ON THE BERGMAN METRIC OF PSEUDOCONVEX DOMAINS IN A COMPLEX PROJECTIVE SPACE

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Abstract. We prove a localization principle of the Bergman kernel form and metric for \( C^2 \) pseudoconvex domains in the complex projective space. An estimate of the Bergman distance is also given.

1. Introduction

Let \( M \) be a complex \( n \)-dimensional manifold. Let \( \mathcal{H} \) be the Hilbert space of holomorphic \( n \)-forms on \( M \) such that \( \int_M |f| < \infty \). Let \( h_0, h_1, \ldots \) be a complete orthonormal basis for \( \mathcal{H} \). Then the \( 2n \)-form defined on \( M \times M \) given by \( K_M = \sum_{j=0}^{\infty} h_j \wedge \bar{h}_j \) is called the Bergman kernel of \( M \). Let \( (z_1, \ldots, z_n) \) be a local coordinate system in \( M \) and let \( K_M^*(z) = K_M^*(z)dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \) where \( K_M^* \) is a locally defined function. Thus \( \beta_M := \partial \bar{\partial} \log K_M^* \) is a well-defined Hermitian form of bidegree \((1,1)\), whenever \( K_M^* \) is nonzero. We say that \( M \) possesses a Bergman metric iff \( \beta_M \) is everywhere positive definite. The following localization principle for the Bergman kernel (metric) is well known.

Theorem (cf. [11], [9]). Let \( \Omega \subset \subset \mathbb{C}^n \) be pseudoconvex. Then for any two neighborhoods \( V \subset \subset U \) of a boundary point \( p \) there is a constant \( C > 0 \) such that

\[
K_{\Omega}(x) \geq C \cdot K_{\Omega \cap U}(x),
\]

\[
\beta_{\Omega}(x; X) \geq C \cdot \beta_{\Omega \cap U}(x; X)
\]

for all \( x \in \Omega \cap V \) and \( X \in T_x^{1,0}(\Omega) \).

It is natural to ask on which pseudoconvex domains in \( \mathbb{P}^n \) the localization principle holds. (Notice that the complement of a hypersurface in \( \mathbb{P}^n \) is pseudoconvex; however, the Bergman kernel form vanishes.) In this direction, Diederich and Ohsawa [11] proved the localization principle for another Bergman kernel (metric), which is induced by square-integrable holomorphic functions with respect to the Fubini-Study metric on any pseudoconvex domain \( \Omega \) in \( \mathbb{P}^n \) such that the interior of its complement is not empty. Such a Bergman metric is not invariant under biholomorphic mappings. In this note, we will show...
Theorem 1. Let $\Omega$ be a pseudoconvex domain with $C^2$ boundary in $\mathbb{P}^n$. Then there is a constant $\gamma > 0$ such that for any two neighborhoods $V \subset \subset U$ of a boundary point $p$, one has

$$|K_\Omega(x)|_{FS} \geq C \cdot |K_{\Omega \cap U}(x)|_{FS} \left(\frac{1}{\log \delta_{\Omega}(x)^\gamma}\right),$$

$$\beta_{\Omega}(x; X) \geq C \cdot \beta_{\Omega \cap U}(x; X) \left(\frac{1}{\log \delta_{\Omega}(x)^\gamma}\right)$$

for all $x \in \Omega \cap V$ and $X \in T^1_{\Omega}(\Omega)$. Here $\delta_{\Omega}$ (resp. $|\cdot|_{FS}$) denotes the boundary distance (resp. length) w.r.t. the Fubini-Study metric $ds^2_{FS}$.

As an immediate consequence of Theorem 1 and the Ohsawa-Takegoshi extension theorem [19], we obtain

Corollary. Let $\Omega$ be as above. Then

$$|K_\Omega(x)|_{FS} \geq C \cdot \delta_{\Omega}(x)^{-2} \cdot \log \delta_{\Omega}^{-\gamma}.$$ 

We mention that Theorem 1 cannot be generalized to an arbitrary fat pseudoconvex domain in $\mathbb{P}^n$ as the following example shows.

Example. We consider the following domain:

$$\Omega = \{[z_0, z_1, z_2] \in \mathbb{P}^2 : |z_1| < |z_0|\}$$

where $(z_0, z_1, z_2)$ denote homogeneous coordinates in $\mathbb{P}^2$. Note that $\Omega$ is biholomorphic to the product of the unit disc and the complex plane via

$$\zeta_1 = z_1/z_0, \quad \zeta_2 = z_2/z_0.$$ 

Hence $\Omega$ is a fat pseudoconvex domain in $\mathbb{P}^n$, but the Bergman metric is degenerate.

However, the estimate of the Bergman metric in Theorem 1 does not imply the completeness. It was shown in [5] that every hyperconvex manifold is Bergman complete, while every $C^2$ pseudoconvex domain in $\mathbb{P}^n$ is hyperconvex [18]; hence it is Bergman complete. We mention that there are a lot of works concerning the completeness of $\beta_M$ (cf. [4], [5], [6], [12], [13], [14], [15], [17], [20]). On the other hand, Diederich and Ohsawa [10] proved that the Bergman distance for a bounded $C^2$ pseudoconvex domain in $\mathbb{C}^n$ has a lower bound of a constant multiple $\log |\log \delta_{\Omega}^\gamma|$. This lower bound was improved by Blocki [3] to $|\log \delta_{\Omega}|/\log |\log \delta_{\Omega}|$.

We extend his result to the complex projective space.

Theorem 2. Let $\Omega \subset \mathbb{P}^n$ be a pseudoconvex domain with $C^2$ boundary. Then

$$\text{dist}_{\Omega}(x_0, x) \geq C \cdot \frac{\log \delta_{\Omega}(x)}{\log |\log \delta_{\Omega}(x)|},$$

where $\text{dist}_{\Omega}(x_0, x)$ denotes the Bergman distance between $x_0$ and $x$.

2. Proof of Theorem 2

Let $y \in \Omega$ be arbitrary fixed. Take a smooth function $\kappa : \mathbb{R} \to [0, 1]$ such that $\kappa|_{(-\infty, 1/2]} = 1$ and $\kappa|_{[1, \infty)} = 0$. Let $d_{FS}(x, y)$ denote the Fubini-Study distance between two points $x, y$ in $\mathbb{P}^n$. Let $r$ be the injectivity radius of $\mathbb{P}^n$ and $c$ be the upper bound of the sectional curvature. Set

$$\phi_y(x) = \kappa(d_{FS}(x, y)/r_0) \log(d_{FS}(x, y)/r_0) - 1.$$
where \( r_0 = \min\{r, \pi/2c\} \). By the Hessian comparison theorem (cf. [12]), there is a positive constant \( C_1 \) independent of \( y \) such that
\[
\partial \bar{\partial} \phi_y(x) \geq -C_1 ds^2_{FS}.
\]
From now on we assume \( r_0 = 1 \) for the sake of simplicity. By [18], there exists a smooth and strictly psh function \( \rho : \Omega \to (-1, 0) \) such that \( |\rho| \approx \delta^\tau \) for some \( 0 < \tau < 1 \) and
\[
\partial \bar{\partial} \rho \geq C_2 |\rho| ds^2_{FS}
\]
where \( C_2 \) is a positive constant. Choose a cut-off function \( \chi : \mathbb{R} \to [0, 1] \) such that \( \chi \equiv 1 \) on \([-1/2, 1/2]\) and \( \chi \equiv 0 \) on \([1, +\infty) \cup (-\infty, -1]\). For any \( y \in M \), we set
\[
\varphi_y = \chi \left( \log(-\log(-\rho)) - \log(-\log(-\rho(y))) \right) \phi_y.
\]
Following an idea of [10], we will show

**Lemma 3.** There exists a sufficiently large constant \( b \) such that for any \( y \in \Omega \) satisfying \( |\rho(y)| < 2^{-\epsilon} \), we have

(i) \( \varphi_y - \log(-\varphi_y) - b \log(-\rho) \) is \( C^2 \) strictly psh on \( \Omega \setminus \{y\} \);

(ii) \( \varphi_y \) has a logarithmic pole at \( y \);

(iii) \( \text{supp} \varphi_y \subset \{x \in \Omega : |\rho(x)|^\epsilon \leq |\rho(x)| \leq |\rho(x)|^{1/\epsilon} \} \).

**Proof.** (ii), (iii) are trivial. We only need to show (i). By a straightforward computation, we obtain on \( \Omega \setminus \{y\} \),
\[
\partial \bar{\partial} \varphi_y = \frac{\phi_y}{\log(-\rho)} \left( \chi''(\cdot) \frac{\partial \log(-\rho) \bar{\partial} \log(-\rho)}{\log(-\rho)} \right.
- \chi'(\cdot) \frac{\partial \log(-\rho) \bar{\partial} \log(-\rho)}{\log(-\rho)} + \chi'(\cdot) \bar{\partial} \log(-\rho) \left.ight)
+ \frac{\chi'(\cdot) \phi_y}{\log(-\rho)} \left( \partial \log(-\rho) \frac{\partial \phi_y}{\phi_y} + \frac{\partial \phi_y}{\phi_y} \bar{\partial} \log(-\rho) \right)
+ \chi(\cdot) \partial \bar{\partial} \phi_y.
\]
By the Cauchy-Schwarz inequality, for any constant \( \theta > 0 \) one has
\[
\pm 2 \text{Re} \left\{ \partial \log(-\rho) \frac{\bar{\partial} \phi_y}{\phi_y} \right\}
\leq \theta \partial \log(-\rho) \bar{\partial} \log(-\rho) + \theta^{-1} \partial \log(-\phi_y) \bar{\partial} \log(-\phi_y).
\]
Since \( -\rho(y) < 2^{-\epsilon} \), it follows from (iii) that \( -\rho(x) < 1/2 \) on \( \text{supp} \varphi_y \). Hence there is a positive constant \( C_3 \) depending only on the choice of \( \chi \) such that
\[
\partial \bar{\partial} \varphi_y \geq -\frac{C_3 |\phi_y|}{|\log(-\rho)|} \left\{ \bar{\partial} \log(-\rho) + \theta \log(-\rho) \partial \log(-\rho) \right. \\
+ \theta^{-1} \partial \log(-\phi_y) \bar{\partial} \log(-\phi_y) \left. \right\} - C_1 ds^2_{FS}.
\]
Note that
\[
\text{supp} \chi'(\cdot) \subset \{x \in \Omega : |\rho(y)|^\epsilon \leq |\rho(x)| \leq |\rho(y)|^{1/\epsilon} \}
\text{ or } |\rho(y)|^{1/\epsilon} \leq |\rho(x)| \leq |\rho(y)|^{1/\epsilon}.
\]
This implies
\[
|\rho(x) - \rho(y)| \geq \frac{|\rho(x)|}{2}
\]
on supp χ′(·). Since |ρ| ≈ δ|Ω|, |ρ(x) − ρ(y)| = O(dFS(x, y)^r). By the above inequality, there is a constant C_4 > 0 independent of y so that

|φ_y| ≤ C_4|log(−ρ)|

holds on supp χ′(·). On the other hand, one has

\[ \partial \bar{\partial}(- \log(-\rho)) = \frac{\partial \bar{\partial}ρ}{|ρ|} + \partial \log(-\rho) \bar{\partial} \log(-\rho) \]
\[ \geq C_2d^2FS + \partial \log(-\rho) \bar{\partial} \log(-\rho), \]
\[ \partial \bar{\partial}(- \log(-φ_y)) = \frac{\partial \bar{\partial}φ_y}{|φ_y|} + \partial \log(-φ_y) \bar{\partial} \log(-φ_y) \]
\[ \geq -C_1d^2FS + \partial \log(-φ_y) \bar{\partial} \log(-φ_y) \]

since φ_y ≤ −1. Hence after fixing sufficiently large θ, we can choose b > 0 such that (i) holds. The proof is complete.

Let g_Ω be the pluricomplex Green function on Ω, i.e., g_Ω(x, y) = sup{u(x)} where the supremum is taken over all negative functions u ∈ PSH(Ω) satisfying the property that the function u − log|z| is bounded from above in a deleted neighborhood of y for some holomorphic local coordinates z centered at y, that is, z(y) = 0.

**Lemma 4.** There is a constant c > 0 such that

\[ g_Ω(x, y) ≥ \frac{c\rho(x)}{ρ(y)} |\log(-ρ(y))| \]

for any x, y ∈ Ω with |ρ(x)| ≤ |ρ(y)|/2.

**Proof.** Let ε = |ρ(y)|^ε and set

λ_y = φ_y − log(-φ_y) − b log(-ρ) + 2b log ε.

By Lemma 3, λ_y is a negative psh function on Ω_z = \{x ∈ Ω : |ρ(x)| > ε^2\}. Set

η_y = \left\{ \begin{array}{ll}
\max\{λ_y, c_y(ρ + ε^2)\}, & |ρ| ≤ ε, \\
λ_y, & |ρ| > ε,
\end{array} \right.

where

c_y = \frac{1}{ε - ε^2} \inf_{|ρ(x)| = ε} λ_y(x).

Then η_y is a well-defined negative psh function on Ω_z such that

η_y(x) ≥ c_yρ(x) ≥ C_5ρ(x)|\log ε(ε - ε^2) ≥ -C_6

for all |ρ(x)| ≤ ε^3/2. Here C_5, C_6 are positive constants independent of y. Set

\hat{ρ} = -C_6 \frac{\log(-ρ + ε^2) - \log(2ε^2)}{\log(ε^4 + ε^2) - \log(2ε^2)}.

Clearly, \hat{ρ} is a psh function on Ω such that \hat{ρ} ≤ C_7 where C_7 is independent of ε. Note that

\hat{ρ}(x) = -C_6 ≤ η_y(x), \quad \text{if} \quad |ρ(x)| = ε^3/2,
\hat{ρ}(x) = 0 = η_y(x), \quad \text{if} \quad |ρ(x)| = ε^2.
Hence the function defined by
\[
\mu_y = \begin{cases} 
\eta_y, & \text{on } \{ |\rho| > e^{3/2} \}, \\
\max\{\eta_y, \tilde{\rho}\}, & \text{on } \{ e^{2} \leq |\rho| \leq e^{3/2} \}, \\
\tilde{\rho}, & \text{on } \{ |\rho| < e^{2} \}
\end{cases}
\]
is well-defined, psh on \(\Omega\) and has a logarithmic pole at \(y\). Set
\[
\nu_y = \begin{cases} 
\max\{\mu_y - C_7, \tilde{c}_y \rho \}, & \rho(x) \geq \frac{1}{2}\rho(y), \\
\mu_y - C_7, & \rho(x) < \frac{1}{2}\rho(y),
\end{cases}
\]
where
\[
\tilde{c}_y = \frac{2}{\rho(y)} \inf_{\rho(x) = \frac{1}{2}\rho(y)} (\mu_y(x) - C_7).
\]
If \(\rho(x) \geq \frac{1}{2}\rho(y)\), then we have
\[
\left|\frac{\rho(y)}{2}\right| \leq |\rho(x) - \rho(y)| \leq c' \cdot d_{FS}(x, y)^7
\]
for a suitable constant \(c' > 0\), which implies
\[
\tilde{c}_y \leq C_8 \frac{\log(-\rho(y))}{|\rho(y)|}.
\]
Hence
\[
g_{\Omega}(x, y) \geq \tilde{c}_y \rho(x) \geq -C_8 \frac{\rho(x)}{\rho(y)} |\log(-\rho(y))|.
\]

The following is the key step in proving Theorem 2. The main idea comes from [3].

**Proposition 5.** There is a constant \(C > 0\) such that for any \(y \in \Omega\) with \(|\rho(y)| < e^{-1}\) one has
\[
\{ x \in \Omega : g_{\Omega}(x, y) < -1 \} \subset \{ x \in \Omega : C^{-1} |\rho(y)| \cdot |\log(-\rho(y))|^{-1} \leq |\rho(x)| \leq C |\rho(y)| \cdot |\log(-\rho(y))|^n \}.
\]

**Proof.** From [2], we know that for any nonnegative psh functions \(u, v\) defined on a smooth bounded domain \(D\) in a Stein manifold with \(u|_{\partial D} = 0\), then
\[
\int_D |u|^n (dd^c v)^n \leq n! \|v\|_{\infty}^{n-1} \int_D |v|(dd^c u)^n
\]
where \(d^c = i(\bar{\partial} - \partial)\). Fix arbitrary \(x, y \in \Omega\) with \(\rho(x) \leq 2\rho(y)\). Set \(\epsilon = |\rho(y)|^{\alpha}\) and \(\alpha = -\frac{2}{7}(b+1) \log \epsilon\). We exhaust \(\Omega\) by a sequence of smooth strongly pseudoconvex domains \(\Omega_j, j = 1, 2, \ldots\). By the above inequality, we have
\[
\int_{\Omega_j} \|g_{\Omega_j}(\cdot, y)\|^n (dd^c \max\{g_{\Omega_j}(\cdot, x), -\alpha\})^n \leq n! \alpha^{n-1} \int_{\Omega_j} \max\{g_{\Omega_j}(\cdot, x), -\alpha\}|(dd^c g_{\Omega_j}(\cdot, y)|^n \leq n!(2\pi)^n \alpha^{n-1} |g_{\Omega_j}(y, x)|
\]
since \((dd^c g_{\Omega_j}(\cdot, y))^n = (2\pi)^n \delta_y\) (cf. [8]). It is also known from [8] that the measure \((dd^c \max\{g_{\Omega_j}(\cdot, x), -\alpha\})^n\) is supported on \(\{g_{\Omega_j}(\cdot, x) = -\alpha\}\) with total mass \((2\pi)^n\).
Hence
\[ \inf_{\{g_{\Omega}(\cdot, \cdot) = -\alpha\}} |g_{\Omega}(\cdot, y)|^n \leftarrow \inf_{\{g_{\Omega}(\cdot, x) = -\alpha\}} |g_{\Omega}(\cdot, y)|^n \]
\[ \leq n^{1/n} |g_{\Omega}(y, x)| \]
\[ \rightarrow n^{1/n} |g_{\Omega}(y, x)| \]
(1)
as \( j \to \infty \). According to Lemma 4, one has \( g_{\Omega}(z, x) > -1 \) provided \( |\rho(z)| \leq |\rho(x)|^e \).

On the other hand, for any \( z, x \in \Omega \) with \( |\rho(z)| > |\rho(x)|^e \) one has
\[ g_{\Omega}(z, x) \geq \mu_x - C_7 = \lambda_x - C_7 \]
\[ \geq \varphi_x(z) - \log(-\varphi_x(z)) + 2m \log |\rho(x)| - C_7 \]
\[ \geq \varphi_x(z) - \log(-\varphi_x(z)) + \frac{2b}{\tau} \log \epsilon - C_8, \]
which implies
\[ \{g_{\Omega}(\cdot, \cdot) = -\alpha\} \subset B(x, \epsilon) := \{d_{FS}(\cdot, x) < \epsilon^{1/\tau}\} \]
provided \( \epsilon \) is sufficiently small. Hence, by (1) there exists \( \bar{x} \in B(x, \epsilon^{1/\tau}) \) such that
(2)
\[ |g_{\Omega}(\bar{x}, y)|^n \leq C_9 |\log \epsilon|^{n-1} |g_{\Omega}(y, x)|. \]

By Bertini’s Lemma, one can take a generic hyperplane \( H \) such that it does not contain \( x, \bar{x}, y \). On \( P^n \setminus H \) one can introduce inhomogeneous coordinates \( w = (w_1, \cdots, w_n) \). We can also choose \( H \) such that \( |w(x) - w(\bar{x})| \approx d_{FS}(x, \bar{x}) \) where the implicit constants depend only on \( \Omega \). One can regard \( \Omega \setminus H \) as an unbounded \( C^2 \) pseudoconvex domain in \( C^n \). Set
\[ \tilde{\Omega} = \{w \in \Omega \setminus H : w + w(\bar{x}) - w(x) \in \Omega \setminus H\}. \]

Since \( ds_{FS}^2 = \partial \bar{\partial} \log(1 + |w|^2) \leq \partial \bar{\partial}|w|^2 \) on \( P^n \setminus H \), there is a constant \( C_{10} > 0 \) such that
\[ \partial \bar{\Omega} \cap (\Omega \setminus H) \subset \{\delta_{\Omega} < C_{10} \epsilon^{1/\tau}\}. \]

Therefore,
\[ h(w) = \begin{cases} 
\max \{g_{\Omega}(w, w(y)), g_{\Omega}(w + w(\bar{x}) - w(x), w(y)) - \delta\}, \quad w \in \tilde{\Omega}, \\
g_{\Omega}(w, w(y)), \quad w \in (\Omega \setminus H) \setminus \tilde{\Omega},
\end{cases} \]
where \( \delta = \sup_{\delta_{\Omega} < C_{10} \epsilon^{1/\tau}, \{g_{\Omega}(\cdot, w(y))\} \text{ is a well-defined negative psh function with a logarithmic pole at } w(y) \text{ on } \Omega \setminus H. \) Since \( H \) is an analytic subset, \( h \) extends to a psh function on the whole of \( \Omega \). Therefore,
(3)
\[ g_{\Omega}(x, y) \geq h(w(x)) \geq g_{\Omega}(\bar{x}, y) - \delta. \]

By (2), (3), for any \( x, y \in \Omega \) with \( \rho(x) \leq 2\rho(y) \), one has
\[ |g_{\Omega}(x, y)| \leq \delta + C_{11} |\log \epsilon|^{1-\frac{1}{n}} |g_{\Omega}(y, x)|^{1/n} \]
\[ \leq \frac{1}{2} + C_{12} \left( \frac{\rho(y)}{\rho(x)} \right)^{1/n} |\log(-\rho(y))| \]
according to Lemma 4. The proof is complete.

Proof of Theorem 2. We follow the argument as in [10]. Let \( y_1, y_2 \in \Omega \) be two arbitrary points satisfying
\[ |\rho(y_2)| < 2^{-\epsilon}, \quad C|\rho(y_1)| \cdot |\log(-\rho(y_1))|^{1/n} \leq C^{-1} |\rho(y_2)| \cdot |\log(-\rho(y_2))|^{-1}. \]
We take a complete orthonormal basis \( \{h_j\}_{j=0}^\infty \) for \( \mathcal{H} \) such that \( h_j(y_2) = 0 \) for all \( j \geq 1 \). According to Kobayashi [15], we can immerse \( M \) into the infinite-dimensional complex projective space \( \mathbb{CP}(\mathcal{H}) \) via the map

\[
\sigma : x \mapsto (h_0(x) : h_1(x) : \cdots).
\]

Since each point \( P = (\zeta_0 : \zeta_1 : \cdots) \) in the projective space corresponds to an entire great circle of the unit sphere consisting of points \( (\zeta_0 e^{i\theta_0}, \xi_1 e^{i\theta_0}, \cdots) \), then the Fubini-Study distance between two points \( P, Q \) is equal to the distance in the spherical geometry between the corresponding great circles. By the choice of the basis, we have \( \sigma(y_2) = (1 : 0 : \cdots) \) and \( \sigma(y_1) = (a_0 : a_1 : \cdots) \) where \( a_j = h_j^*(y_1)/\sqrt{K_\Omega(y_1)} \).

Hence,

\[
\text{dist}_\Omega(y_1, y_2) \geq \text{dist}_{FS}(\sigma(y_1), \sigma(y_2)) \geq \inf_{\theta_1, \theta_2} |e^{i\theta_1}(a_0, a_1, \cdots) - e^{i\theta_2}(1, 0, \cdots)| = \sqrt{(1 - |a_0|)^2 + \sum_{j=1}^{\infty} |a_j|^2}.
\]

Therefore, if \( |a_0| \leq 1/2 \), then \( \text{dist}_{FS}(y_1, y_2) \geq 1/2 \). Otherwise, take a smooth function \( \lambda \) on \( \mathbb{R} \) such that \( \lambda = 1 \) on \( (-\infty, -1] \) and \( \lambda = 0 \) on \( [0, \infty) \). Set

\[
\eta = \lambda(-\log(-g\Omega(\cdot, y_1) + 1) + 2)h_0, \\
\varphi = 2n(g\Omega(\cdot, y_1) + g\Omega(\cdot, y_2)) - \log(-g\Omega(\cdot, y_1) + 1).
\]

By Proposition 5, we see that \( \{g\Omega(\cdot, y_1) < -1\} \cap \{g\Omega(\cdot, y_2) < -1\} = \emptyset \). By the well-known \( L^2 \) estimates (cf. [7], [16]), we can solve the equation \( \bar{\partial}u = \varphi \) in such a way that

\[
\left| \int_\Omega u \wedge \bar{u} e^{-\varphi} \right| \leq \left| \int_\Omega |\bar{\partial}\lambda|^2 \bar{\partial}u \wedge \bar{h}_0 e^{-\varphi} \right| \leq C_{12} \quad \text{since} \quad \bar{\partial}\lambda \varphi \geq \varphi.
\]

Therefore, \( F = \eta - u \) is holomorphic on \( \Omega \) and satisfies \( F(y_1) = h_0(y_1), F(y_2) = 0 \) and

\[
\left| \int_\Omega F \wedge F \right| \leq C_{13}.
\]

Hence

\[
\text{dist}_\Omega(y_1, y_2) \geq \frac{\sum_{j=1}^{\infty} |a_j|^2}{\sqrt{K_\Omega(y_1)}} \geq \frac{|a_0|}{\sqrt{C_{13}}} \geq \frac{1}{2\sqrt{C_{13}}}. 
\]

Now if \( c_0, c_1, \cdots, c_k \) are finite increasing positive numbers such that \( c_k \leq 2^{-k+1} \) and

\[
C^{-1}c_k |\log c_k|^{-1} = C_{k-1} |\log c_{k-1}|^{n_k},
\]

then

\[
c_k \leq C^{k-1} |\log c_{k-1}|^{n_k} \leq C^{k} c_{k-2} |\log c_{k-2}|^{2n_k} \leq \cdots \leq C^{2k} c_0 |\log c_0|^{n_k}.
\]
Given \( y \in \Omega \), fix a point \( y_0 \) with \( |\rho(y_0)| = 2^{-\epsilon} \). Take a Bergman geodesic \( l \) connecting \( y_0, y \). Let \( c_0 = |\rho(y)|, c_k = 2^{-\epsilon} \). Take \( y_i \in l \) with \( |\rho(y_i)| = c_i, i = 0, 1, \cdots, k \). Then

\[
\text{dist}_\Omega(y_0, y) \geq \sum_{i=0}^{k-1} \text{dist}_\Omega(y_i, y_{i+1}) \geq C_{14} k,
\]

from which the desired estimate follows.

### 3. Proof of Theorem 1

Let \( y \in \Omega \cap V \) be an arbitrary point with \( |\rho(y)| < e^{-\epsilon} \). Set \( \tilde{\epsilon} = C^{-1}|\rho(y)| \cdot |\log(-\rho(y))|^{-1} \) and \( \Omega_{\tilde{\epsilon}} = \{ x \in \Omega : |\rho(x)| > \tilde{\epsilon} \} \). Here \( C \) is the constant in Proposition 5. Set

\[
\tilde{\lambda}_y = \varphi_y - \log(-\phi_y) - b \log(-\rho) + b \log \tilde{\epsilon}, \\
\psi_y = \max\{\tilde{\lambda}_y, gM(\cdot, y)\}.
\]

Then \( \psi_y \) is a negative psh function with a logarithmic pole at \( y \) on \( \Omega_{\tilde{\epsilon}} \) such that

\[
\{ x \in \Omega_{\tilde{\epsilon}} : \psi_y(x) < -1 \} \subset \{ x \in \Omega_{\tilde{\epsilon}} : gM(x, y) < -1 \}
\]

(4)

\[
\subset \{ x \in \Omega_{\tilde{\epsilon}} : |\rho(x)| \leq C|\rho(y)| \cdot |\log(-\rho(y))|^n \}.
\]

On the other hand, for any \( 0 < \tilde{r} < r_0 \), there is a positive constant \( \tilde{C} \) depending only on \( \tilde{r} \) such that

(5)

\[
\psi_y(x) \geq \tilde{\lambda}_y(x) \geq -\tilde{C} - b(n + 1) \log |\log(-\rho(y))|, \quad \forall x \in \{ \psi_y < -1 \} \setminus B(y, \tilde{r}).
\]

Without loss of generality, we assume that \( B(y, 2\tilde{r}) \subset U \). Choose a holomorphic \( n \)-form \( f \) on \( \Omega \cap U \) with unit \( L^2 \)-norm such that \( f \wedge f(y) = K_{\Omega \cap U}(y) \). Let \( \kappa, \lambda \) be the cut-off functions as above. Set

\[
\tilde{\varphi}_y = 2n\psi_y - \log(-\psi_y + 1), \\
u = \bar{\partial}(\lambda(-\log(-\psi_y + 1) + \log 1) + \log 2) \kappa(d_{FS}(\cdot, y)/2\tilde{r})f)
\]

on \( \Omega_{\tilde{\epsilon}} \). We recall the following \( L^2 \) estimate:

**Theorem (cf. [3], [1]).** Let \( M \) be a Stein manifold. Let \( \varphi, \psi \) be psh functions such that \( r\bar{\partial}\psi \geq \bar{\partial}\psi \) holds in the sense of distribution for some \( 0 < r < 1 \). Then for any \( \bar{\partial} \)-closed \((n, 1)\) form \( v \) with \( \int_M |v|_{\bar{\partial}(\varphi + \psi)}^2 e^{\varphi - \psi} < +\infty \), there exists an \((n, 0)\) form \( u \) on \( M \) such that \( \bar{\partial} u = v \) and

\[
\left| \int_M u \wedge \bar{u} e^{\varphi - \psi} \right| \leq C_1 \int_M |v|_{\bar{\partial}(\varphi + \psi)}^2 e^{\varphi - \psi}.
\]

We apply this theorem with \( \psi = -\frac{1}{2} \log(-\rho), \varphi = \tilde{\varphi}_y + \psi \) to get a solution \( u \) of \( \bar{\partial} u = v \) such that

\[
\left| \int_{\Omega_{\tilde{\epsilon}}} u \wedge \bar{u} e^{-\tilde{\varphi}_y} \right| \leq \tilde{C}_2 |\log(-\rho(y))| \leq \tilde{C}_3 |\log \delta_{\Omega}(y)| \leq \tilde{C}_4 |\log \delta_{\Omega}(y)| \leq \tilde{C}_5 |\log \delta_{\Omega}(y)|
\]

because of (4), (5) and

\[
\bar{\partial}\bar{\partial}(\tilde{\varphi}_y + \psi) \geq \bar{\partial} \log(-\psi_y + 1) \bar{\partial} \log(-\psi_y + 1) + \frac{C_1}{2} d_{FS}^2.
\]
Here $\gamma > 0$ depends only on $b$, $n$ and $\tilde{C}_2$, $\tilde{C}_3$ are independent of $y$. Thus we obtain a holomorphic $n$-form

$$F = \lambda ( - \log ( - \psi_y + 1 ) + \log 2 ) n ( d F_S ( \cdot, y ) / 2 \tilde{r} ) f - u$$

on $\Omega$, such that $F \wedge \bar{F} ( y ) = K_{\tilde{\Omega} \setminus U} ( y )$ and its $L^2$-norm is bounded above by a constant multiple of $| \log \delta_{\Omega} ( y ) |^\gamma$. Finally, we apply the above theorem with

$$\psi = - \frac{1}{2} \log ( - g_{\Omega} ( \cdot, y ) + 1 ),$$

$$\varphi = 2 n g_{\Omega} ( \cdot, y ) + \psi$$

to get a solution of

$$\bar{\partial} u = v := \bar{\partial} ( \lambda ( - \log ( - g_{\Omega} ( \cdot, y ) + 1 ) + \log 2 ) F )$$

on $\Omega$ such that

$$\left| \int_\Omega u \wedge \bar{u} e^{-2 n g_{\Omega} ( \cdot, y )} \right| \leq \tilde{C}_4 \left| \int_\Omega \bar{F} \wedge F \right|.$$

Set $\tilde{F} = \lambda ( - \log ( - g_{\Omega} ( \cdot, y ) + 1 ) + \log 2 ) F - u$. Then $\tilde{F}$ is a holomorphic $n$-form on $\Omega$ satisfying $\tilde{F} ( y ) = F ( y )$ and the $L^2$-norm bounded above by a constant multiple of $| \log \delta_{\Omega} ( y ) |^\gamma$, from which we obtain the estimate of the Bergman kernel. The argument for the Bergman metric is similar.

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