdots\times}_{\ell}T,\mathbb{R}P^2\underbrace{\times\cdots\times}_{\ell}\mathbb{R}P^2)$$

of ℓ copies of $(T,\mathbb{R}P^2)$ forms a new involution, and its fixed point set is $\bigsqcup_{i=0}^{\ell}\binom{\ell}{i}\mathbb{R}P^1\underbrace{\times\cdots\times}_{i}\mathbb{R}P^1$, where $\mathbb{R}P^1\underbrace{\times\cdots\times}_{i}\mathbb{R}P^1$ means a point if $i=0$. This new involution is cobordant to an involution $(\Phi_1,M_1^{2\ell})$ having fixed set $F=F^\ell\sqcup F^{\ell-1}\sqcup\cdots\sqcup F^0$ with $\dim M^{2\ell}=2\dim F$ and $w(F)=1$, where

$$F^p=\begin{cases}\mathbb{R}P^1\underbrace{\times\cdots\times}_{p}\mathbb{R}P^1 & \text{if } \binom{\ell}{p}\not\equiv 0 \mod 2,\\ \text{empty} & \text{if } \binom{\ell}{p}\equiv 0 \mod 2.\end{cases}$$

Consider $M_1^{2\ell}\times M_1^{2\ell}$ with two involutions $t_1=$ twist and $t_2=\Phi_1\times\Phi_1$. The fixed point set of this $(\mathbb{Z}_2)^2$-action $(\Phi_2,M_2^{4\ell})$ is the fixed point set of Φ_1 in the diagonal copy of $M_1^{2\ell}$ which is $F=F^\ell\sqcup F^{\ell-1}\sqcup\cdots\sqcup F^0$, which has $w(F)=1$ and $\dim M_2^{4\ell}=2^2\dim F$. Squaring this example gives examples for all $(\mathbb{Z}_2)^k$-actions. Actually, if $(\Phi_{k-1},M_{k-1}^{2^{k-1}\ell})$ is a $(\mathbb{Z}_2)^{k-1}$-action fixing $F=F^\ell\sqcup F^{\ell-1}\sqcup\cdots\sqcup F^0$, then the twist and the diagonal $(\mathbb{Z}_2)^{k-1}$-action induced by Φ_{k-1} on $M_{k-1}^{2^{k-1}\ell}\times M_{k-1}^{2^{k-1}\ell}$ produce a $(\mathbb{Z}_2)^k$-action $(\Phi_k,M_k^{2^k\ell})$ whose fixed set is still $F=F^\ell\sqcup F^{\ell-1}\sqcup\cdots\sqcup F^0$, and $\dim M_k^{2^k\ell}=2^k\dim F$. Also, the linear independence for the fixed point set is obvious since F^p is connected for each p. However, $(\Phi_k,M_k^{2^k\ell})$ is nonbounding for every value of $\dim F=\ell$ and every k.

Example 2. Consider the standard $(\mathbb{Z}_2)^2$-action $(\Phi_0,\mathbb{R}P^2)$ given by

$$[x_0,x_1,x_2]\longmapsto[x_0,g_1x_1,g_2x_2],$$

which fixes three isolated points, where $(g_1,g_2)\in(\mathbb{Z}_2)^2$. Then the diagonal action on the product of $2\ell'$ copies of $(\Phi_0,\mathbb{R}P^2)$ is also a $(\mathbb{Z}_2)^2$-action denoted by $(\Phi,M^{4\ell'})$, and the fixed point set of this action is formed by $3^{2\ell'}$ isolated points. Furthermore, by using the construction as in Example 1 to $(\Phi,M^{2\ell'})$, one may obtain a $(\mathbb{Z}_2)^k$-action $(\Psi,M^{2^k\ell'})$, which fixes $3^{2\ell'}$ isolated points. Now, the diagonal action on the product of $(\Psi,M^{2^k\ell'})$ and $(\Phi_k,M_k^{2^k\ell})$ in Example 1 produces a $(\mathbb{Z}_2)^k$-action $(\Phi',M^{2^k(\ell+\ell')})$ fixing the disjoint union F' of $3^{2\ell'}$ copies of $F=F^\ell\sqcup F^{\ell-1}\sqcup\cdots\sqcup F^0$.

However, $(\Phi', M^{2^k(\ell+\ell')})$ never bounds, although $\dim M^{2^k(\ell+\ell')} = 2^k(\ell + \ell') > 2^k \dim F'$ for $\ell' > 0$ and $w(F') = 1$. This is because each p-dimensional part of F' does not satisfy the linear independence property.

4. The case $\dim M = 2^k \dim F$

Suppose that (Φ, M^n) is a $(\mathbb{Z}_2)^k$-action with $w(F) = 1$. Now let us consider the case in which $\dim M^n = 2^k \dim F$.

When $k = 1$, one has

Proposition 4.1. *Let (Φ, M^n) is an involution with $w(F) = 1$. If $\dim M^n = 2\dim F$, then (Φ, M^n) either bounds or is cobordant to $(T \underbrace{\times \cdots \times}_{\dim F} T, \mathbb{R}P^2 \underbrace{\times \cdots \times}_{\dim F} \mathbb{R}P^2)$, where the involution $(T, \mathbb{R}P^2)$ is given by $[x_0, x_1, x_2] \longmapsto [-x_0, x_1, x_2]$.*

Proof. It suffices to show that if (Φ, M^n) is nonbounding, then it is cobordant to

$$(T \underbrace{\times \cdots \times}_{\dim F} T, \mathbb{R}P^2 \underbrace{\times \cdots \times}_{\dim F} \mathbb{R}P^2).$$

Suppose that (Φ, M^n) is nonbounding. Then the $n/2$-dimensional component $F^{n/2}$ with normal bundle $\nu^{n/2}$ must not bound. For any a nondyadic partition $\omega = (j_1, ..., j_t)$ of n, one must have $t \leq n/2$ since each j_α is not of the form $2^s - 1$. Take $f_\omega(x) = \sum x_1(x_1 + 1)^{j_1-1} \cdots x_t(x_t + 1)^{j_t-1}$. One then has from [KS1] that

$$s_{(j_1,\ldots,j_t)}[M^n] = \begin{cases} 0 & \text{if } t \neq n/2, \\ s_{(j_1-1,\ldots,j_{n/2}-1)}(\nu^{n/2})[F^{n/2}] & \text{if } t = n/2. \end{cases}$$

However, $t = n/2$ forces ω to be $(2, \underbrace{\ldots}_{n/2}, 2)$, and so if (Φ, M^n) is nonbounding, then all Stiefel-Whitney numbers of M^n are zero except $s_{(2, \underbrace{\ldots}_{n/2}, 2)}[M^n]$. Thus one must have that M^n is cobordant to $\mathbb{R}P^2 \underbrace{\times \cdots \times}_{n/2} \mathbb{R}P^2$. By [KS1], since the cobordism class of M^n determines that of (Φ, M^n), (Φ, M^n) is cobordant to

$$(T \underbrace{\times \cdots \times}_{\dim F} T, \mathbb{R}P^2 \underbrace{\times \cdots \times}_{\dim F} \mathbb{R}P^2).$$

□

Proposition 4.1 shows that there is only one nonbounding equivariant cobordism class of involutions (Φ, M^n) with $\dim M^n = 2\dim F$ and $w(F) = 1$.

For the case $k > 1$, generally there is no such uniqueness result for nonbounding $(\mathbb{Z}_2)^k$-actions with $w(F) = 1$, and $\dim M = 2^k \dim F$ even if each p-dimensional part of the fixed point set has the linear independence property. (Note: Pergher showed in [P] that if (Φ, M^n) is a $(\mathbb{Z}_2)^k$-action with $\dim M^n = 2^k \dim F$ and F being connected (here F does not necessarily satisfy $w(F) = 1$), then (Φ, M^n) is cobordant to the twist on $F \underbrace{\times \cdots \times}_{2^k} F$. Thus, if F is connected with $w(F) = 1$ and $\dim M = 2^k \dim F$, then (Φ, M^n) bounds equivariantly.) The question seems to be quite complicated. Two kinds of examples are stated as follows.

First, let us look at the $(\Phi_k, M_k^{2^k\ell})$ in Example 1. By applying automorphisms of $(\mathbb{Z}_2)^k$ to Φ_k to switch normal representations around, one may obtain more $(\mathbb{Z}_2)^k$-actions, each of which is not cobordant to $(\Phi_k, M_k^{2^k\ell})$, and the disjoint union of any two such actions also gives the example with the linear independence for the fixed point set. Except for these examples, one may also construct other examples as follows.

Consider the $(\mathbb{Z}_2)^2$-action $(\Psi_0, \mathbb{R}P^4)$ defined by

$$t_1 : [x_0, x_1, x_2, x_3, x_4] \longmapsto [-x_0, -x_1, x_2, x_3, x_4],$$
$$t_2 : [x_0, x_1, x_2, x_3, x_4] \longmapsto [-x_0, x_1, -x_2, x_3, x_4],$$

whose fixed point set F_0 is the disjoint union of three isolated points

$$p_1 = [1, 0, 0, 0, 0], \quad p_2 = [0, 1, 0, 0, 0], \quad p_3 = [0, 0, 1, 0, 0]$$

and a real projective 1-space $\mathbb{R}P^1$. Obviously, the 1-dimensional part $\mathbb{R}P^1$ of F_0 possesses the linear independence property. Now let us show that the 0-dimensional part of F_0 also possesses the linear independence property. Let ρ_1, ρ_2, ρ_3 be three irreducible 1-dimensional real $(\mathbb{Z}_2)^2$-representations defined as follows:

$$\rho_1(t_1) = -1, \rho_1(t_2) = 1; \quad \rho_2(t_1) = 1, \rho_2(t_2) = -1; \quad \rho_3(t_1) = -1, \rho_3(t_2) = -1,$$

where t_1, t_2 are two generators of $(\mathbb{Z}_2)^2$. Then it is easy to see that the normal representations at three isolated points $[1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0]$ are

$$\rho_1\rho_2\rho_3^2, \ \rho_1^2\rho_2\rho_3, \ \rho_1\rho_2^2\rho_3,$$

respectively. Furthermore, one knows that the equivariant Euler classes at three isolated points p_1, p_2, p_3 are

$$a_1a_2(a_1 + a_2)^2, \ a_1^2a_2(a_1 + a_2), \ a_1a_2^2(a_1 + a_2),$$

respectively. Since

$$\frac{1}{a_1a_2(a_1 + a_2)^2}, \ \frac{1}{a_1^2a_2(a_1 + a_2)}, \ \frac{1}{a_1a_2^2(a_1 + a_2)}$$

are linearly independent in the quotient field of $\mathbb{Z}_2[a_1, a_2]$, one obtains that the 0-dimensional part of F_0 also possesses the linear independence property. Thus, $(\Psi_0, \mathbb{R}P^4)$ is a $(\mathbb{Z}_2)^2$-action with the linear independence for the fixed point set. Furthermore, the diagonal action on the product of ℓ copies of $(\Psi_0, \mathbb{R}P^4)$ is also a $(\mathbb{Z}_2)^2$-action $(\Psi_2, N_2^{2^2\ell})$, and the fixed point set F is the product of ℓ copies of F_0, which has $w(F) = 1$ and $\dim N_2^{2^2\ell} = 2^2 \dim F$. In particular, the r-dimensional part F^r of F is a disjoint union

$$\bigsqcup_{r_1+r_2+r_3+r=\ell} \binom{\ell}{r_1, r_2, r_3, r} p_1^{r_1} p_2^{r_2} p_3^{r_3} (\mathbb{R}P^1)^r,$$

where the number $\binom{\ell}{r_1,r_2,r_3,r} = \frac{\ell!}{r_1!r_2!r_3!r!}$ is the multinomial coefficient, and $p_i^{r_i}$ means $p_i \underbrace{\times \cdots \times}_{r_i} p_i$. Thus, $(\Psi_2, N_2^{2^2\ell})$ is actually cobordant to a $(\mathbb{Z}_2)^2$-action such that the r-dimensional part of its fixed point set is a union

$$\bigsqcup_{\substack{r_1+r_2+r_3+r=\ell \\ \binom{\ell}{r_1,r_2,r_3,r} \equiv 1 \mod 2}} p_1^{r_1} p_2^{r_2} p_3^{r_3} (\mathbb{R}P^1)^r. \tag{4.1}$$

With this understood, for a convenience in the following discussion one will regard the r-dimensional part F^r of the fixed point set of $(\Psi_2, N_2^{2^2\ell})$ as being the form (4.1). Also, it is easy to see that all elements of the normal-dimensional sequence set of the form (4.1) are distinct and nonbounding. Specifically, $(\mathbb{R}P^1)^r$, with its nontrivial normal bundle is nonbounding, and the normal representation for $p_1^{r_1}p_2^{r_2}p_3^{r_3}$ is $\rho_1^{r_1+2r_2+r_3}\rho_2^{r_1+r_2+2r_3}\rho_3^{2r_1+r_2+r_3} = (\rho_1\rho_2\rho_3)^{\ell-r}\rho_1^{r_1}\rho_2^{r_2}\rho_3^{r_3}$; these are distinct representations.

Generally, $(\Psi_2, N_2^{2^2\ell})$ is not a $(\mathbb{Z}_2)^2$-action with the linear independence for the fixed point set. For example, taking $\ell = 3$, one has that the 0-dimensional part F^0 of the fixed point set consists of nine isolated points

$$p_1^3, p_2^3, p_3^3, p_1^2p_2, p_1^2p_3, p_1p_2^2, p_2^2p_3, p_1p_3^2, p_2p_3^2.$$

It is easy to see that the equivariant Euler classes of these nine isolated points are

$$a_1^3a_2^3(a_1+a_2)^6, a_1^6a_2^3(a_1+a_2)^3, a_1^3a_2^6(a_1+a_2)^3, a_1^4a_2^3(a_1+a_2)^5, a_1^3a_2^4(a_1+a_2)^5,$$
$$a_1^5a_2^3(a_1+a_2)^4, a_1^5a_2^4(a_1+a_2)^3, a_1^3a_2^5(a_1+a_2)^4, a_1^4a_2^5(a_1+a_2)^3,$$

respectively. Now let us look at four isolated points $p_1^2p_2, p_1^2p_3, p_1p_3^2, p_2p_3^2$. One has that

$$\begin{aligned}
&\frac{1}{a_1^4a_2^3(a_1+a_2)^5} + \frac{1}{a_1^3a_2^4(a_1+a_2)^5} + \frac{1}{a_1^3a_2^5(a_1+a_2)^4} + \frac{1}{a_1^4a_2^5(a_1+a_2)^3} \\
= \quad & \frac{a_1^2a_2^3(a_1+a_2) + a_1^3a_2^2(a_1+a_2) + a_1^3a_2(a_1+a_2)^2 + a_1^2a_2(a_1+a_2)^3}{a_1^6a_2^6(a_1+a_2)^6} \\
= \quad & 0
\end{aligned}$$

in the quotient field of $\mathbb{Z}_2[a_1, a_2]$, so linear independence fails.

Now let us discuss when $(\Psi_2, N_2^{2^2\ell})$ is a $(\mathbb{Z}_2)^2$-action with the linear independence for the fixed point set. Note that $\binom{\ell}{r_1,r_2,r_3,r} = \binom{\ell}{r}\binom{\ell-r}{r_1,r_2,r_3}$, so $\binom{\ell}{r_1,r_2,r_3,r} \equiv 1 \mod 2$ if and only if $\binom{\ell}{r} \equiv 1 \mod 2$ and $\binom{\ell-r}{r_1,r_2,r_3} \equiv 1 \mod 2$.

For the r-dimensional part F^r with $\binom{\ell}{r} \equiv 1 \mod 2$, since each component $p_1^{r_1}p_2^{r_2}p_3^{r_3}(\mathbb{R}P^1)^r$ with $\binom{\ell-r}{r_1,r_2,r_3} \equiv 1 \mod 2$ contains $(\mathbb{R}P^1)^r$ as a factor with the same normal bundle, and since the equivariant Euler class of

$$p_1^{r_1}p_2^{r_2}p_3^{r_3}(\mathbb{R}P^1)^r|_{p_1^{r_1}p_2^{r_2}p_3^{r_3}}$$

restricted to $p_1^{r_1}p_2^{r_2}p_3^{r_3}$ is

$$a_1^{r_1+2r_2+r_3}a_2^{r_1+r_2+2r_3}(a_1+a_2)^{2r_1+r_2+r_3},$$

the linear independence of F^r is equivalent to that of

$$B_r = \{\frac{1}{a_1^{r_1+2r_2+r_3}a_2^{r_1+r_2+2r_3}(a_1+a_2)^{2r_1+r_2+r_3}} | r_1+r_2+r_3 = \ell - r, \binom{\ell-r}{r_1,r_2,r_3} \equiv 1 \mod 2\}$$

in the quotient field of $\mathbb{Z}_2[a_1, a_2]$. Thus one has

Claim 1. *The linear independence of F^r is equivalent to that of B_r in the quotient field of $\mathbb{Z}_2[a_1, a_2]$.*

On the other hand, all elements of B_r are actually all nonzero monomials formed by $\frac{1}{a_1a_2(a_1+a_2)^2}$, $\frac{1}{a_1^2a_2(a_1+a_2)}$, $\frac{1}{a_1a_2^2(a_1+a_2)}$ of the expression of

$$(\frac{1}{a_1a_2(a_1+a_2)^2}+\frac{1}{a_1^2a_2(a_1+a_2)}+\frac{1}{a_1a_2^2(a_1+a_2)})^{\ell-r}$$

over $\mathbb{Z}_2$. Since

$$(\frac{1}{a_1a_2(a_1+a_2)^2}+\frac{1}{a_1^2a_2(a_1+a_2)}+\frac{1}{a_1a_2^2(a_1+a_2)})^{\ell-r} = \frac{(a_1a_2+a_2(a_1+a_2)+a_1(a_1+a_2))^{\ell-r}}{(a_1a_2(a_1+a_2))^{2\ell-2r}},$$

the problem is reduced to determining when all elements of

$$D_r^{\ell-r}=\{a_1^{r_1+r_3}a_2^{r_1+r_2}(a_1+a_2)^{r_2+r_3}|r_1+r_2+r_3=\ell-r,\binom{\ell-r}{r_1,r_2,r_3}\equiv 1 \mod 2\}$$

are linearly independent in $\mathbb{Z}_2[a_1,a_2]$, where $D_r^{\ell-r}$ just consists of all nonzero monomials formed by $a_1a_2, a_1(a_1+a_2)$, and $a_2(a_1+a_2)$ of the expression of $(a_1a_2+a_2(a_1+a_2)+a_1(a_1+a_2))^{\ell-r}$ over $\mathbb{Z}_2$.

Definition. Let $\ell=2^{s_1}+\cdots+2^{s_u}$ with $s_1<\cdots<s_u$ be the 2-adic expansion of ℓ. Then ℓ has the *gap property* if $u=1$ or $u>1$ and $s_{i+1}-s_i>1$ for each $1\le i\le u-1$.

Claim 2. *ℓ has the property that all elements of $D_r^{\ell-r}$ are linearly independent in $\mathbb{Z}_2[a_1,a_2]$ for each $0\le r\le\ell$ with $\binom{\ell}{r}\equiv 1 \mod 2$ if and only if ℓ has the gap property.*

Proof. Suppose ℓ has the gap property. One uses induction on u. If $u=1$, then ℓ is of the form 2^s so

$$\binom{\ell}{r_1,r_2,r_3,r}\equiv 0 \mod 2$$

except that one of r_1,r_2,r_3,r is equal to ℓ. This means that $r=0$ or ℓ. When $r=0$, D_0^ℓ contains three nonzero terms $a_1^\ell a_2^\ell, a_1^\ell(a_1+a_2)^\ell, a_2^\ell(a_1+a_2)^\ell$, which are obviously linearly independent in $\mathbb{Z}_2[a_1,a_2]$. When $r=\ell$, D_ℓ^0 contains only the term 1, which is also linearly independent in $\mathbb{Z}_2[a_1,a_2]$. If $u\le v$, suppose inductively that for each $0\le r\le\ell$ with $\binom{\ell}{r}\equiv 1 \mod 2$, all elements of $D_r^{\ell-r}$ are linearly independent in $\mathbb{Z}_2[a_1,a_2]$. Consider the case $u=v+1$, for $r>0$ with $\binom{\ell}{r}\equiv 1 \mod 2$. Since $\binom{\ell}{r}\equiv 1 \mod 2$, one knows that the 2-adic expansion of r is part of that of ℓ, so $\ell-r$ has the gap property and the number of terms of the 2-adic expansion of $\ell-r$ is at most v. Thus, by induction, if $r>0$ with $\binom{\ell}{r}\equiv 1 \mod 2$, then all elements of $D_r^{\ell-r}$ are linearly independent in $\mathbb{Z}_2[a_1,a_2]$. As for $r=0$, write $\ell=\ell'+2^{s_{v+1}}$. One then has that

$$\begin{aligned}&\{a_1a_2+a_1(a_1+a_2)+a_2(a_1+a_2)\}^\ell\\ =\ &\{a_1a_2+a_1(a_1+a_2)+a_2(a_1+a_2)\}^{\ell'}\\ &\times(a_1^{2^{s_{v+1}}}a_2^{2^{s_{v+1}}}+a_1^{2^{s_{v+1}}}(a_1+a_2)^{2^{s_{v+1}}}+a_2^{2^{s_{v+1}}}(a_1+a_2)^{2^{s_{v+1}}})\end{aligned}$$

so

$$D_0^\ell=\{a_1^{2^{s_{v+1}}}a_2^{2^{s_{v+1}}}\}\times D_0^{\ell'}\sqcup\{a_1^{2^{s_{v+1}}}(a_1+a_2)^{2^{s_{v+1}}}\}\times D_0^{\ell'}\sqcup\{a_2^{2^{s_{v+1}}}(a_1+a_2)^{2^{s_{v+1}}}\}\times D_0^{\ell'}.$$

Let

$$\begin{aligned}0 =& a_1^{2^{s_v+1}} a_2^{2^{s_v+1}} \sum_{X\in D_0^{\ell'}} l_X^{(1)} X + a_1^{2^{s_v+1}} (a_1+a_2)^{2^{s_v+1}} \sum_{X\in D_0^{\ell'}} l_X^{(2)} X \\ &+ a_2^{2^{s_v+1}} (a_1+a_2)^{2^{s_v+1}} \sum_{X\in D_0^{\ell'}} l_X^{(3)} X \\ =& a_1^{2^{s_{v+1}+1}} \sum_{X\in D_0^{\ell'}} l_X^{(2)} X + a_2^{2^{s_{v+1}+1}} \sum_{X\in D_0^{\ell'}} l_X^{(3)} X \\ &+ a_1^{2^{s_v+1}} a_2^{2^{s_v+1}} \sum_{X\in D_0^{\ell'}} (l_X^{(1)} + l_X^{(2)} + l_X^{(3)}) X,\end{aligned}$$

where $l_X^{(1)}, l_X^{(2)}, l_X^{(3)} \in \mathbb{Z}_2$. Since $s_{v+1} - s_i > 1$ for any $i \le v$, this forces

$$\sum_{X\in D_0^{\ell'}} l_X^{(1)} X = 0, \sum_{X\in D_0^{\ell'}} l_X^{(2)} X = 0, \sum_{X\in D_0^{\ell'}} l_X^{(3)} X = 0.$$

By induction, one has that all $l_X^{(1)}, l_X^{(2)}, l_X^{(3)}$ are zero, so all elements of D_0^ℓ are linearly independent in $\mathbb{Z}_2[a_1, a_2]$. This completes the induction.

Now suppose ℓ does not have the gap property. There is at least one i with $s_{i+1} - s_i = 1$. Take $r_0 = \ell - (2^s + 2^{s+1})$ where $s_i = s$, and then $D_{r_0}^{\ell - r_0}$ contains the nine terms of $(a_1a_2 + a_1(a_1+a_2) + a_2(a_1+a_2))^{3\cdot 2^s}$ or the 2^s powers of the elements of D_0^3. Then

$$0 = (a_1^2 a_2^3 (a_1+a_2) + a_1^3 a_2^2 (a_1+a_2) + a_1^3 a_2 (a_1+a_2)^2 + a_1^2 a_2 (a_1+a_2)^3)^{2^s}$$

gives a linear dependence relation, exactly as in the case $\ell = 3$. □

Combining Claims 1 and 2, one has

Fact. *$(\Psi_2, N_2^{2^2\ell})$ is a $(\mathbb{Z}_2)^2$-action with the linear independence for the fixed point set if and only if ℓ has the gap property.*

Now suppose that ℓ has the gap property. Using the construction in Example 1, one can obtain a $(\mathbb{Z}_2)^k$-action $(\Psi_k, N_k^{2^k\ell})$, and the fixed point set F is still the same as that of $(\Psi_2, N_2^{2^2\ell})$. Obviously, $(\Psi_k, N_k^{2^k\ell})$ is nonbounding and is a $(\mathbb{Z}_2)^k$-action with $w(F) = 1$ and $\dim N_k^{2^k\ell} = 2^k \dim F$. The linear independence of each r-dimensional part F^r of the fixed point set of $(\Psi_k, N_k^{2^k\ell})$ follows from the following Lemma 4.1, so $(\Psi_k, N_k^{2^k\ell})$ is also a $(\mathbb{Z}_2)^k$-action with the linear independence for the fixed point set.

Both $(\Phi_k, M_k^{2^k\ell})$ and $(\Psi_k, N_k^{2^k\ell})$ give two different kinds of examples since $M_k^{2^k\ell}$ is never cobordant to $N_k^{2^k\ell}$.

Lemma 4.1. *Suppose that (Φ, M^n) is a $(\mathbb{Z}_2)^k$-action satisfying linear independence for the fixed point set. Then $(\Psi, M^n \times M^n)$, which is the $(\mathbb{Z}_2)^{k+1}$-action formed by the diagonal action $\Phi \times \Phi$ and twist on $M^n \times M^n$, also satisfies the linear independence for its fixed point set.*

Proof. The fixed point set of the $(\mathbb{Z}_2)^{k+1}$-action Ψ on $M^n \times M^n$ is the same as the fixed point set of (Φ, M^n), but the normal representation changes. Along a

fixed connected component C^p the tangent bundle of M^n restricts to a $(\mathbb{Z}_2)^k$-representation

$$\prod_{\rho\in\mathrm{Hom}((\mathbb{Z}_2)^k,\mathbb{Z}_2)} \rho^{q_{C,\rho}}.$$

Note that $\rho_0^{q_{C,\rho_0}}$ corresponds to the tangent bundle of C^p, so $q_{C,\rho_0}=p$. Let Λ and $\bar{\Lambda}$ be two subsets of $\mathrm{Hom}((\mathbb{Z}_2)^{k+1},\mathbb{Z}_2)$ such that both Λ and $\bar{\Lambda}$ are isomorphic to $\mathrm{Hom}((\mathbb{Z}_2)^k,\mathbb{Z}_2)$ as $\mathbb{Z}_2$ vector spaces, and each δ_ρ in Λ is $\rho\in\mathrm{Hom}((\mathbb{Z}_2)^k,\mathbb{Z}_2)$ on $(\mathbb{Z}_2)^k$ and 1 on the new $\mathbb{Z}_2$ generator t_{k+1} and each $\bar{\delta}_\rho$ in $\bar{\Lambda}$ is $\rho\in\mathrm{Hom}((\mathbb{Z}_2)^k,\mathbb{Z}_2)$ on $(\mathbb{Z}_2)^k$ and -1 on the new $\mathbb{Z}_2$ generator t_{k+1}. Then, the tangent bundle of $M^n\times M^n$ along C^p restricts to a $(\mathbb{Z}_2)^{k+1}$-representation

$$\prod_{\delta_\rho\in\Lambda}\delta_\rho^{q_{C,\rho}}\prod_{\bar{\delta}_\rho\in\bar{\Lambda}}\bar{\delta}_\rho^{q_{C,\rho}}.$$

In particular, the normal $(\mathbb{Z}_2)^{k+1}$-representation of C^p in $M^n\times M^n$ is

$$\prod_{\delta_\rho\in\Lambda,\delta_\rho\neq\delta_{\rho_0}}\delta_\rho^{q_{C,\rho}}\prod_{\bar{\delta}_\rho\in\bar{\Lambda}}\bar{\delta}_\rho^{q_{C,\rho}}.$$

As $\mathbb{Z}_2$ vector spaces, $\mathrm{Hom}((\mathbb{Z}_2)^{k+1},\mathbb{Z}_2)\cong H^1(B(\mathbb{Z}_2)^{k+1};\mathbb{Z}_2)$, and $\Lambda\cong\bar{\Lambda}\cong\mathrm{Hom}((\mathbb{Z}_2)^k,\mathbb{Z}_2)$. Without loss of generality, one may assume that each nontrivial element δ_ρ in Λ corresponds to a polynomial of degree one in $a_1,...,a_k$, denoted by $\alpha_\rho\in H^1(B(\mathbb{Z}_2)^{k+1};\mathbb{Z}_2)$, and of course, the trivial element δ_{ρ_0} in Λ corresponds to zero element of $H^1(B(\mathbb{Z}_2)^{k+1};\mathbb{Z}_2)$, where $H^*(B(\mathbb{Z}_2)^{k+1};\mathbb{Z}_2)=\mathbb{Z}_2[a_1,...,a_{k+1}]$. Then one sees that each $\bar{\delta}_\rho$ in $\bar{\Lambda}$ corresponds to the polynomial $\alpha_\rho+a_{k+1}$ in $H^1(B(\mathbb{Z}_2)^{k+1};\mathbb{Z}_2)$. Thus, the equivariant Euler class of the normal $(\mathbb{Z}_2)^{k+1}$-representation of C^p is

$$\begin{aligned}&\prod_{\delta_\rho\in\Lambda,\delta_\rho\neq\delta_{\rho_0}}\alpha_\rho^{q_{C,\rho}}\cdot a_{k+1}^p\cdot\prod_{\delta_\rho\in\Lambda,\delta_\rho\neq\delta_{\rho_0}}(\alpha_\rho+a_{k+1})^{q_{C,\rho}}\\&=a_{k+1}^p\prod_{\delta_\rho\in\Lambda,\delta_\rho\neq\delta_{\rho_0}}[\alpha_\rho(\alpha_\rho+a_{k+1})]^{q_{C,\rho}}.\end{aligned}$$

Now consider the p-dimensional part F^p of the fixed point set of $(\Psi,M^n\times M^n)$. Let

$$\sum_{C^p\subset F^p}\frac{l_C}{a_{k+1}^p\prod_{\delta_\rho\in\Lambda,\delta_\rho\neq\delta_{\rho_0}}[\alpha_\rho(\alpha_\rho+a_{k+1})]^{q_{C,\rho}}}=0,$$

where $l_C\in\mathbb{Z}_2$. Since

$$\begin{aligned}&\sum_{C^p\subset F^p}\frac{l_C}{a_{k+1}^p\prod_{\delta_\rho\in\Lambda,\delta_\rho\neq\delta_{\rho_0}}[\alpha_\rho(\alpha_\rho+a_{k+1})]^{q_{C,\rho}}}\\&=\frac{1}{a_{k+1}^p}\sum_{C^p\subset F^p}\frac{l_C}{\prod_{\delta_\rho\in\Lambda,\delta_\rho\neq\delta_{\rho_0}}[\alpha_\rho(\alpha_\rho+a_{k+1})]^{q_{C,\rho}}},\end{aligned}$$

one has

$$\sum_{C^p\subset F^p}\frac{l_C}{\prod_{\delta_\rho\in\Lambda,\delta_\rho\neq\delta_{\rho_0}}[\alpha_\rho(\alpha_\rho+a_{k+1})]^{q_{C,\rho}}}=0.$$

Since $\prod_{\delta_\rho\in\Lambda,\delta_\rho\neq\delta_{\rho_0}}\alpha_\rho^{q_{C,\rho}}\neq 0$, after reducing modulo a_{k+1} one has that

$$\sum_{C^p\subset F^p}\frac{l_C}{\prod_{\delta_\rho\in\Lambda,\delta_\rho\neq\delta_{\rho_0}}\alpha_\rho^{2q_{C,\rho}}}=(\sum_{C^p\subset F^p}\frac{l_C}{\prod_{\delta_\rho\in\Lambda,\delta_\rho\neq\delta_{\rho_0}}\alpha_\rho^{q_{C,\rho}}})^2=0.$$

Squaring is a monomorphism in the quotient field of $\mathbb{Z}_2[a_1, ..., a_{k+1}]$, and thus one has

$$\sum_{C^p \subset F^p} \frac{l_C}{\prod_{\delta_\rho \in \Lambda, \delta_\rho \neq \delta_{\rho_0}} \alpha_\rho^{q_{C,\rho}}} = 0$$

so $l_C = 0$ for all C^p in F^p, since $\prod_{\delta_\rho \in \Lambda, \delta_\rho \neq \delta_{\rho_0}} \alpha_\rho^{q_{C,\rho}}$ is actually identified with the equivariant Euler class of the normal $(\mathbb{Z}_2)^k$-representation of C^p as a fixed component of (Φ, M^n). □

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