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LINEAR FUNCTIONALS ON THE CUNTZ ALGEBRA

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ABSTRACT. For a pure state p' on $\mathcal{O}_n$, which is an extension of a pure state p on UHF_n with the property that if $(\mathcal{H}_{p'}, \pi_{p'}, \omega_{p'})$ is a corresponding representation, then $\pi_{p'}(\mathrm{UHF}_n) = B(\mathcal{H}_{p'})$, p' induces a unital shift of $B(\mathcal{H})$ of the Powers index n. We describe states p on UHF_n by using sequences of unit vectors in $\mathbb{C}^n$. We study the linear functionals on the Cuntz algebra $\mathcal{O}_n$ whose restrictions are the product pure state on UHF_n. We find conditions on the sequence of unit vectors for which the corresponding linear functionals on $\mathcal{O}_n$ become states under these conditions.

1. INTRODUCTION

A non-degenerate representation of the Cuntz algebra $\mathcal{O}_n$ induces a unital endomorphism of $B(\mathcal{H})$ of Powers index n. We consider the uniformly hyperfinite algebra UHF_n as a subalgebra of the Cuntz algebra $\mathcal{O}_n$. For a representation π of the Cuntz algebra $\mathcal{O}_n$, $\pi(\mathrm{UHF}_n)$ is weakly dense in $B(\mathcal{H})$ if and only if the corresponding unital endomorphism of $B(\mathcal{H})$ is a shift of Powers index n [BJP]. By GNS construction, a study of the representation of the C^*-algebra $\mathcal{O}_n$ is equivalent to a study of its state. Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and denote by $B(\mathcal{H})$ the von Neumann algebra of all linear operators on $\mathcal{H}$. The term endomorphism will denote a $*$-homomorphism of $B(\mathcal{H})$ into itself. The study of endomorphisms of the von Neumann algebras, especially of $B(\mathcal{H})$, has received increased attention in connection with several related areas such as the Jones index for subfactor duality for compact groups [Jon], [Wor]. Let $\varphi\colon B(\mathcal{H}) \to B(\mathcal{H})$ be a unital endomorphism. The Powers index $n \in \{1, 2, \cdots, \infty\}$ of φ is defined by n such that $\varphi(B(\mathcal{H}))' \cap B(\mathcal{H})$ is isomorphic to the factor of type I_n [Pow]. A unital endomorphism $\varphi\colon B(\mathcal{H}) \to B(\mathcal{H})$ is said to be a shift if $\bigcap_{k=1}^{\infty} \varphi^k(B(\mathcal{H})) = \mathbb{C}1$. In [BJP] and [Lac], the authors study the program initiated by Powers of analyzing the conjugacy classes of discrete shifts on $B(\mathcal{H})$. The main theme is to describe the correspondence between endomorphisms of Powers index $n \in \{1, 2, \cdots, \infty\}$ and representations of the Cuntz algebra $\mathcal{O}_n$ which implement the endomorphisms. Theorem 1.1 in [BJP] and Theorem 4.5 in [Lac] show a characterization of the conjugacy study that is focused on the pure states on UHF_n and their extensions on $\mathcal{O}_n$ arising in the study of the correspondence between endomorphisms, especially the shift of Powers index n of $B(\mathcal{H})$, and as representations of $\mathcal{O}_n$ as described above. We restrict n as finite throughout this paper.

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Definition 1.1 ([Pow]). A uniformly hyperfinite algebra UHF is a C^*-algebra which is the norm closure of an increasing sequence of type I_{n_i}-factors that can be identified with $\bigotimes_{i=1}^{\infty} M_{n_i}$. A UHF$_n$ algebra is the UHF algebra Glimm type n^∞, that is, $\bigotimes M_n$.

Definition 1.2 ([Cun]). The Cuntz algebra $\mathcal{O}_n$, $n = 2, 3, \ldots$, is the universal C^*-algebra generated by n isometries $s_1, s_2, \cdots, s_n$ subject to the relations

$$s_i^* s_j = \delta_{ij} 1 \quad \text{and} \quad \sum_{i=1}^{n} s_i s_i^* = 1. \tag{1.1}$$

In this paper, we consider UHF$_n$ as a subalgebra of $\mathcal{O}_n$. That is, UHF$_n$ is the closure of the linear span of all Wick-ordered monomials of the form

$$s_{i_1} s_{i_2} \cdots s_{i_k} s_{j_k}^* s_{j_{k-1}}^* \cdots s_{j_1}^* \tag{1.2}$$

over $k = 0, 1, 2, \ldots$.

Theorem 1.3 (Theorem 2.1 in [Arv], Proposition 2.2 in [Lac], and Theorem 3.1 in [BJP]). *Let φ be a unital endomorphism of $B(\mathcal{H})$ of Powers index $n = 2, 3, \ldots$. It follows that there exists a non-degenerate representation of $\mathcal{O}_n$ on $\mathcal{H}$ such that*

$$\varphi(X) = \sum_{i=1}^{n} S_i X S_i^* \tag{1.3}$$

where S_i is the representative of s_i. Conversely, any non-degenerate representation of $\mathcal{O}_n$ on $\mathcal{H}$ defines an endomorphism of index n by (1.3). *The representation is unique up to the canonical action of the unitary group $\mathcal{U}(n)$.*

A canonical representation of the n-dimensional unitary group $\mathcal{U}(n)$ in the automorphism group of $\mathcal{O}_n$ is defined by $\mathcal{T}_g(s_i) = \sum_{j=1}^{n} g_{ji} s_j$ for $g = [g_{ij}]_{i,j=1}^{n} \in \mathcal{U}(n)$. The canonical action of $\mathcal{U}(n)$ on $\mathcal{O}_n$ gives rise to an action of $\mathcal{U}(n)$ on each of these spaces. Also, the unitary group $\mathcal{U}(n)$ on $\mathcal{H}$ acts on each of these spaces where $U \in U(\mathcal{H})$, and π is the map

$$\pi(\cdot) \to U\pi(\cdot)U*$$

for non-degenerate representations of $\mathcal{O}_n$ on $\mathcal{H}$. Details are in [BJP].

Theorem 1.4 (Theorem 3.3 in [BJP]). *Let $\pi \to \varphi(\pi)$ be the surjective map from the set of non-degenerate representations of $\mathcal{O}_n$ on $\mathcal{H}$ onto the set of unital endomorphisms of $B(\mathcal{H})$ of Powers index n. Then $\pi(\mathrm{UHF}_n)$ is weakly dense in $B(\mathcal{H})$ if and only if $\varphi(\pi)$ is a unital shift on $B(\mathcal{H})$ of Powers index n. These representations identify with the cycle representation defined by any vector state, defined by a unit vector in $\mathcal{H}$. The corresponding states on $\mathcal{O}_n$ can be characterized abstractly by the restriction of those states to* UHF$_n$. *Let P denote the set of pure states p on* UHF$_n$ *such that it has a pure extension p' to $\mathcal{O}_n$ with the property that, if $(\mathcal{H}_{p'}, \pi_{p'}, \Omega_{p'})$ is the corresponding representation, then $\pi_{p'}(\mathrm{UHF}_n)'' = B(\mathcal{H}_{p'})$.*

2. An extension on $\mathcal{O}_n$ of the pure state on UHF$_n$

With the identification UHF$_n$ as an infinite tensor product $\bigotimes M_n$, an element

$$s_{i_1} \cdots s_{i_k} s_{j_k}^* \cdots s_{j_1}^* \in \mathrm{UHF}_n \subset \mathcal{O}_n$$

can be identified with

$$E_{i_1 j_1} \otimes \cdots \otimes E_{i_1 j_1} \otimes I \otimes I \otimes \cdots,$$

where E_{ij} is the matrix in M_n whose (i,j)-element is 1 and the others are 0. Let P be the set of pure states p on UHF_n which has a pure state extension p' to $\mathcal{O}_n$ with the property that if $(\mathcal{H}_{p'}, \pi_{p'}, \Omega_{p'})$ is the corresponding representation, then $\pi_{p'}(\mathrm{UHF}_n)'' = B(\mathcal{H}_{p'})$. Let $p_i = p|_{M_{n_i}}$, where M_{n_i} is the i-th M_n in the infinite tensor product $\bigotimes M_n$. Then p_i is a pure state on M_n. Since $p = \bigotimes_{i=1}^{\infty} p_i$, the pure state p is a product pure state. We are going to study pure states p on UHF_n for a p_i vector state on $\mathbb{C}^n$ and find their pure extension p' on $\mathcal{O}_n$.

Definition 2.1. Let $a_m = (a_m^1, \ldots, a_m^n)$ be a unit vector in $\mathbb{C}^n$ and let $\{a_m\}_{m=1}^{\infty}$ be a sequence of unit vectors. A pure state μ_{a_m} on UHF_n is defined by

$$\mu_{a_m}(E_{i_1 j_1} \otimes E_{i_2 j_2} \otimes \cdots \otimes E_{i_l j_l} \otimes I \otimes \cdots) = a_1^{i_1} a_1^{\overline{j}_1} a_2^{i_2} a_2^{\overline{j}_1} \cdots a_l^{i_l} a_2^{\overline{j}_l} \tag{2.1}$$

for $D_{i_1 j_1} \otimes E_{i_2 j_2} \otimes \cdots \otimes E_{i_l j_l} \otimes I \otimes \cdots \in \mathrm{UHF}_n$.

Definition 2.2. For a sequence $\{a_m\}_{m=1}^{\infty}$ of unit vectors in $\mathbb{C}^n$, the linear functional μ'_{a_m} on the Cuntz algebra $\mathcal{O}_n$ is defined by

$$\mu'_{a_m}(s_{i_1} \cdots s_{i_l} s_{i_k}^* \cdots s_{j_1}^*) = a_1^{i_1} \cdots a_l^{i_l} a_k^{\overline{j}_k} \cdots a_2^{\overline{j}_l}. \tag{2.2}$$

Since $\mu'_{a_m}|\mathrm{UHF}_n = \mu_{a_m}$ is a pure state on UHF_n, $\mu'_{a_m} \in P$ if μ'_{a_m} is a pure state on $\mathcal{O}_n$. The linear functional μ'_{a_m} on $\mathcal{O}_n$ coming from (2.2) is not a state in general, but a linear functional on $\mathcal{O}_n$.

The following is an example of this.

Example 2.3. Let $n = 2$, $a_1 = (-1, 0)$ and $a_n = (1, 0)$ for all $m = 2, 3, \ldots$. If $\sqrt{3}x = s_1^*(1 + s_1 + s_1 s_1)$, then

$$\begin{aligned} \mu'_{a_m}(x^* x) =& \frac{1}{3} \mu'_{a_m}(2 + 2s_1 + 2s_1^* + s_1 s_1 + s_1 s_1^* + s_1^* s_1^*) \\ =& \frac{1}{3}\{2 + 2(-1) + 2(-1) + (-1)1 + (-1)(-1) + (-1)(1)\} \\ =& -1. \end{aligned}$$

If $a \sim b$ means $a = \lambda b$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, then $\sim$ is an equivalence relation on $\mathbb{C}$. Let $\mathbf{a}$ be a representative of an equivalence class of a.

Theorem 2.4. *There is a one-to-one correspondence between the set of sequences $\{\mathbf{a}_m\}_m$ for unit vectors $a_m \in \mathbb{C}^n$ and the set of product pure states on the* UHF_n *algebra.*

Proof. If we set $p_i(x) = \langle x a_m, a_m \rangle$, then the proof follows straightforwardly with $p(\cdot) = \otimes \langle \cdot a_m, a_m \rangle$ and $\langle \cdot a_m, a_m \rangle = \langle \cdot \lambda a_m, \lambda a_m \rangle$ for $|\lambda| = 1$.

3. Conditions of $\{a_m\}$ to be a state on $\mathcal{O}_n$

As we showed in Example 2.3, the linear functional on $\mathcal{O}_n$ defined in (2.2) is not a state on $\mathcal{O}_n$ in general. However, the state $\mu_{\{a_n\}}|_{\mathrm{UHF}_n} = \mu_{a_m}$ on UHF_n always has a pure state extension on $\mathcal{O}_n$ [BJP]. A pure state extension of the product pure state μ_{a_m} on UHF_n may differ from the one defined in (2.2). We first look for conditions of the sequence $\{a_m\}$ so that the linear functional $\mu'_{\{a_m\}}$ becomes a pure state on the Cuntz algebra $\mathcal{O}_n$.

Theorem 3.1. *For the state μ on $\mathcal{O}_n$, $\mu(s_i s_i^*) = \mu(s_i)\mu(s_i^*)$ if $\sum_{i=1}^n |\mu(s_i)|^2 = 1$.*

Proof. Let $N = \begin{pmatrix} \mu(s_1^*) \cdots \mu(s_n)^* \\ 0 \cdots 0 \\ 0 \cdots 0 \end{pmatrix}$ be an $n \times n$ matrix in $M_n(\mathcal{O}_n)$. For two unit vectors a and b in $\mathbb{C}^n$, let c and d be the real numbers satisfying

$$aNb = ce^{id}.$$

For any real number x, the quadratic equation in x,

$$\begin{aligned} &x^2 + 2cx + a \\ &= (xe^{id}a, b) \begin{pmatrix} 1 & & & 0 & \mu(s_1^*) & \cdots & \mu(s_n^*) \\ & \ddots & & & 0 & \cdots & 0 \\ 0 & & & 1 & 0 & \cdots & 0 \\ \mu(s_1) & 0 & \cdots & 0 & \mu(s_1 s_1^*) & \cdots & \mu(s_1 s_n^*) \\ \ddots & & & & \ddots & & \ddots \\ \mu(s_n) & 0 & \cdots & 0 & \mu(s_n s_1^*) & \cdots & \mu(s_n s_n^*) \end{pmatrix} (xe^{id}a, b)^* \\ &= (xe^{id}a, b) \begin{pmatrix} I_n & N \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} I_n & N \\ 0 & \end{pmatrix} (xe^{id}a, b)^*, \end{aligned}$$

is non-negative with $\alpha = b(\mu(s_i s_j^*))_{i,j=1}^n b^*$ and $c = e^{id}aNb$. Since $x^2 + 2cx + \alpha \geq 0$, we have $D = 4C^2 - 4\alpha \leq 0$. Then

$$|c|^2 = |e^{id}aNb|^2 \leq b(\mu(s_i s_j^*))b^* = \alpha,$$

and

$$|\langle aN, b\rangle|^2 \leq \langle b(\mu(s_i s_j^*)), b\rangle.$$

When $aN \neq 0$, if we replace a by the normalized unit vector $\frac{bN^*}{\|bN*\|}$, then

$$\langle bN^*N, b\rangle \leq \langle b(\mu(s_1 s_j^*)), b\rangle.$$

We have $N^*N \leq \mu(s_i s_j^*)$ which implies

$$\begin{pmatrix} \mu(\bar{s}_1^*) & 0 & \cdots & 0 \\ \cdots & & & \\ \mu(\bar{s}_n^*) & 0 & \ddots & 0 \end{pmatrix} \begin{pmatrix} \mu(s_1^*) & \cdots & 0 \\ 0 & \cdots & \\ \ddots & & \ddots \\ 0 & \cdots & 0 \end{pmatrix} \leq \begin{pmatrix} \mu(s_1 s_1^*) & \cdots & \mu(s_1 s_n^*) \\ \ddots & & \ddots \\ \mu(s_n s_1^*) & \cdots & \mu(s_n s_n^*) \end{pmatrix}.$$

Thus

$$\mu(s_i)\mu(\bar{s}_j) \leq \mu(s_i s_j^*).$$

Now,

$$\begin{aligned} \sum_{i=1}^n |\mu(s_i)|^2 = \operatorname{tr}(\mu(s_i)\mu(\bar{s}_j)) &\leq \operatorname{tr}(\mu(s_i s_j^*)) \\ &= \sum_{i=1}^n \mu(s_i s_i^*) \\ &= \mu \sum_{i=1}^n (s_i s_i^*) \\ &= 1. \end{aligned}$$

If $\sum_{i=1}^{n}|\mu(s_i)|^2 = 1$, then $(\mu(s_i)\mu(\bar{s}_j) - \mu(s_i s_j^*))$ is a positive matrix with its trace 0. Thus $(\mu(s_i)\mu(\bar{s}_j) - \mu(s_i s_j^*))$ is a zero matrix. Hence we have

$$(\mu(s_i)\mu(\bar{s}_j)) = (\mu(s_i s_j^*)),$$

for $i,\ j - 1, \ldots, n$.

Theorem 3.2. *If $(\mu_1(s_1), \ldots, \mu(s_n))$ and $(\mu_2(s_1), \ldots, \mu_2(s_n))$ are unit vectors in $\mathbb{C}^n$ for states μ_1 and μ_2 on $\mathcal{O}_n$ satisfying $\mu_1|_{\mathrm{UHF}_n} = \mu_2|_{\mathrm{UHF}_n}$, then there exists a complex number λ, $|\lambda| = 1$, such that*

$$\begin{aligned}(\mu_1(s_1), \ldots, \mu_1(s_n)) &= \lambda(\mu_2(s_1), \ldots, \mu_2(s_n)) \\ &= 1.\end{aligned}$$

Proof. Since $s_i s_i^* \in \mathrm{UHF}_n$, we have $\mu_1(s_i s_i^*) = \mu_2(s_i s_i^*)$ for $i = 1, \ldots, n$, and

$$\begin{aligned}&(\mu_1(s_1), \ldots, \mu_1(s_n))(\mu_1(s_1), \ldots, \mu_1(s_n))^* \\ &\quad = (\mu_2(s_1), \ldots, \mu_2(s_n))(\mu_2(s_1), \ldots, \mu_2(s_n))^*\end{aligned}$$

by Theorem 3.1. Note that

$$(\mu_1(s_1), \ldots, \mu_1(s_n))^*(\mu_1(s_1), \ldots, \mu_1(s_n))$$

and

$$(\mu_2(s_1), \ldots, \mu_2(s_n))^*(\mu_2(s_1), \ldots, \mu_2(s_n))$$

are projections of rank 1 and their eigenvectors are $(\mu_1(s_1), \ldots, \mu_1(s_n))$ and $(\mu_2(s_1), \ldots, \mu_2(s_n))$, respectively. Since the eigenspace dimension is one, two unit vectors are linear dependent. Hence, for some complex number λ such that $|\lambda| = 1$, we have $(\mu(s_1), \ldots, \mu_1(s_n)) = \lambda(\mu_2(s_1), \ldots, \mu_2(s_n))$.

Theorem 3.3. *Let $a_m = \lambda_m a$ for unit vectors a, a_m, and λ_m in $\mathbb{C}$ for $m \in \mathbb{N}$. If μ_{a_m} is a state on $\mathcal{O}_n$, then $\lambda_m = 1$ for $m \in \mathbb{N}$.*

Proof. Let $\varphi(x) = \sum_{i=1}^{n} s_i x s_i^*$ for $x \in \mathcal{O}_n$. For state $\mu \in \mathcal{O}_n$,

$$\begin{aligned}&\mu\left(\sum_{i=1}^{n}(\varphi^{k-1}(s_i) - \varphi^k(s_i))(\varphi^{k-1}(s_i) - \varphi^k(s_i))^*\right) \\ &\quad = \mu\left(\sum_{i=1}^{n}(\varphi^{k-1}(s_i s_i^*) + \varphi^k(s_i s_i^*) - \varphi^{k-1}(\varphi(s_i)s_i^*) - \varphi^{k-1}(s_i\varphi(s_i^*)))\right)^* \\ &\quad = \mu(1+1) - 2\varphi^{k-1}\left(\sum_{i,j=1}^{n} s_j s_i s_i^* s_i\right) \\ &\quad = 0.\end{aligned}$$

Thus we have

$$\mu(\varphi^{k-1}(s_i)) = \mu(\varphi^k(s_i))$$

for $i = 1, \ldots, n$. If we set $\mu = \mu_{a_m}$, then $a_m^i = a_{m+1}^i$. Hence we have $\lambda_m = 1$ for $m \in \mathbb{N}$.

According to Theorem 3.3, the linear functional μ_{a_m} defined in (2.2) is a state only for the constant sequence $a_m = a$ for the unit vector a_m for all $m = 1, 2, \ldots$. This type of linear functional appeared in [BJP]. Theorem 3.3 is proof of the positivity, which implies that it is a pure state on $\mathcal{O}_n$.

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