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ON THE BERGMAN METRIC OF PSEUDOCONVEX DOMAINS IN A COMPLEX PROJECTIVE SPACE

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ABSTRACT. We prove a localization principle of the Bergman kernel form and metric for C^2 pseudoconvex domains in the complex projective space. An estimate of the Bergman distance is also given.

1. INTRODUCTION

Let M be a complex n-dimensional manifold. Let $\mathcal{H}$ be the Hilbert space of holomorphic n-forms on M such that $|\int_M f \wedge \bar{f}| < \infty$. Let $h_0, h_1, \cdots$ be a complete orthonormal basis for $\mathcal{H}$. Then the $2n$-form defined on $M \times M$ given by $K_M = \sum_{j=0}^{\infty} h_j \wedge \bar{h}_j$ is called the *Bergman kernel* of M. Let $(z_1, \cdots, z_n)$ be a local coordinate system in M and let $K_M(z) = K_M^*(z) dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$ where K_M^* is a locally defined function. Thus $\beta_M := \partial\bar{\partial} \log K_M^*$ is a well-defined Hermitian form of bidegree (1,1), whenever K_M^* is nonzero. We say that M possesses a *Bergman metric* iff β_M is everywhere positive definite. The following localization principle for the Bergman kernel (metric) is well known.

Theorem (cf. [17], [9])**.** *Let $\Omega \subset\subset \mathbf{C}^n$ be pseudoconvex. Then for any two neighborhoods $V \subset\subset U$ of a boundary point p there is a constant $C > 0$ such that*

$$\begin{aligned} K_\Omega(x) &\geq C \cdot K_{\Omega\cap U}(x), \\ \beta_\Omega(x; X) &\geq C \cdot \beta_{\Omega\cap U}(x; X) \end{aligned}$$

for all $x \in \Omega \cap V$ and $X \in T_x^{1,0}(\Omega)$.

It is natural to ask on which pseudoconvex domains in $\mathbf{P}^n$ the localization principle holds. (Notice that the complement of a hypersurface in $\mathbf{P}^n$ is pseudoconvex; however, the Bergman kernel form vanishes.) In this direction, Diederich and Ohsawa [11] proved the localization principle for another Bergman kernel (metric), which is induced by square-integrable holomorphic functions with respect to the Fubini-Study metric on any pseudoconvex domain Ω in $\mathbf{P}^n$ such that the interior of its complement is not empty. Such a Bergman metric is not invariant under biholomorphic mappings. In this note, we will show

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Theorem 1. *Let Ω be a pseudoconvex domain with C^2 boundary in $\mathbf{P}^n$. Then there is a constant $\gamma > 0$ such that for any two neighborhoods $V \subset\subset U$ of a boundary point p, one has*

$$\begin{aligned} |K_\Omega(x)|_{FS} &\geq C \cdot \frac{|K_{\Omega\cap U}(x)|_{FS}}{|\log \delta_\Omega(x)|^\gamma}, \\ \beta_\Omega(x;X) &\geq C \cdot \frac{\beta_{\Omega\cap U}(x;X)}{|\log \delta_\Omega(x)|^\gamma} \end{aligned}$$

for all $x \in \Omega \cap V$ and $X \in T_x^{1,0}(\Omega)$. Here δ_Ω (resp. $|\cdot|_{FS}$) denotes the boundary distance (resp. length) w.r.t. the Fubini-Study metric ds^2_{FS}.

As an immediate consequence of Theorem 1 and the Ohsawa-Takegoshi extension theorem [19], we obtain

Corollary. *Let Ω be as above. Then*

$$|K_\Omega(x)|_{FS} \geq C \cdot \delta_\Omega(x)^{-2} |\log \delta_\Omega|^{-\gamma}.$$

We mention that Theorem 1 cannot be generalized to an arbitrary fat pseudoconvex domain in $\mathbf{P}^n$ as the following example shows.

Example. We consider the following domain:

$$\Omega = \{[z_0, z_1, z_2] \in \mathbf{P}^2 : |z_1| < |z_0|\}$$

where (z_0, z_1, z_2) denote homogeneous coordinates in $\mathbf{P}^2$. Note that Ω is biholomorphic to the product of the unit disc and the complex plane via

$$\zeta_1 = z_1/z_0, \ \zeta_2 = z_2/z_0.$$

Hence Ω is a fat pseudoconvex domain in $\mathbf{P}^n$, but the Bergman metric is degenerate.

However, the estimate of the Bergman metric in Theorem 1 does not imply the completeness. It was shown in [5] that every hyperconvex manifold is Bergman complete, while every C^2 pseudoconvex domain in $\mathbf{P}^n$ is hyperconvex [18]; hence it is Bergman complete. We mention that there are a lot of works concerning the completeness of β_M (cf. [4], [5], [6], [12], [13], [14], [15], [17], [20]). On the other hand, Diederich and Ohsawa [10] proved that the Bergman distance for a bounded C^2 pseudoconvex domain in $\mathbf{C}^n$ has a lower bound of a constant multiple $\log|\log \delta_\Omega|$. This lower bound was improved by Blocki [3] to $|\log \delta_\Omega| / \log|\log \delta_\Omega|$. We extend his result to the complex projective space.

Theorem 2. *Let $\Omega \subset \mathbf{P}^n$ be a pseudoconvex domain with C^2 boundary. Then*

$$\mathrm{dist}_\Omega(x_0, x) \geq \frac{C|\log \delta_\Omega(x)|}{\log|\log \delta_\Omega(x)|},$$

where $\mathrm{dist}_\Omega(x_0, x)$ denotes the Bergman distance between x_0 and x.

2. Proof of Theorem 2

Let $y \in \Omega$ be arbitrary fixed. Take a smooth function $\kappa : \mathbf{R} \to [0,1]$ such that $\kappa|_{(-\infty,1/2]} = 1$ and $\kappa|_{[1,\infty)} = 0$. Let $d_{FS}(x,y)$ denote the Fubini-Study distance between two points x, y in $\mathbf{P}^n$. Let r be the injectivity radius of $\mathbf{P}^n$ and c be the upper bound of the sectional curvature. Set

$$\phi_y(x) = \kappa(d_{FS}(x,y)/r_0) \log(d_{FS}(x,y)/r_0) - 1$$

where $r_0 = \min\{r, \pi/2c\}$. By the Hessian comparison theorem (cf. [12]), there is a positive constant C_1 independent of y such that

$$\partial\bar\partial\phi_y(x) \geq -C_1 ds^2_{FS}.$$

From now on we assume $r_0 = 1$ for the sake of simplicity. By [18], there exists a smooth and strictly psh function $\rho : \Omega \to (-1, 0)$ such that $|\rho| \approx \delta_\Omega^\tau$ for some $0 < \tau < 1$ and

$$\partial\bar\partial\rho \geq C_2|\rho| ds^2_{FS}$$

where C_2 is a positive constant. Choose a cut-off function $\chi : \mathbf{R} \to [0, 1]$ such that $\chi \equiv 1$ on $[-1/2, 1/2]$ and $\chi \equiv 0$ on $[1, +\infty) \cup (-\infty, -1]$. For any $y \in M$, we set

$$\varphi_y = \chi\left(\log(-\log(-\rho)) - \log(-\log(-\rho(y)))\right)\phi_y.$$

Following an idea of [10], we will show

Lemma 3. *There exists a sufficiently large constant b such that for any $y \in \Omega$ satisfying $|\rho(y)| < 2^{-e}$, we have*
(i) $\varphi_y - \log(-\phi_y) - b\log(-\rho)$ is C^2 strictly psh on $\Omega\backslash\{y\}$;
(ii) φ_y has a logarithmic pole at y;
(iii) $\operatorname{supp}\varphi_y \subset \{x \in \Omega : |\rho(y)|^e \leq |\rho(x)| \leq |\rho(y)|^{1/e}\}$.

Proof. (ii), (iii) are trivial. We only need to show (i). By a straightforward computation, we obtain on $\Omega\backslash\{y\}$,

$$\begin{aligned}
\partial\bar\partial\varphi_y \;=\; & \frac{\phi_y}{\log(-\rho)}\left(\chi''(\cdot)\frac{\partial\log(-\rho)\bar\partial\log(-\rho)}{\log(-\rho)}\right. \\
& \left. -\chi'(\cdot)\frac{\partial\log(-\rho)\bar\partial\log(-\rho)}{\log(-\rho)} + \chi'(\cdot)\partial\bar\partial\log(-\rho)\right) \\
& +\frac{\chi'(\cdot)\phi_y}{\log(-\rho)}\left(\partial\log(-\rho)\frac{\bar\partial\phi_y}{\phi_y} + \frac{\partial\phi_y}{\phi_y}\bar\partial\log(-\rho)\right) \\
& +\chi(\cdot)\partial\bar\partial\phi_y.
\end{aligned}$$

By the Cauchy-Schwarz inequality, for any constant $\theta > 0$ one has

$$\begin{aligned}
& \pm 2\,\mathrm{Re}\left\{\partial\log(-\rho)\frac{\bar\partial\phi_y}{\phi_y}\right\} \\
\leq\;& \theta\partial\log(-\rho)\bar\partial\log(-\rho) + \theta^{-1}\partial\log(-\phi_y)\bar\partial\log(-\phi_y).
\end{aligned}$$

Since $-\rho(y) < 2^{-e}$, it follows from (iii) that $-\rho(x) < 1/2$ on $\operatorname{supp}\varphi_y$. Hence there is a positive constant C_3 depending only on the choice of χ such that

$$\begin{aligned}
\partial\bar\partial\varphi_y \;\geq\; & -\frac{C_3|\phi_y|}{|\log(-\rho)|}\{\partial\bar\partial(-\log(-\rho)) + \theta\partial\log(-\rho)\bar\partial\log(-\rho) \\
& +\theta^{-1}\partial\log(-\phi_y)\bar\partial\log(-\phi_y)\} - C_1 ds^2_{FS}.
\end{aligned}$$

Note that

$$\begin{aligned}
\operatorname{supp}\chi'(\cdot) \;\subset\; & \{x \in \Omega : |\rho(y)|^e \leq |\rho(x)| \leq |\rho(y)|^{\sqrt{e}} \\
& \text{or } |\rho(y)|^{1/\sqrt{e}} \leq |\rho(x)| \leq |\rho(y)|^{1/e}\}.
\end{aligned}$$

This implies

$$|\rho(x) - \rho(y)| \geq \frac{|\rho(x)|}{2}$$

on $\operatorname{supp}\chi'(\,\cdot\,)$. Since $|\rho| \approx \delta_\Omega^\tau$, $|\rho(x)-\rho(y)| = O(d_{FS}(x,y)^\tau)$. By the above inequality, there is a constant $C_4 > 0$ independent of y so that

$$|\phi_y| \leq C_4 |\log(-\rho)|$$

holds on $\operatorname{supp}\chi'(\,\cdot\,)$. On the other hand, one has

$$\begin{aligned} \partial\bar\partial(-\log(-\rho)) &= \frac{\partial\bar\partial\rho}{|\rho|} + \partial\log(-\rho)\bar\partial\log(-\rho) \\ &\geq C_2 ds_{FS}^2 + \partial\log(-\rho)\bar\partial\log(-\rho), \\ \partial\bar\partial(-\log(-\phi_y)) &= \frac{\partial\bar\partial\phi_y}{|\phi_y|} + \partial\log(-\phi_y)\bar\partial\log(-\phi_y) \\ &\geq -C_1 ds_{FS}^2 + \partial\log(-\phi_y)\bar\partial\log(-\phi_y) \end{aligned}$$

since $\phi_y \leq -1$. Hence after fixing sufficiently large θ, we can choose $b > 0$ such that (i) holds. The proof is complete.

Let g_Ω be the pluricomplex Green function on Ω, i.e., $g_\Omega(x,y) = \sup\{u(x)\}$ where the supremum is taken over all negative functions $u \in PSH(\Omega)$ satisfying the property that the function $u - \log|z|$ is bounded from above in a deleted neighborhood of y for some holomorphic local coordinates z centered at y, that is, $z(y) = 0$.

Lemma 4. *There is a constant $c > 0$ such that*

$$g_\Omega(x,y) > -\frac{c\rho(x)}{\rho(y)} \cdot |\log(-\rho(y))|$$

for any $x, y \in \Omega$ with $|\rho(x)| \leq |\rho(y)|/2$.

Proof. Let $\epsilon = |\rho(y)|^e$ and set

$$\lambda_y = \varphi_y - \log(-\phi_y) - b\log(-\rho) + 2b\log\epsilon.$$

By Lemma 3, λ_y is a negative psh function on $\Omega_{\epsilon^2} = \{x \in \Omega : |\rho(x)| > \epsilon^2\}$. Set

$$\eta_y = \begin{cases} \max\{\lambda_y, c_y(\rho+\epsilon^2)\}, & |\rho| \leq \epsilon, \\ \lambda_y, & |\rho| > \epsilon, \end{cases}$$

where

$$c_y = -\frac{1}{\epsilon - \epsilon^2} \cdot \inf_{|\rho(x)|=\epsilon} \lambda_y(x).$$

Then η_y is a well-defined negative psh function on Ω_{ϵ^2} such that

$$\eta_y(x) \geq c_y\rho(x) \geq C_5\rho(x)|\log\epsilon|/(\epsilon-\epsilon^2) \geq -C_6$$

for all $|\rho(x)| \leq \epsilon^{3/2}$. Here C_5, C_6 are positive constants independent of y. Set

$$\tilde\rho = -C_6 \frac{\log(-\rho+\epsilon^2) - \log(2\epsilon^2)}{\log(\epsilon^{3/2}+\epsilon^2) - \log(2\epsilon^2)}.$$

Clearly, $\tilde\rho$ is a psh function on Ω such that $\tilde\rho \leq C_7$ where C_7 is independent of ϵ. Note that

$$\begin{aligned} \tilde\rho(x) = -C_6 \leq \eta_y(x), &\quad \text{if} \quad |\rho(x)| = \epsilon^{3/2}, \\ \tilde\rho(x) = 0 = \eta_y(x), &\quad \text{if} \quad |\rho(x)| = \epsilon^2. \end{aligned}$$

Hence the function defined by

$$\mu_y = \begin{cases} \eta_y, & \text{on } \{|\rho| > \epsilon^{3/2}\}, \\ \max\{\eta_y, \tilde{\rho}\}, & \text{on } \{\epsilon^2 \le |\rho| \le \epsilon^{3/2}\}, \\ \tilde{\rho}, & \text{on } \{|\rho| < \epsilon^2\} \end{cases}$$

is well-defined, psh on Ω and has a logarithmic pole at y. Set

$$\nu_y = \begin{cases} \max\{\mu_y - C_7, \tilde{c}_y \rho\}, & \rho(x) \ge \frac{1}{2}\rho(y), \\ \mu_y - C_7, & \rho(x) < \frac{1}{2}\rho(y), \end{cases}$$

where

$$\tilde{c}_y = \frac{2}{\rho(y)} \inf_{\rho(x) = \frac{1}{2}\rho(y)} (\mu_y(x) - C_7).$$

If $\rho(x) \ge \frac{1}{2}\rho(y)$, then we have

$$\frac{|\rho(y)|}{2} \le |\rho(x) - \rho(y)| \le c' \cdot d_{FS}(x,y)^\tau$$

for a suitable constant $c' > 0$, which implies

$$\tilde{c}_y \le \frac{C_8 |\log(-\rho(y))|}{|\rho(y)|}.$$

Hence

$$g_\Omega(x,y) \ge \tilde{c}_y \rho(x) \ge -C_8 \frac{\rho(x)}{\rho(y)} |\log(-\rho(y))|.$$

The following is the key step in proving Theorem 2. The main idea comes from [3].

Proposition 5. *There is a constant $C > 0$ such that for any $y \in \Omega$ with $|\rho(y)| < e^{-1}$ one has*

$$\begin{aligned} & \{x \in \Omega : g_\Omega(x,y) < -1\} \\ \subset \; & \{x \in \Omega : C^{-1}|\rho(y)| \cdot |\log(-\rho(y))|^{-1} \le |\rho(x)| \le C|\rho(y)| \cdot |\log(-\rho(y))|^n\}. \end{aligned}$$

Proof. From [2], we know that for any nonnegative psh functions u, v defined on a smooth bounded domain D in a Stein manifold with $u|_{\partial D} = 0$, then

$$\int_D |u|^n (dd^c v)^n \le n! \|v\|_\infty^{n-1} \int_D |v| (dd^c u)^n$$

where $d^c = i(\bar{\partial} - \partial)$. Fix arbitrary $x, y \in \Omega$ with $\rho(x) \le 2\rho(y)$. Set $\epsilon = |\rho(y)|^e$ and $\alpha = -\frac{2}{\tau}(b+1)\log\epsilon$. We exhaust Ω by a sequence of smooth strongly pseudoconvex domains Ω_j, $j = 1, 2, \cdots$. By the above inequality, we have

$$\begin{aligned} & \int_{\Omega_j} |g_{\Omega_j}(\cdot, y)|^n (dd^c \max\{g_{\Omega_j}(\cdot, x), -\alpha\})^n \\ \le \; & n! \alpha^{n-1} \int_{\Omega_j} |\max\{g_{\Omega_j}(\cdot, x), -\alpha\}| (dd^c g_{\Omega_j}(\cdot, y))^n \\ \le \; & n!(2\pi)^n \alpha^{n-1} |g_{\Omega_j}(y, x)| \end{aligned}$$

since $(dd^c g_{\Omega_j}(\cdot, y))^n = (2\pi)^n \delta_y$ (cf. [8]). It is also known from [8] that the measure $(dd^c \max\{g_{\Omega_j}(\cdot, x), -\alpha\})^n$ is supported on $\{g_{\Omega_j}(\cdot, x) = -\alpha\}$ with total mass $(2\pi)^n$.

Hence

$$\begin{aligned}\inf_{\{g_\Omega(\cdot,x)=-\alpha\}}|g_\Omega(\cdot,y)|^n &\leftarrow \inf_{\{g_{\Omega_j}(\cdot,x)=-\alpha\}}|g_{\Omega_j}(\cdot,y)|^n\\ &\le n!\alpha^{n-1}|g_{\Omega_j}(y,x)|\\ &\to n!\alpha^{n-1}|g_\Omega(y,x)| \qquad (1)\end{aligned}$$

as $j\to\infty$. According to Lemma 4, one has $g_\Omega(z,x)>-1$ provided $|\rho(z)|\le|\rho(x)|^e$. On the other hand, for any $z,x\in\Omega$ with $|\rho(z)|>|\rho(x)|^e$ one has

$$\begin{aligned}g_\Omega(z,x) &\ge \mu_x - C_7 = \lambda_x - C_7\\ &\ge \varphi_x(z)-\log(-\phi_x(z))+2be\log|\rho(x)|-C_7\\ &\ge \varphi_x(z)-\log(-\phi_x(z))+\frac{2b}{\tau}\log\epsilon - C_8,\end{aligned}$$

which implies

$$\{g_\Omega(\cdot,x)=-\alpha\}\subset B(x,\epsilon):=\{d_{FS}(\cdot,x)<\epsilon^{1/\tau}\}$$

provided ϵ is sufficiently small. Hence, by (1) there exists $\widetilde{x}\in B(x,\epsilon^{1/\tau})$ such that

$$|g_\Omega(\widetilde{x},y)|^n\le C_9|\log\epsilon|^{n-1}|g_\Omega(y,x)|. \qquad (2)$$

By Bertini's Lemma, one can take a generic hyperplane H such that it does not contain $x,\tilde{x},y$. On $\mathbf{P}^n\backslash H$ one can introduce inhomogeneous coordinates $w=(w_1,\cdots,w_n)$. We can also choose H such that $|w(x)-w(\tilde{x})|\approx d_{FS}(x,\tilde{x})$ where the implicit constants depend only on Ω. One can regard $\Omega\backslash H$ as an unbounded C^2 pseudoconvex domain in $\mathbf{C}^n$. Set

$$\widetilde{\Omega}=\{w\in\Omega\backslash H: w+w(\widetilde{x})-w(x)\in\Omega\backslash H\}.$$

Since $ds^2_{FS}=\partial\bar\partial\log(1+|w|^2)\le\partial\bar\partial|w|^2$ on $\mathbf{P}^n\backslash H$, there is a constant $C_{10}>0$ such that

$$\partial\widetilde{\Omega}\cap(\Omega\backslash H)\subset\{\delta_\Omega<C_{10}\epsilon^{1/\tau}\}.$$

Therefore,

$$h(w)=\begin{cases}\max\{g_\Omega(w,w(y)),g_\Omega(w+w(\widetilde{x})-w(x),w(y))-\delta\}, & w\in\widetilde{\Omega},\\ g_\Omega(w,w(y)), & w\in(\Omega\backslash H)\backslash\widetilde{\Omega},\end{cases}$$

where $\delta=\sup_{\delta_\Omega<C_{10}\epsilon^{1/\tau}}|g_\Omega(\cdot,w(y))|$ is a well-defined negative psh function with a logarithmic pole at $w(y)$ on $\Omega\backslash H$. Since H is an analytic subset, h extends to a psh function on the whole of Ω. Therefore,

$$g_\Omega(x,y)\ge h(w(x))\ge g_\Omega(\widetilde{x},y)-\delta. \qquad (3)$$

By (2), (3), for any $x,y\in\Omega$ with $\rho(x)\le 2\rho(y)$, one has

$$\begin{aligned}|g_\Omega(x,y)| &\le \delta+C_{11}|\log\epsilon|^{1-\frac{1}{n}}|g_\Omega(y,x)|^{1/n}\\ &\le \frac12+C_{12}\left(\frac{\rho(y)}{\rho(x)}\right)^{1/n}|\log(-\rho(y))|\end{aligned}$$

according to Lemma 4. The proof is complete.

Proof of Theorem 2. We follow the argument as in [10]. Let $y_1,y_2\in\Omega$ be two arbitrary points satisfying

$$|\rho(y_2)|<2^{-e},\ C|\rho(y_1)|\cdot|\log(-\rho(y_1))|^n\le C^{-1}|\rho(y_2)|\cdot|\log(-\rho(y_2))|^{-1}.$$

We take a complete orthonormal basis $\{h_j\}_{j=0}^{\infty}$ for $\mathcal{H}$ such that $h_j(y_2)=0$ for all $j\ge 1$. According to Kobayashi [15], we can immerse M into the infinite-dimensional complex projective space $\mathbf{CP}(\mathcal{H})$ via the map

$$\sigma : x \mapsto (h_0(x) : h_1(x) : \cdots).$$

Since each point $P=(\zeta_0:\zeta_1:\cdots)$ in the projective space corresponds to an entire great circle of the unit sphere consisting of points $(\zeta_0 e^{i\theta}, \zeta_1 e^{i\theta}, \cdots)$, then the Fubini-Study distance between two points P, Q is equal to the distance in the spherical geometry between the corresponding great circles. By the choice of the basis, we have $\sigma(y_2)=(1:0:\cdots)$ and $\sigma(y_1)=(a_0:a_1:\cdots)$ where $a_j = h_j^*(y_1)/\sqrt{K_\Omega^*(y_1)}$. Hence,

$$\begin{aligned}
\mathrm{dist}_\Omega(y_1,y_2) &\ge \mathrm{dist}_{FS}(\sigma(y_1),\sigma(y_2)) \\
&\ge \inf_{\theta_1,\theta_2} |e^{i\theta_1}(a_0,a_1,\cdots) - e^{i\theta_2}(1,0,\cdots)| \\
&= \sqrt{(1-|a_0|)^2 + \sum_{j=1}^{\infty} |a_j|^2}.
\end{aligned}$$

Therefore, if $|a_0|\le 1/2$, then $\mathrm{dist}_\beta(y_1,y_2)\ge 1/2$. Otherwise, take a smooth function λ on $\mathbf{R}$ such that $\lambda=1$ on $(-\infty,-1]$ and $\lambda=0$ on $[0,\infty)$. Set

$$\begin{aligned}
\eta &= \lambda\left(-\log(-g_\Omega(\cdot,y_1)+1)+\log 2\right) h_0, \\
\varphi &= 2n(g_\Omega(\cdot,y_1)+g_\Omega(\cdot,y_2)) - \log(-g_\Omega(\cdot,y_1)+1).
\end{aligned}$$

By Proposition 5, we see that $\{g_\Omega(\cdot,y_1)<-1\}\cap\{g_\Omega(\cdot,y_2)<-1\}=\emptyset$. By the well-known L^2 estimates (cf. [7], [16]), we can solve the equation $\overline{\partial} u = \overline{\partial}\eta$ in such a way that

$$\begin{aligned}
\left|\int_\Omega u\wedge \bar{u} e^{-\varphi}\right| &\le \left|\int_\Omega |\bar\partial\lambda|^2_{\partial\bar\partial\varphi}\, h_0\wedge \bar h_0 e^{-\varphi}\right| \\
&\le C_{12}
\end{aligned}$$

since $\partial\bar\partial\varphi \ge (-g_\Omega(\cdot,y_1)+1)^{-2}\partial g_\Omega(\cdot,y_1)\bar\partial g_\Omega(\cdot,y_1)$ holds in the sense of distribution. Therefore, $F=\eta-u$ is holomorphic on Ω and satisfies $F(y_1)=h_0(y_1)$, $F(y_2)=0$ and

$$\left|\int_\Omega F\wedge\overline{F}\right| \le C_{13}.$$

Hence

$$\mathrm{dist}_\Omega(y_1,y_2) \ge \sqrt{\sum_{j=1}^{\infty}|a_j|^2} \ge \sqrt{\frac{F(y_1)\wedge\bar F(y_1)}{C_{13}K_\Omega(y_1)}}$$

$$= \sqrt{\frac{h_0(y_1)\wedge\bar h_0(y_1)}{C_{13}K_\Omega(y_1)}} = \frac{|a_0|}{\sqrt{C_{13}}} \ge \frac{1}{2\sqrt{C_{13}}}.$$

Now if $c_0, c_1, \cdots, c_k$ are finite increasing positive numbers such that $c_k \le 2^{-e}$ and

$$C^{-1}c_k|\log c_k|^{-1} = Cc_{k-1}|\log c_{k-1}|^n,$$

then

$$c_k \le C^2 c_{k-1}|\log c_{k-1}|^n \le C^4 c_{k-2}|\log c_{k-2}|^{2n} \le \cdots \le C^{2k}c_0|\log c_0|^{nk}.$$

Given $y \in \Omega$, fix a point y_0 with $|\rho(y_0)| = 2^{-e}$. Take a Bergman geodesic l connecting y_0, y. Let $c_0 = |\rho(y)|$, $c_k = 2^{-e}$. Take $y_i \in l$ with $|\rho(y_i)| = c_i$, $i = 0, 1, \cdots, k$. Then

$$\mathrm{dist}_\Omega(y_0, y) \geq \sum_{i=0}^{k-1} \mathrm{dist}_\Omega(y_i, y_{i+1}) \geq C_{14} k,$$

from which the desired estimate follows.

3. Proof of Theorem 1

Let $y \in \Omega \cap V$ be an arbitrary point with $|\rho(y)| < e^{-1}$. Set $\tilde{\epsilon} = C^{-1}|\rho(y)| \cdot |\log(-\rho(y))|^{-1}$ and $\Omega_{\tilde{\epsilon}} = \{x \in \Omega : |\rho(x)| > \tilde{\epsilon}\}$. Here C is the constant in Proposition 5. Set

$$\begin{aligned} \tilde{\lambda}_y &= \varphi_y - \log(-\phi_y) - b\log(-\rho) + b\log\tilde{\epsilon}, \\ \psi_y &= \max\{\tilde{\lambda}_y, g_\Omega(\cdot, y)\}. \end{aligned}$$

Then ψ_y is a negative psh function with a logarithmic pole at y on $\Omega_{\tilde{\epsilon}}$ such that

$$\begin{aligned} &\{x \in \Omega_{\tilde{\epsilon}} : \psi_y(x) < -1\} \subset \{x \in \Omega_{\tilde{\epsilon}} : g_\Omega(x, y) < -1\} \\ &\subset \{x \in \Omega_{\tilde{\epsilon}} : |\rho(x)| \leq C|\rho(y)| \cdot |\log(-\rho(y))|^n\}. \end{aligned} \tag{4}$$

On the other hand, for any $0 < \tilde{r} < r_0$, there is a positive constant $\tilde{C}$ depending only on $\tilde{r}$ such that

$$\psi_y(x) \geq \tilde{\lambda}_y(x) \geq -\tilde{C} - b(n+1)\log|\log(-\rho(y))|, \ \forall\, x \in \{\psi_y < -1\} \backslash B(y, \tilde{r}). \tag{5}$$

Without loss of generality, we assume that $B(y, 2\tilde{r}) \subset U$. Choose a holomorphic n-form f on $\Omega \cap U$ with unit L^2-norm such that $f \wedge \bar{f}(y) = K_{\Omega\cap U}(y)$. Let κ, λ be the cut-off functions as above. Set

$$\begin{aligned} \tilde{\varphi}_y &= 2n\psi_y - \log(-\psi_y + 1), \\ v &= \bar{\partial}(\lambda\,(-\log(-\psi_y + 1) + \log 2)\,\kappa(d_{FS}(\cdot, y)/2\tilde{r}) f) \end{aligned}$$

on $\Omega_{\tilde{\epsilon}}$. We recall the following L^2 estimate:

Theorem (cf. [10], [1]). *Let M be a Stein manifold. Let φ, ψ be psh functions such that $r\partial\bar{\partial}\psi \geq \partial\psi\bar{\partial}\psi$ holds in the sense of distribution for some $0 < r < 1$. Then for any $\bar{\partial}$-closed $(n,1)$ form v with $\int_M |v|^2_{\partial\bar{\partial}(\varphi+\psi)} e^{\psi-\varphi} < +\infty$, there exists an $(n,0)$ form u on M such that $\bar{\partial}u = v$ and*

$$\left|\int_M u \wedge \bar{u} e^{\psi-\varphi}\right| \leq C_r \int_M |v|^2_{\partial\bar{\partial}(\varphi+\psi)} e^{\psi-\varphi}.$$

We apply this theorem with $\psi = -\frac{1}{2}\log(-\rho)$, $\varphi = \tilde{\varphi}_y + \psi$ to get a solution u of $\bar{\partial}u = v$ such that

$$\left|\int_{\Omega_{\tilde{\epsilon}}} u \wedge \bar{u} e^{-\tilde{\varphi}_y}\right| \leq \tilde{C}_2|\log(-\rho(y))|^\gamma \leq \tilde{C}_3|\log\delta_\Omega(y)|^\gamma$$

because of $(4), (5)$ and

$$\partial\bar{\partial}(\tilde{\varphi}_y + \psi) \geq \partial\log(-\psi_y + 1)\bar{\partial}\log(-\psi_y + 1) + \frac{C_1}{2} ds^2_{FS}.$$

Here $\gamma > 0$ depends only on b, n and $\tilde{C}_2$, $\tilde{C}_3$ are independent of y. Thus we obtain a holomorphic n-form

$$F = \lambda\,(-\log(-\psi_y + 1) + \log 2)\,\kappa(d_{FS}(\cdot, y)/2\tilde{r})f - u$$

on $\Omega_{\tilde{\epsilon}}$ such that $F \wedge \bar{F}(y) = K_{\Omega\cap U}(y)$ and its L^2-norm is bounded above by a constant multiple of $|\log \delta_\Omega(y)|^\gamma$. Finally, we apply the above theorem with

$$\begin{aligned} \psi &= -\frac{1}{2}\log(-g_\Omega(\cdot, y) + 1), \\ \varphi &= 2ng_\Omega(\cdot, y) + \psi \end{aligned}$$

to get a solution of

$$\bar{\partial}u = v := \bar{\partial}(\lambda\,(-\log(-g_\Omega(\cdot, y) + 1) + \log 2)\,F)$$

on Ω such that

$$\left|\int_\Omega u \wedge \bar{u}e^{-2ng_\Omega(\cdot,y)}\right| \le \tilde{C}_4 \left|\int_{\Omega_{\tilde{\epsilon}}} F \wedge \overline{F}\right|.$$

Set $\tilde{F} = \lambda\,(-\log(-g_\Omega(\cdot, y) + 1) + \log 2)\,F - u$. Then $\tilde{F}$ is a holomorphic n-form on Ω satisfying $\tilde{F}(y) = F(y)$ and the L^2-norm bounded above by a constant multiple of $|\log \delta_\Omega(y)|^\gamma$, from which we obtain the estimate of the Bergman kernel. The argument for the Bergman metric is similar.

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References

[1] B. Berndtsson and Ph. Charpentier, *A Sobolev mapping property of the Bergman kernel*, Math. Z. **235** (2000), 1–10. MR1785069 (2002a:32039)

[2] Z. Blocki, *Estimates for the complex Monge-Ampère operator*, Bull. Pol. Acad. Sci. **41** (1993), 151–157. MR1414762 (97j:32009)

[3] ———, *The Bergman metric and the pluricomplex Green function*, MPI (Leipzig) preprint no: 85 (2002).

[4] Z. Blocki and P. Pflug, *Hyperconvexity and Bergman completeness*, Nagoya Math. J. **151** (1998), 221–225. MR1650305 (2000b:32065)

[5] B. Y. Chen, *Bergman completeness of hyperconvex manifolds*, preprint.

[6] B. Y. Chen and J. H. Zhang, *The Bergman metric on a Stein manifold with a bounded plurisubharmonic function*, Trans. Amer. Math. Soc. **354** (2002), 2997–3009. MR1897387 (2003c:32014)

[7] J. P. Demailly, *Estimations L^2 pour l'opérateur d'un fibré vectoriel holomorphe semi-positiv au dessus d'une variété kählérienne complète*, Ann. Sci. Éc. Norm. Sup. **15** (1982), 457–511. MR0690650 (85d:32057)

[8] ———, *Mesures de Monge-Ampère et mesures pluriharmoniques*, Math. Z. **194** (1987), 519–564. MR0881709 (88g:32034)

[9] K. Diederich, J. E. Fornaess and G. Herbort, *Boundary behavior of the Bergman metric*, Complex Analysis of Several Variables (Madison, Wis., 1982), 59–67, Proc. Sympos. Pure Math. **41**, Amer. Math. Soc., Providence, RI, 1984. MR0740872 (85j:32039)

[10] K. Diederich and T. Ohsawa, *An estimate for the Bergman distance on pseudoconvex domains*, Ann. of Math. **141** (1995), 181–190. MR1314035 (95j:32039)

[11] ———, *On pseudoconvex domains in* $\mathbf{P}^n$, Tokyo J. Math. **21** (1998), 353–358. MR1663574 (99k:32024)

[12] R. E. Greene and H. Wu, *Function theory on manifolds which possess a pole*, Lecture Notes in Mathematics **699**, Springer-Verlag, 1979. MR0521983 (81a:53002)
[13] G. Herbort, *The Bergman metric on hyperconvex domains*, Math. Z. **232** (1999), 183–196. MR1714284 (2000i:32020)
[14] M. Jarnicki and P. Pflug, *Bergman completeness of complete circular domains*, Ann. Pol. Math. Vol 50 (1989), 219–222. MR1044868 (91f:32026)
[15] S. Kobayashi, *Geometry of bounded domains*, Trans. Amer. Math. Soc. **92** (1959), 267–290. MR0112162 (22:3017)
[16] T. Ohsawa, *Boundary behavior of the Bergman kernel function on pseudoconvex domains*, Publ. RIMS, Kyoto Univ. **20** (1984), 897–902. MR0764336 (86d:32025)
[17] ———-, *On the completeness of the Bergman metric*, Proc. Jap. Acad. Sci. 57, Ser. A (1981), 238–240. MR0618233 (82j:32053)
[18] T. Ohsawa and N. Sibony, *Bounded p.s.h. functions and pseudoconvexity in Kähler manifolds*, Nagoya Math. J. **149** (1998), 1–8. MR1619572 (2000b:32062)
[19] T. Ohsawa and K. Takegoshi, *On the extension of L^2 holomorphic functions*, Math. Z. **195** (1987), 197–204. MR0892051 (88g:32029)
[20] W. Zwonek, *Completeness, Reinhardt domains and the method of complex geodesics in the theory of invariant functions*, Dissert. Math. **388** (2000), 103 pp. MR1785672 (2001h:32016)

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