ON THE BERGMAN METRIC OF PSEUDOCONVEX DOMAINS IN A COMPLEX PROJECTIVE SPACE

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(Communicated by Mei-Chi Shaw)

Abstract. We prove a localization principle of the Bergman kernel form and metric for \( C^2 \) pseudoconvex domains in the complex projective space. An estimate of the Bergman distance is also given.

1. Introduction

Let \( M \) be a complex \( n \)-dimensional manifold. Let \( \mathcal{H} \) be the Hilbert space of holomorphic \( n \)-forms on \( M \) such that \( \int_M f \wedge \bar{f} < \infty \). Let \( h_0, h_1, \cdots \) be a complete orthonormal basis for \( \mathcal{H} \). Then the \( 2n \)-form defined on \( M \times M \) given by \( K_M = \sum_{j=0}^{\infty} h_j \wedge \bar{h}_j \) is called the Bergman kernel of \( M \). Let \( (z_1, \cdots, z_n) \) be a local coordinate system in \( M \) and let \( K_M(z) = K_M^*(z)dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \) where \( K_M^* \) is a locally defined function. Thus \( \beta_M := \partial \bar{\partial} \log K_M^* \) is a well-defined Hermitian form of bidegree \((1,1)\), whenever \( K_M^* \) is nonzero. We say that \( M \) possesses a Bergman metric iff \( \beta_M \) is everywhere positive definite. The following localization principle for the Bergman kernel (metric) is well known.

Theorem (cf. [17], [9]). Let \( \Omega \subset \subset C^n \) be pseudoconvex. Then for any two neighborhoods \( V \subset \subset U \) of a boundary point \( p \) there is a constant \( C > 0 \) such that

\[
K_{\Omega}(x) \geq C \cdot K_{\Omega \cap U}(x),
\]

\[
\beta_{\Omega}(x; X) \geq C \cdot \beta_{\Omega \cap U}(x; X)
\]

for all \( x \in \Omega \cap V \) and \( X \in T^1_{x,0}(\Omega) \).

It is natural to ask on which pseudoconvex domains in \( \mathbb{P}^n \) the localization principle holds. (Notice that the complement of a hypersurface in \( \mathbb{P}^n \) is pseudoconvex; however, the Bergman kernel form vanishes.) In this direction, Diederich and Ohsawa [11] proved the localization principle for another Bergman kernel (metric), which is induced by square-integrable holomorphic functions with respect to the Fubini-Study metric on any pseudoconvex domain \( \Omega \) in \( \mathbb{P}^n \) such that the interior of its complement is not empty. Such a Bergman metric is not invariant under biholomorphic mappings. In this note, we will show...
Theorem 1. Let $\Omega$ be a pseudoconvex domain with $C^2$ boundary in $\mathbb{P}^n$. Then there is a constant $\gamma > 0$ such that for any two neighborhoods $V \subset \subset U$ of a boundary point $p$, one has

$$|K_\Omega(x)|_{FS} \geq C \cdot \frac{|K_{\Omega \cap U}(x)|_{FS}}{\log \delta_{\Omega}(x)^\gamma},$$

$$\beta_\Omega(x; X) \geq C \cdot \frac{\beta_{\Omega \cap U}(x; X)}{\log \delta_{\Omega}(x)^\gamma},$$

for all $x \in \Omega \cap V$ and $X \in T^1_{\Omega}(\Omega)$. Here $\delta_{\Omega}$ (resp. $|\cdot|_{FS}$) denotes the boundary distance (resp. length) w.r.t. the Fubini-Study metric $ds^2_{FS}$.

As an immediate consequence of Theorem 1 and the Ohsawa-Takegoshi extension theorem [19], we obtain

Corollary. Let $\Omega$ be as above. Then

$$|K_\Omega(x)|_{FS} \geq C \cdot \delta_{\Omega}(x)^{-2} |\log \delta_{\Omega}|^{-\gamma}.$$

We mention that Theorem 1 cannot be generalized to an arbitrary fat pseudoconvex domain in $\mathbb{P}^n$ as the following example shows.

Example. We consider the following domain:

$$\Omega = \{[z_0, z_1, z_2] \in \mathbb{P}^2 : |z_1| < |z_0|\}$$

where $(z_0, z_1, z_2)$ denote homogeneous coordinates in $\mathbb{P}^2$. Note that $\Omega$ is biholomorphic to the product of the unit disc and the complex plane via

$$\zeta_1 = z_1/z_0, \quad \zeta_2 = z_2/z_0.$$ 

Hence $\Omega$ is a fat pseudoconvex domain in $\mathbb{P}^n$, but the Bergman metric is degenerate.

However, the estimate of the Bergman metric in Theorem 1 does not imply the completeness. It was shown in [5] that every hyperconvex manifold is Bergman complete, while every $C^2$ pseudoconvex domain in $\mathbb{P}^n$ is hyperconvex [18]; hence it is Bergman complete. We mention that there is a lot of works concerning the completeness of $\beta_M$ (cf. [4], [5], [6], [12], [13], [14], [15], [17], [20]). On the other hand, Diederich and Ohsawa [10] proved that the Bergman distance for a bounded $C^2$ pseudoconvex domain in $\mathbb{C}^n$ has a lower bound of a constant multiple $\log |\log \delta_{\Omega}|$. This lower bound was improved by Blocki [3] to $|\log \delta_{\Omega}|/\log |\log \delta_{\Omega}|$.

We extend his result to the complex projective space.

Theorem 2. Let $\Omega \subset \mathbb{P}^n$ be a pseudoconvex domain with $C^2$ boundary. Then

$$\text{dist}_{\Omega}(x_0, x) \geq C \frac{|\log \delta_{\Omega}(x)|}{|\log \delta_{\Omega}(x)|},$$

where $\text{dist}_{\Omega}(x_0, x)$ denotes the Bergman distance between $x_0$ and $x$.

2. Proof of Theorem 2

Let $y \in \Omega$ be arbitrary fixed. Take a smooth function $\kappa : \mathbb{R} \to [0, 1]$ such that $\kappa|_{(-\infty, 1/2]} = 1$ and $\kappa|_{[1, \infty)} = 0$. Let $d_{FS}(x, y)$ denote the Fubini-Study distance between two points $x, y$ in $\mathbb{P}^n$. Let $r$ be the injectivity radius of $\mathbb{P}^n$ and $c$ be the upper bound of the sectional curvature. Set

$$\phi_y(x) = \kappa(d_{FS}(x, y)/r_0) \log(d_{FS}(x, y)/r_0) - 1.$$
where $r_0 = \min\{r, \pi/2c\}$. By the Hessian comparison theorem (cf. [12]), there is a positive constant $C_1$ independent of $y$ such that

$$\partial \bar{\partial} \phi_y(x) \geq -C_1 ds^2_{FS}.$$  

From now on we assume $r_0 = 1$ for the sake of simplicity. By [18], there exists a smooth and strictly psh function $\rho : \Omega \to (-1, 0)$ such that $|\rho| \approx \delta_{\Omega}$ for some $0 < \tau < 1$ and

$$\partial \bar{\partial} \rho \geq C_2|\rho|ds^2_{FS}$$

where $C_2$ is a positive constant. Choose a cut-off function $\chi : \mathbb{R} \to [0, 1]$ such that $\chi \equiv 1$ on $[-1/2, 1/2]$ and $\chi \equiv 0$ on $[1, +\infty) \cup (-\infty, -1]$. For any $y \in M$, we set

$$\varphi_y = \chi \left( \log(-\log(-\rho)) - \log(-\log(-\rho(y))) \right) \phi_y.$$  

Following an idea of [10], we will show

**Lemma 3.** There exists a sufficiently large constant $b$ such that for any $y \in \Omega$ satisfying $|\rho(y)| < 2^{-c}$, we have

(i) $\varphi_y - \log(-\varphi_y) - b \log(-\rho)$ is $C^2$ strictly psh on $\Omega \setminus \{y\}$;

(ii) $\varphi_y$ has a logarithmic pole at $y$;

(iii) $\text{supp} \varphi_y \subset \{x \in \Omega : |\rho(x)|^c \leq |\rho(x)| \leq |\rho(y)|^c \}.$

**Proof.** (ii), (iii) are trivial. We only need to show (i). By a straightforward computation, we obtain on $\Omega \setminus \{y\}$,

$$\partial \bar{\partial} \varphi_y = \frac{\partial \phi_y}{\log(-\rho)} \left( \chi''(\cdot) \frac{\partial \log(-\rho) \bar{\partial} \log(-\rho)}{\log(-\rho)} \right.$$  

$$- \chi'(\cdot) \frac{\partial \log(-\rho) \bar{\partial} \log(-\rho)}{\log(-\rho)} + \chi'(\cdot) \bar{\partial} \log(-\rho) \right)$$

$$+ \frac{\chi'(\cdot) \phi_y}{\log(-\rho)} \left( \partial \log(-\rho) \frac{\bar{\partial} \phi_y}{\phi_y} + \frac{\partial \phi_y}{\phi_y} \right) \bar{\partial} \log(-\rho)$$

$$+ \chi(\cdot) \bar{\partial} \phi_y.$$  

By the Cauchy-Schwarz inequality, for any constant $\theta > 0$ one has

$$\pm 2 \Re \left\{ \partial \log(-\rho) \frac{\bar{\partial} \phi_y}{\phi_y} \right\}$$

$$\leq \theta \partial \log(-\rho) \bar{\partial} \log(-\rho) + \theta^{-1} \partial \log(-\phi_y) \bar{\partial} \log(-\phi_y).$$

Since $-\rho(y) < 2^{-c}$, it follows from (iii) that $-\rho(x) < 1/2$ on $\text{supp} \varphi_y$. Hence there is a positive constant $C_3$ depending only on the choice of $\chi$ such that

$$\partial \bar{\partial} \varphi_y \geq -C_3 \frac{\phi_y}{\log(-\rho)} \left\{ \partial \bar{\partial} \left( \log(-\rho) \right) + \theta \partial \log(-\rho) \bar{\partial} \log(-\rho) \right.$$  

$$+ \theta^{-1} \partial \log(-\phi_y) \bar{\partial} \log(-\phi_y) \right\} - C_1 ds^2_{FS}.$$  

Note that

$$\text{supp} \chi'(\cdot) \subset \{x \in \Omega : |\rho(y)|^c \leq |\rho(x)| \leq |\rho(y)|^c \}$$

or $|\rho(y)|^{1/c} \leq |\rho(x)| \leq |\rho(y)|^{1/c}$. This implies

$$|\rho(x) - \rho(y)| \geq \frac{|\rho(x)|}{2}.$$
Let \( \lambda \) where \( |\phi_y| \leq C_4 |\log(-\rho)| \) holds on \( \text{supp} \chi' \). On the other hand, one has
\[
\partial \bar{\partial}(- \log(-\rho)) = \frac{\partial \bar{\partial} \rho}{|\rho|} + \partial \log(-\rho) \bar{\partial} \log(-\rho)
\geq C_2 dS_{FS}^2 + \partial \log(-\rho) \bar{\partial} \log(-\rho),
\]
\[
\partial \bar{\partial}(- \log(-\phi_y)) = \frac{\partial \bar{\partial} \phi_y}{|\phi_y|} + \partial \log(-\phi_y) \bar{\partial} \log(-\phi_y)
\geq -C_1 dS_{FS}^2 + \partial \log(-\phi_y) \bar{\partial} \log(-\phi_y)
\]
since \( \phi_y \leq -1 \). Hence after fixing sufficiently large \( \theta \), we can choose \( b > 0 \) such that \( (i) \) holds. The proof is complete.

Let \( g_\Omega \) be the pluricomplex Green function on \( \Omega \), i.e., \( g_\Omega(x, y) = \sup \{ u(x) \} \) where the supremum is taken over all negative functions \( u \in PSH(\Omega) \) satisfying the property that the function \( u - \log |z| \) is bounded from above in a deleted neighborhood of \( y \) for some holomorphic local coordinates \( z \) centered at \( y \), that is, \( z(y) = 0 \).

**Lemma 4.** There is a constant \( c > 0 \) such that
\[
g_\Omega(x, y) > \frac{c \rho(x)}{\rho(y)} \cdot |\log(-\rho(y))|
\]
for any \( x, y \in \Omega \) with \( |\rho(x)| \leq |\rho(y)|/2 \).

**Proof.** Let \( \epsilon = |\rho(y)|^c \) and set
\[
\lambda_y = \varphi_y - \log(-\phi_y) - b \log(-\rho) + 2b \log \epsilon.
\]
By Lemma 3, \( \lambda_y \) is a negative psh function on \( \Omega_{x, z} = \{ x \in \Omega : |\rho(x)| > \epsilon^2 \} \). Set
\[
\eta_y = \left\{ \begin{array}{ll}
\max\{ \lambda_y, c_y (\rho + \epsilon^2) \}, & |\rho| \leq \epsilon, \\
\lambda_y, & |\rho| > \epsilon,
\end{array} \right.
\]
where
\[
c_y = \frac{1}{\epsilon - \epsilon^2} \cdot \inf_{|\rho(x)| = \epsilon} \lambda_y(x).
\]
Then \( \eta_y \) is a well-defined negative psh function on \( \Omega_{x, z} \) such that
\[
\eta_y(x) \geq c_y \rho(x) \geq C_5 \rho(x) |\log \epsilon|/|\epsilon - \epsilon^2| \geq -C_6
\]
for all \( |\rho(x)| \leq \epsilon^{3/2} \). Here \( C_5, C_6 \) are positive constants independent of \( y \). Set
\[
\tilde{\rho} = -C_6 \frac{\log(-\rho + \epsilon^2) - \log(2\epsilon^2)}{\log(\epsilon^{3/2} + \epsilon^2) - \log(2\epsilon^2)}.
\]
Clearly, \( \tilde{\rho} \) is a psh function on \( \Omega \) such that \( \tilde{\rho} \leq C_7 \) where \( C_7 \) is independent of \( \epsilon \). Note that
\[
\tilde{\rho}(x) = -C_6 \leq \eta_y(x), \quad \text{if} \quad |\rho(x)| = \epsilon^{3/2},
\]
\[
\tilde{\rho}(x) = 0 = \eta_y(x), \quad \text{if} \quad |\rho(x)| = \epsilon^2.
\]
Hence the function defined by
\[
\mu_y = \begin{cases} 
\eta_y, & \text{on } \{|\rho| \geq e^{3/2}\}, \\
\max\{\eta_y, \tilde{\rho}\}, & \text{on } \{e^2 \leq |\rho| \leq e^{3/2}\}, \\
\tilde{\rho}, & \text{on } \{|\rho| < e^2\}
\end{cases}
\]
is well-defined, psh on \(\Omega\) and has a logarithmic pole at \(y\). Set
\[
\nu_y = \begin{cases} 
\max\{\mu_y - C_7, \tilde{c}_y \rho\}, & \rho(x) \geq \frac{1}{2} \rho(y), \\
\mu_y - C_7, & \rho(x) < \frac{1}{2} \rho(y),
\end{cases}
\]
where
\[
\tilde{c}_y = \frac{2}{\rho(y)} \inf_{\rho(x) = \frac{1}{2} \rho(y)} (\mu_y(x) - C_7).
\]
If \(\rho(x) \geq \frac{1}{2} \rho(y)\), then we have
\[
\frac{1}{2} |\rho(x)| \leq |\rho(x) - \rho(y)| \leq c' \cdot d_{FS}(x, y)^7
\]
for a suitable constant \(c' > 0\), which implies
\[
\tilde{c}_y \leq C_8 \frac{|\log(-\rho(y))|}{|\rho(y)|}.
\]
Hence
\[
g_\Omega(x, y) \geq \tilde{c}_y \rho(x) \geq -C_8 \frac{\rho(x)}{\rho(y)} |\log(-\rho(y))|.
\]

The following is the key step in proving Theorem 2. The main idea comes from [3].

**Proposition 5.** There is a constant \(C > 0\) such that for any \(y \in \Omega\) with \(|\rho(y)| < e^{-1}\) one has
\[
\{x \in \Omega : g_\Omega(x, y) < -1\} \subset \{x \in \Omega : C^{-1}|\rho(y)| \cdot |\log(-\rho(y))|^{-1} \leq |\rho(x)| \leq C|\rho(y)| \cdot |\log(-\rho(y))|^n\}.
\]

**Proof.** From [2], we know that for any nonnegative psh functions \(u, v\) defined on a smooth bounded domain \(D\) in a Stein manifold with \(u|_{\partial D} = 0\), then
\[
\int_D |u^n((d^c v)^n) \leq n! \|v\|_\infty^{-1} \int_D |v|^n d^c u^n
\]
where \(d^c = i(\bar{\partial} - \partial)\). Fix arbitrary \(x, y \in \Omega\) with \(\rho(x) \leq 2\rho(y)\). Set \(\epsilon = |\rho(y)|\) and \(\alpha = -\frac{8}{7}(b + 1) \log \epsilon\). We exhaust \(\Omega\) by a sequence of smooth strongly pseudoconvex domains \(\Omega_j, j = 1, 2, \cdots\). By the above inequality, we have
\[
\int_{\Omega_j} |g_{\Omega_j}(\cdot, y)|^n (d^c \max\{g_{\Omega_j}(\cdot, x), -\alpha\})^n \\
\leq n! \alpha^{n-1} \int_{\Omega_j} |\max\{g_{\Omega_j}(\cdot, x), -\alpha\}| (d^c g_{\Omega_j}(\cdot, y))^n \\
\leq n!(2\pi)^n \alpha^{n-1} |g_{\Omega_j}(y, x)|
\]
since \((d^c g_{\Omega_j}(\cdot, y))^n = (2\pi)^n \delta_y\) (cf. [8]). It is also known from [8] that the measure \((d^c \max\{g_{\Omega_j}(\cdot, x), -\alpha\})^n)\) is supported on \(\{g_{\Omega_j}(\cdot, x) = -\alpha\}\) with total mass \((2\pi)^n\).
Hence
\[
\inf_{\{g_\Omega(\cdot, x) = -\alpha\}} |g_\Omega(\cdot, y)|^n \leftarrow \inf_{\{g_\Omega(\cdot, x) = -\alpha\}} |g_\Omega(\cdot, y)|^n
\leq n!\alpha^{n-1}|g_\Omega(y, x)|
\rightarrow n!\alpha^{n-1}|g_\Omega(y, x)|
\]

(1)
as \(j \to \infty\). According to Lemma 4, one has \(g_\Omega(z, x) > -1\) provided \(|\rho(z)| \leq |\rho(x)|^\epsilon\).

On the other hand, for any \(z, x, y \in \Omega\) with \(|\rho(z)| > |\rho(x)|^\epsilon\) one has
\[
g_\Omega(z, x) \geq \mu_x - C_7 = \lambda_x - C_7
\geq \varphi_x(z) - \log(-\phi_x(z)) + 2b \log |\rho(x)| - C_7
\geq \varphi_x(z) - \log(-\phi_x(z)) + \frac{2b}{\tau} \log \epsilon - C_8,
\]

which implies
\[
\{g_\Omega(\cdot, x) = -\alpha\} \subset B(x, \epsilon) := \{d_{FS}(\cdot, x) < \epsilon^{1/\tau}\}
\]
provided \(\epsilon\) is sufficiently small. Hence, by (1) there exists \(\bar{x} \in B(x, \epsilon^{1/\tau})\) such that
\[
|g_\Omega(\bar{x}, y)|^n \leq C_9 |\log \epsilon|^{n-1}|g_\Omega(y, x)|.
\]

By Bertini’s Lemma, one can take a generic hyperplane \(H\) such that it does not contain \(x, \bar{x}, y\). On \(P^n \setminus H\) one can introduce inhomogeneous coordinates \(w = (w_1, \cdots, w_n)\). We can also choose \(H\) such that \(|w(x) - w(\bar{x})| \approx d_{FS}(x, \bar{x})\) where the implicit constants depend only on \(\Omega\). One can regard \(\Omega \setminus H\) as an unbounded \(C^2\) pseudoconvex domain in \(C^n\). Set
\[
\tilde{\Omega} = \{w \in \Omega \setminus H : w + w(\bar{x}) - w(x) \in \Omega \setminus H\}.
\]

Since \(ds_{FS}^2 = \partial \bar{\partial} \log(1 + |w|^2) \leq \partial \bar{\partial} |w|^2\) on \(P^n \setminus H\), there is a constant \(C_{10} > 0\) such that
\[
\partial \bar{\partial} \tilde{\Omega} \cap (\Omega \setminus H) \subset \{\delta_\Omega < C_{10} \epsilon^{1/\tau}\}.
\]

Therefore,
\[
h(w) = \left\{ \begin{array}{ll}
\max \{g_\Omega(w, w(y)), g_\Omega(w + w(\bar{x}) - w(x), w(y)) - \delta\}, & w \in \tilde{\Omega},
\ h_\Omega(w, w(y)), & w \in (\Omega \setminus H) \setminus \tilde{\Omega},
\end{array} \right.
\]

where \(\delta = \sup_{0 < \rho < C_{10} \epsilon^{1/\tau}} |g_\Omega(\cdot, w(y))|\) is a well-defined negative psh function with a logarithmic pole at \(w(y)\) on \(\Omega \setminus H\). Since \(H\) is an analytic subset, \(h\) extends to a psh function on the whole of \(\Omega\). Therefore,
\[
g_\Omega(x, y) \geq h(w(x)) \geq g_\Omega(\bar{x}, y) - \delta.
\]

By (2), (3), for any \(x, y \in \Omega\) with \(\rho(x) \leq 2\rho(y)\), one has
\[
|g_\Omega(x, y)| \leq \delta + C_{11} |\log \epsilon|^{1-\frac{1}{\tau}} |g_\Omega(y, x)|^{1/n}
\leq \frac{1}{2} + C_{12} \left( \frac{\rho(y)}{\rho(x)} \right)^{1/n} |\log(-\rho(y))|
\]
according to Lemma 4. The proof is complete.

Proof of Theorem 2. We follow the argument as in [10]. Let \(y_1, y_2 \in \Omega\) be two arbitrary points satisfying
\[
|\rho(y_2)| < 2^{-\epsilon}, \ C|\rho(y_1)| \cdot |\log(-\rho(y_1))|^n \leq C^{-1}|\rho(y_2)| \cdot |\log(-\rho(y_2))|^{-1}.
\]
We take a complete orthonormal basis \( \{ h_j \}_{j=0}^{\infty} \) for \( \mathcal{H} \) such that \( h_j(y_2) = 0 \) for all \( j \geq 1 \). According to Kobayashi [15], we can immerse \( M \) into the infinite-dimensional complex projective space \( \mathbb{CP}(\mathcal{H}) \) via the map
\[
\sigma : x \mapsto (h_0(x) : h_1(x) : \cdots).
\]
Since each point \( P = (\zeta_0 : \zeta_1 : \cdots) \) in the projective space corresponds to an entire great circle of the unit sphere consisting of points \( (\zeta_0 e^{i\theta}, \zeta_1 e^{i\theta}, \cdots) \), then the Fubini-Study distance between two points \( P, Q \) is equal to the distance in the spherical geometry between the corresponding great circles. By the choice of the basis, we have \( \sigma(y_2) = (1 : 0 : \cdots) \) and \( \sigma(y_1) = (a_0 : a_1 : \cdots) \) where \( a_j = h_j^*(y_1)/\sqrt{K_{\Omega}(y_1)} \).

Hence,
\[
\text{dist}_{\Omega}(y_1, y_2) \geq \text{dist}_{FS}(\sigma(y_1), \sigma(y_2)) \geq \inf_{\theta_1, \theta_2} |e^{i\theta_1}(a_0, a_1, \cdots) - e^{i\theta_2}(1, 0, \cdots)| = \sqrt{(1 - |a_0|^2 + \sum_{j=1}^{\infty} |a_j|^2)}.
\]
Therefore, if \( |a_0| \leq 1/2 \), then \( \text{dist}_{\beta}(y_1, y_2) \geq 1/2 \). Otherwise, take a smooth function \( \lambda \) on \( \mathbb{R} \) such that \( \lambda = 1 \) on \( (-\infty, -1] \) and \( \lambda = 0 \) on \( [0, \infty) \). Set
\[
\begin{align*}
\eta &= \lambda(-\log(-g_{\Omega}(\cdot, y_1) + 1) + \log 2) h_0, \\
\varphi &= 2n(g_{\Omega}(\cdot, y_1) + g_{\Omega}(\cdot, y_2)) - \log(-g_{\Omega}(\cdot, y_1) + 1).
\end{align*}
\]
By Proposition 5, we see that \( \{ g_{\Omega}(\cdot, y_1) < -1 \} \cap \{ g_{\Omega}(\cdot, y_2) < -1 \} = \emptyset \). By the well-known \( L^2 \) estimates (cf. [7, 10]), we can solve the equation \( \bar{\partial}u = \bar{\partial}\eta \) in such a way that
\[
\int_{\Omega} |u \wedge \bar{\partial}e^{-\varphi} | \leq \int_{\Omega} |\bar{\partial}\lambda|^2 |\bar{\partial}\varphi| h_0 \wedge \bar{\partial}h_0 e^{-\varphi} | \leq C_{12}
\]
since \( \bar{\partial}\varphi \geq (-g_{\Omega}(\cdot, y_1) + 1)^2 \bar{\partial}g_{\Omega}(\cdot, y_1) \bar{\partial}g_{\Omega}(\cdot, y_1) \) holds in the sense of distribution. Therefore, \( F = \eta - u \) is holomorphic on \( \Omega \) and satisfies \( F(y_1) = h_0(y_1), F(y_2) = 0 \) and
\[
\int_{\Omega} |F| \leq C_{13}.
\]
Hence,
\[
\text{dist}_{\Omega}(y_1, y_2) \geq \sqrt{\sum_{j=1}^{\infty} |a_j|^2} \geq \frac{\sqrt{F(y_1) \wedge \bar{\partial}F(y_1)}}{C_{13}K_{\Omega}(y_1)} = \frac{|a_0|}{\sqrt{C_{13}}} \geq \frac{1}{2C_{13}}.
\]
Now if \( c_0, c_1, \cdots, c_k \) are finite increasing positive numbers such that \( c_k \leq 2^{-c} \) and
\[
C^{-1}c_k |\log c_k|^{-1} = Cc_{k-1} |\log c_{k-1}|^n,
\]
then
\[
c_k \leq C^2c_{k-1} |\log c_{k-1}|^n \leq C^4c_{k-2} |\log c_{k-2}|^{2n} \leq \cdots \leq C^{2k}c_0 |\log c_0|^{nk}.
\]
Given $y \in \Omega$, fix a point $y_0$ with $|\rho(y_0)| = 2^{-\varepsilon}$. Take a Bergman geodesic $l$ connecting $y_0, y$. Let $c_0 = |\rho(y)|, c_k = 2^{-\varepsilon}$. Take $y_i \in l$ with $|\rho(y_i)| = c_i$, $i = 0, 1, \cdots, k$. Then

$$\text{dist}_\Omega(y_0, y) \geq \sum_{i=0}^{k-1} \text{dist}_\Omega(y_i, y_{i+1}) \geq C_{14}k,$$

from which the desired estimate follows.

3. Proof of Theorem 1

Let $y \in \Omega \cap V$ be an arbitrary point with $|\rho(y)| < e^{-1}$. Set $\varepsilon = C^{-1}|\rho(y)| \cdot |\log(-\rho(y))|^{-1}$ and $\Omega_\varepsilon = \{x \in \Omega : |\rho(x)| > \varepsilon\}$. Here $C$ is the constant in Proposition 5. Set

$$\lambda_y = \varphi_y - \log(-\phi_y) - b\log(-\rho) + b\log \varepsilon,$$

$$\psi_y = \max\{\lambda_y, g_M(\cdot, y)\}.$$

Then $\psi_y$ is a negative psh function with a logarithmic pole at $y$ on $\Omega_\varepsilon$ such that

$$\{x \in \Omega_\varepsilon : \psi_y(x) < -1\} \subset \{x \in \Omega_\varepsilon : g_M(x, y) < -1\}$$

$$\subset \{x \in \Omega_\varepsilon : |\rho(x)| \leq C|\rho(y)| \cdot |\log(-\rho(y))|^{n}\}. \quad (4)$$

On the other hand, for any $0 < \tilde{r} < r_0$, there is a positive constant $\tilde{C}$ depending only on $\tilde{r}$ such that

$$\psi_y(x) \geq \tilde{\lambda}_y(x) \geq -\tilde{C} - b(n + 1) \log |\log(-\rho(y))|, \forall x \in \{\psi_y < -1\}\setminus B(y, \tilde{r}). \quad (5)$$

Without loss of generality, we assume that $B(y, 2\tilde{r}) \subset U$. Choose a holomorphic $n$-form $f$ on $\Omega \cap U$ with unit $L^2$-norm such that $f \wedge \bar{f}(y) = K_{\Omega \cap U}(y)$. Let $\kappa, \lambda$ be the cut-off functions as above. Set

$$\tilde{\varphi}_y = 2n\psi_y - \log(-\psi_y + 1),$$

$$\nu = \partial(\lambda(-\log(-\psi_y + 1) + \log 2)\kappa(d_{FS}(\cdot, y)/2\tilde{r})f)$$

on $\Omega_\varepsilon$. We recall the following $L^2$ estimate:

**Theorem** (cf. [3], [1]). Let $M$ be a Stein manifold. Let $\varphi, \psi$ be psh functions such that $r\partial\bar{\partial}\psi \geq \partial\bar{\partial}\varphi$ holds in the sense of distribution for some $0 < r < 1$. Then for any $\bar{\partial}$-closed $(n, 1)$ form $\nu$ with $\int_M |v|^2_{\bar{\partial}(\varphi+\psi)}e^{\varphi-\psi} < +\infty$, there exists an $(n, 0)$ form $u$ on $M$ such that $\bar{\partial}u = v$ and

$$\left|\int_M u \wedge \bar{u}e^{\varphi-\psi}\right| \leq C \int_M |v|^2_{\bar{\partial}(\varphi+\psi)}e^{\varphi-\psi}.$$

We apply this theorem with $\psi = -\frac{1}{2} \log(-\rho), \varphi = \tilde{\varphi}_y + \psi$ to get a solution $u$ of $\bar{\partial}u = \nu$ such that

$$\int_{\Omega_\varepsilon} u \wedge \bar{u}e^{-\tilde{\varphi}_y} \leq \tilde{C}_2 |\log(-\rho(y))|^\gamma \leq \tilde{C}_3 |\log \delta_\Omega(y)|^\gamma$$

because of (4), (5) and

$$\bar{\partial}\partial(\tilde{\varphi}_y + \psi) \geq \partial \log(-\psi_y + 1)\bar{\partial} \log(-\psi_y + 1) + \frac{C_1}{2} ds_{FS}^2.$$
Here $\gamma > 0$ depends only on $b, n$ and $\tilde{C}_2, \tilde{C}_3$ are independent of $y$. Thus we obtain a holomorphic $n$-form

$$F = \lambda (-\log(-\psi_y + 1) + \log 2) K_{\mathbb{C}^n}(\cdot, y)$$

on $\Omega$ such that $F \wedge \bar{F}(y) = K_{\mathbb{C}^n U}(y)$ and its $L^2$-norm is bounded above by a constant multiple of $|\log \delta_{\Omega}(y)|^\gamma$. Finally, we apply the above theorem with

$$\psi = -\frac{1}{2} \log(-g_{\Omega}(\cdot, y) + 1),$$

$$\varphi = 2ng_{\Omega}(\cdot, y) + \psi$$

to get a solution of

$$\bar{\partial}u = v := \bar{\partial}(\lambda (-\log(-g_{\Omega}(\cdot, y) + 1) + \log 2) F)$$

on $\Omega$ such that

$$\left| \int_{\Omega} u \wedge \bar{\partial} e^{-2ng_{\Omega}(\cdot, y)} \right| \leq \tilde{C}_4 \left| \int_{\Omega} F \wedge \bar{F} \right|.$$
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