ADDENDUM TO “DENSE SUBSETS OF THE BOUNDARY OF A COXETER SYSTEM”

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ABSTRACT. In this paper, we investigate boundaries of parabolic subgroups of Coxeter groups. Let $(W, S)$ be a Coxeter system and let $T$ be a subset of $S$ such that the parabolic subgroup $W_T$ is infinite. Then we show that if a certain set is quasi-dense in $W$, then $W\partial\Sigma(W_T, T)$ is dense in the boundary $\partial\Sigma(W, S)$ of the Coxeter system $(W, S)$, where $\partial\Sigma(W_T, T)$ is the boundary of $(W_T, T)$.

1. INTRODUCTION AND PRELIMINARIES

The purpose of this paper is to study boundaries of parabolic subgroups of Coxeter groups. In this paper, we use the same notation as [5] and [6]. Every Coxeter system $(W, S)$ determines a Davis-Moussong complex $\Sigma(W, S)$ which is a CAT(0) geodesic space ([2], [3], [4], [7]). If $W$ is infinite, then $\Sigma(W, S)$ is noncompact and $\Sigma(W, S)$ can be compactified by adding its ideal boundary $\partial\Sigma(W, S)$ ([1], [3], [4]). For each subset $T \subset S$, we consider the parabolic subgroup $W_T$ generated by $T$. Then $\Sigma(W_T, T)$ is a subcomplex of $\Sigma(W, S)$ and the boundary $\partial\Sigma(W_T, T)$ of $(W_T, T)$ is a subspace of $\partial\Sigma(W, S)$.

The purpose of this paper is to prove the following theorem.

**Theorem 1.1.** Let $(W, S)$ be a Coxeter system and let $T$ be a subset of $S$ such that $W_T$ is infinite. If the set

$$\bigcup\{W(s) \mid s \in S \text{ such that } o(ss_0) = \infty \text{ and } s_0 t \neq t s_0 \text{ for some } s_0 \in S \setminus T \text{ and } t \in \tilde{T}\}$$

is quasi-dense in $W$ with respect to the word metric, then $W\partial\Sigma(W_T, T)$ is dense in $\partial\Sigma(W, S)$, where $W_T$ is the essential parabolic subgroup of $(W_T, T)$.

Remark. For a Gromov hyperbolic group $G$ and the boundary $\partial G$ of $G$, we can show that $G\alpha$ is dense in $\partial G$ for any $\alpha \in \partial G$ by an easy argument. Hence if $W$ is a hyperbolic Coxeter group, then $W\partial\Sigma(W_T, T)$ is dense in $\partial\Sigma(W, S)$ for any $T \subset S$ such that $W_T$ is infinite.

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As an application of Theorem 1.1 we obtain the following corollary.

**Corollary 1.2.** Let \((W, S)\) be a Coxeter system and let \(T\) be a subset of \(S\) such that \(W_T\) is infinite. Suppose that there exist a maximal spherical subset \(U\) of \(S\) and an element \(s \in S\) such that \(o(su) \geq 3\) for every \(u \in U\) and \(o(su_0) = \infty\) for some \(u_0 \in U\). If

1. \(s \notin T\) and \(u_0 \in \tilde{T}\), or
2. \(u_0 \notin T\) and \(s \in \tilde{T}\),

then \(W \partial \Sigma(W_T, T)\) is dense in \(\partial \Sigma(W, S)\).

Here the following problem is open.

**Problem.** Let \((W, S)\) be a Coxeter system and let \(T\) be a subset of \(S\) such that \(W_T\) is infinite. Is it the case that if \(\partial \Sigma(W_T, T)\) is not \(W\)-invariant, then \(W \partial \Sigma(W_T, T)\) is dense in \(\partial \Sigma(W, S)\)? Particularly, is it the case that if \((W, S)\) is an irreducible Coxeter system, then \(W \partial \Sigma(W_T, T)\) is dense in \(\partial \Sigma(W, S)\) for any subset \(T\) of \(S\) such that \(W_T\) is infinite?

### 2. Proof of the main results

Using some results in \([5]\) and \([6]\), we first prove the following lemma.

**Lemma 2.1.** Let \((W, S)\) be a Coxeter system, let \(T\) be a proper subset of \(S\) such that \(W_T\) is infinite, and let

\[ U = \{s \in S \setminus T \mid W^{(s)}s \cap W_T \text{ is finite}\}. \]

Then \(W_{\tilde{T} \cup U} = W_{\tilde{T}} \times W_U\).

**Proof.** We note that \(S(w) \subset T\) for \(w \in W_T\). Let \(u_0 \in U\) and let \(T(u_0) = \{ t \in T \mid tu_0 \neq ut\}\). We first show that \(W_T \setminus T(u_0)\) is a subgroup of finite index in \(W_T\). Here we note that \(|W_T : W_T \setminus T(u_0)| = |A_{T(u_0)}| \cap W_T|\) by \([5]\) Lemma 2.4. Then

\[
\bigcup_{T' \subset T(u_0)} (W_T)^{T'} = \{ w \in W_T \mid S(w) \subset T(u_0) \}
= \{ w \in W_T \mid T \setminus T(u_0) \subset T \setminus S(w) \}
= A_{T(u_0)} \cap W_T.
\]

We show that \((W_T)^{T'}\) is finite for any \(T' \subset T(u_0)\). Let \(T' \subset T(u_0)\). Since \(tu_0 \neq ut\) for any \(t \in T', (W_T)^{T'}u_0 \subset W^{(u_0)}\) by \([5]\) Lemma 2.7. Hence \((W_T)^{T'} \subset W^{(u_0)}u_0 \cap W_T\), which is finite because \(u_0 \in U\). Thus \((W_T)^{T'}\) is finite for any \(T' \subset T(u_0)\), and \(|W_T : W_T \setminus T(u_0)| = |A_{T(u_0)}| \cap W_T|\) is finite. By \([5]\) Corollary 3.4, \(\tilde{T} \subset T \setminus T(u_0)\). Hence \(T(u_0) \subset T \setminus \tilde{T}\) for any \(u_0 \in U\). Let \(A = \{ t \in T \mid tu_0 \neq ut\} \) for some \(u_0 \in U\). Then \(A = \bigcup_{u_0 \in U} T(u_0) \subset T \setminus \tilde{T}\) and

\[ \tilde{T} \subset T \setminus A = \{ t \in T \mid tu = ut \text{ for every } u \in U\}. \]

Thus \(tu = ut\) for any \(t \in \tilde{T}\) and \(u \in U\). This means that \(W_{\tilde{T} \cup U} = W_{\tilde{T}} \times W_U\). \(\square\)

Using the above lemma, we prove the main results.

**Proof of Theorem 1.1** Suppose that

\[ A := \bigcup \{ W^{(s)} \mid s \in S \text{ such that } o(ss_0) = \infty \text{ and } s_0t \neq ts_0 \}
\text{ for some } s_0 \in S \setminus T \text{ and } t \in \tilde{T}\}

is quasi-dense in \(W\).
We first show that for each \( w \in A \), there exists \( v \in W \) and \( \alpha \in \partial \Sigma(W, T) \) such that \( d(w, \text{Im} \xi_{\alpha}) \leq N \), where \( N \) is the diameter of \( K(W, S) \) in \( \Sigma(W, S) \) and \( \xi_{\alpha} \) is the geodesic ray issuing from 1 such that \( \xi_{\alpha}(\infty) = \alpha \).

Let \( w \in A \). Then \( w \in W^s \), \( o(s_0) = \infty \) and \( s_0 t \neq s_0 \) for some \( s \in S \), \( s_0 \in S \setminus T \) and \( t \in \hat{T} \). By Lemma 2.1, \( W^{(s_0)} \cap W_T \) is infinite. Hence there exists a sequence \( \{x_i\} \subset (W^{(s_0)} \cap W_T)^{-1} \) which converges to some point \( \alpha \in \partial \Sigma(W, T) \). Since \( x_i \in (W^{(s_0)} \cap W_T)^{-1} \), \( (s_0 x_i)^{-1} = x_i^{-1} s_0 \in W^{(s_0)} \). By [6, Lemma 3.3], \( d(w, \text{Im} \xi_{w, s_0}) \leq N \) for any \( i \) because \( w \in W^s \), \( s_0 x_i \in (W^{(s_0)} \cap W_T)^{-1} \) and \( o(s_0) = \infty \). Hence \( d(w, \text{Im} \xi_{w, s_0}) \leq N \).

For each \( \beta \in \partial \Sigma(W, S) \), there exists a sequence \( \{w_i\} \subset A \) which converges to \( \beta \), because \( A \) is quasi-dense in \( W \). By the above argument, there exist sequences \( \{v_i\} \subset W \) and \( \{t_i\} \subset \partial \Sigma(W, T) \) such that \( d(w_i, \text{Im} \xi_{v_i, t_i}) \leq N \) for each \( i \). Then the sequence \( \{w_i t_i\} \) converges to \( \beta \) in \( \partial \Sigma(W, S) \) because \( \{w_i\} \) converges to \( \beta \). Therefore \( W \partial \Sigma(W, T) \) is dense in \( \partial \Sigma(W, S) \).

**Proof of Corollary 1.2.** Suppose that there exist a maximal spherical subset \( U \) of \( S \) and an element \( s \in S \) such that \( o(su) \geq 3 \) for any \( u \in U \) and \( o(su_0) = \infty \) for some \( u_0 \in U \). Then \( W^s \) is quasi-dense in \( W \) by [6, Lemma 2.5].

(1) If \( s \not\in T \) and \( u_0 \in \hat{T} \), then \( W^{(u_0)} \) is quasi-dense in \( W \) because \( W^s u_0 \subset W^{(u_0)} \) by [6, Lemma 2.4]. Hence \( W \partial \Sigma(W, T) \) is dense in \( \partial \Sigma(W, S) \) by Theorem 1.1.

(2) If \( u_0 \not\in T \) and \( s \in \hat{T} \), then by Theorem 1.1 \( W \partial \Sigma(W, T) \) is dense in \( \partial \Sigma(W, S) \), because \( o(su_0) = \infty \), \( u_0 \in S \setminus T \) and \( s \in \hat{T} \).

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**References**


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