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ADDENDUM TO "DENSE SUBSETS OF THE BOUNDARY OF A COXETER SYSTEM"

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ABSTRACT. In this paper, we investigate boundaries of parabolic subgroups of Coxeter groups. Let (W,S) be a Coxeter system and let T be a subset of S such that the parabolic subgroup W_T is infinite. Then we show that if a certain set is quasi-dense in W, then $W\partial\Sigma(W_T,T)$ is dense in the boundary $\partial\Sigma(W,S)$ of the Coxeter system (W,S), where $\partial\Sigma(W_T,T)$ is the boundary of (W_T,T).

1. Introduction and preliminaries

The purpose of this paper is to study boundaries of parabolic subgroups of Coxeter groups. In this paper, we use the same notation as [5] and [6]. Every Coxeter system (W,S) determines a *Davis-Moussong complex* $\Sigma(W,S)$ which is a CAT(0) geodesic space ([2], [3], [4], [7]). If W is infinite, then $\Sigma(W,S)$ is noncompact and $\Sigma(W,S)$ can be compactified by adding its ideal boundary $\partial\Sigma(W,S)$ ([1], [3, §4]). For each subset $T\subset S$, we consider the parabolic subgroup W_T generated by T. Then $\Sigma(W_T,T)$ is a subcomplex of $\Sigma(W,S)$ and the boundary $\partial\Sigma(W_T,T)$ of (W_T,T) is a subspace of $\partial\Sigma(W,S)$.

The purpose of this paper is to prove the following theorem.

Theorem 1.1. *Let (W,S) be a Coxeter system and let T be a subset of S such that W_T is infinite. If the set*

$$\bigcup\{W^{\{s\}} \mid s\in S \text{ such that } o(ss_0)=\infty \text{ and } s_0t\neq ts_0 \text{ for some } s_0\in S\setminus T \text{ and } t\in\tilde{T}\}$$

is quasi-dense in W with respect to the word metric, then $W\partial\Sigma(W_T,T)$ is dense in $\partial\Sigma(W,S)$, where $W_{\tilde{T}}$ is the essential parabolic subgroup of (W_T,T).

Remark. For a Gromov hyperbolic group G and the boundary ∂G of G, we can show that $G\alpha$ is dense in ∂G for any $\alpha\in\partial G$ by an easy argument. Hence if W is a hyperbolic Coxeter group, then $W\partial\Sigma(W_T,T)$ is dense in $\partial\Sigma(W,S)$ for any $T\subset S$ such that W_T is infinite.

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As an application of Theorem 1.1, we obtain the following corollary.

Corollary 1.2. *Let (W,S) be a Coxeter system and let T be a subset of S such that W_T is infinite. Suppose that there exist a maximal spherical subset U of S and an element $s \in S$ such that $o(su) \geq 3$ for every $u \in U$ and $o(su_0) = \infty$ for some $u_0 \in U$. If*

(1) *$s \notin T$ and $u_0 \in \tilde{T}$, or*
(2) *$u_0 \notin T$ and $s \in \tilde{T}$,*

then $W\partial\Sigma(W_T,T)$ is dense in $\partial\Sigma(W,S)$.

Here the following problem is open.

Problem. Let (W,S) be a Coxeter system and let T be a subset of S such that W_T is infinite. Is it the case that if $\partial\Sigma(W_T,T)$ is not W-invariant, then $W\partial\Sigma(W_T,T)$ is dense in $\partial\Sigma(W,S)$? Particularly, is it the case that if (W,S) is an irreducible Coxeter system, then $W\partial\Sigma(W_T,T)$ is dense in $\partial\Sigma(W,S)$ for any subset T of S such that W_T is infinite?

2. Proof of the main results

Using some results in [5] and [6], we first prove the following lemma.

Lemma 2.1. *Let (W,S) be a Coxeter system, let T be a proper subset of S such that W_T is infinite, and let*

$$U = \{s \in S \setminus T \,|\, W^{\{s\}}s \cap W_T \text{ is finite}\}.$$

Then $W_{\tilde{T}\cup U} = W_{\tilde{T}} \times W_U$.

Proof. We note that $S(w) \subset T$ for $w \in W_T$. Let $u_0 \in U$ and let $T(u_0) = \{t \in T \,|\, tu_0 \neq u_0t\}$. We first show that $W_{T\setminus T(u_0)}$ is a subgroup of finite index in W_T. Here we note that $[W_T : W_{T\setminus T(u_0)}] = |A_{T\setminus T(u_0)} \cap W_T|$ by [5, Lemma 2.4]. Then

$$\begin{aligned}\bigcup_{T'\subset T(u_0)} (W_T)^{T'} &= \{w \in W_T \,|\, S(w) \subset T(u_0)\} \\ &= \{w \in W_T \,|\, T \setminus T(u_0) \subset T \setminus S(w)\} \\ &= A_{T\setminus T(u_0)} \cap W_T.\end{aligned}$$

We show that $(W_T)^{T'}$ is finite for any $T' \subset T(u_0)$. Let $T' \subset T(u_0)$. Since $tu_0 \neq u_0t$ for any $t \in T'$, $(W_T)^{T'}u_0 \subset W^{\{u_0\}}$ by [5, Lemma 2.7]. Hence $(W_T)^{T'} \subset W^{\{u_0\}}u_0 \cap W_T$, which is finite because $u_0 \in U$. Thus $(W_T)^{T'}$ is finite for any $T' \subset T(u_0)$, and $[W_T : W_{T\setminus T(u_0)}] = |A_{T\setminus T(u_0)} \cap W_T|$ is finite. By [5, Corollary 3.4], $\tilde{T} \subset T \setminus T(u_0)$. Hence $T(u_0) \subset T \setminus \tilde{T}$ for any $u_0 \in U$. Let $A = \{t \in T \,|\, tu_0 \neq u_0t \text{ for some } u_0 \in U\}$. Then $A = \bigcup_{u_0\in U} T(u_0) \subset T \setminus \tilde{T}$ and

$$\tilde{T} \subset T \setminus A = \{t \in T \,|\, tu = ut \text{ for every } u \in U\}.$$

Thus $tu = ut$ for any $t \in \tilde{T}$ and $u \in U$. This means that $W_{\tilde{T}\cup U} = W_{\tilde{T}} \times W_U$. □

Using the above lemma, we prove the main results.

Proof of Theorem 1.1. Suppose that

$$A := \bigcup\{W^{\{s\}} \,|\, s \in S \text{ such that } o(ss_0) = \infty \text{ and } s_0t \neq ts_0 \text{ for some } s_0 \in S \setminus T \text{ and } t \in \tilde{T}\}$$

is quasi-dense in W.

We first show that for each $w \in A$, there exists $v \in W$ and $\alpha \in \partial\Sigma(W_T, T)$ such that $d(w, \operatorname{Im}\xi_{v\alpha}) \le N$, where N is the diameter of $K(W,S)$ in $\Sigma(W,S)$ and $\xi_{v\alpha}$ is the geodesic ray issuing from 1 such that $\xi_{v\alpha}(\infty) = v\alpha$.

Let $w \in A$. Then $w \in W^{\{s\}}$, $o(ss_0) = \infty$ and $s_0 t \neq t s_0$ for some $s \in S$, $s_0 \in S \setminus T$ and $t \in \tilde{T}$. By Lemma 2.1, $W^{\{s_0\}} s_0 \cap W_T$ is infinite. Hence there exists a sequence $\{x_i\} \subset (W^{\{s_0\}} s_0 \cap W_T)^{-1}$ which converges to some point $\alpha \in \partial\Sigma(W_T, T)$. Since $x_i \in (W^{\{s_0\}} s_0)^{-1}$, $(s_0 x_i)^{-1} = x_i^{-1} s_0 \in W^{\{s_0\}}$. By [6, Lemma 3.3], $d(w, \operatorname{Im}\xi_{ws_0x_i}) \le N$ for any i because $w \in W^{\{s\}}$, $s_0 x_i \in (W^{\{s_0\}})^{-1}$ and $o(ss_0) = \infty$. Hence $d(w, \operatorname{Im}\xi_{ws_0\alpha}) \le N$.

For each $\beta \in \partial\Sigma(W,S)$, there exists a sequence $\{w_i\} \subset A$ which converges to β, because A is quasi-dense in W. By the above argument, there exist sequences $\{v_i\} \subset W$ and $\{\alpha_i\} \subset \partial\Sigma(W_T, T)$ such that $d(w_i, \operatorname{Im}\xi_{v_i\alpha_i}) \le N$ for each i. Then the sequence $\{v_i\alpha_i\}$ converges to β in $\partial\Sigma(W,S)$ because $\{w_i\}$ converges to β. Therefore $W\partial\Sigma(W_T, T)$ is dense in $\partial\Sigma(W,S)$. ☐

Proof of Corollary 1.2. Suppose that there exist a maximal spherical subset U of S and an element $s \in S$ such that $o(su) \ge 3$ for any $u \in U$ and $o(su_0) = \infty$ for some $u_0 \in U$. Then $W^{\{s\}}$ is quasi-dense in W by [6, Lemma 2.5].

(1) If $s \notin T$ and $u_0 \in \tilde{T}$, then $W^{\{u_0\}}$ is quasi-dense in W because $W^{\{s\}} u_0 \subset W^{\{u_0\}}$ by [6, Lemma 2.4]. Hence $W\partial\Sigma(W_T, T)$ is dense in $\partial\Sigma(W,S)$ by Theorem 1.1.

(2) If $u_0 \notin T$ and $s \in \tilde{T}$, then by Theorem 1.1, $W\partial\Sigma(W_T, T)$ is dense in $\partial\Sigma(W,S)$, because $o(su_0) = \infty$, $u_0 \in S \setminus T$ and $s \in \tilde{T}$. ☐

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