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A BASIS THEOREM FOR Π_1^0 CLASSES OF POSITIVE MEASURE AND JUMP INVERSION FOR RANDOM REALS

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Abstract. We extend the Shoenfield jump inversion theorem to the members of any Π_1^0 class $\mathcal{P} \subseteq 2^\omega$ with nonzero measure; i.e., for every Σ_2^0 set $S \geq_T \emptyset'$, there is a Δ_2^0 real $A \in \mathcal{P}$ such that $A' \equiv_T S$. In particular, we get jump inversion for Δ_2^0 1-random reals.

This paper is part of an ongoing program to study the relationship between two fundamental notions of complexity for real numbers. The first is the computational complexity of a real as captured, for example, by its Turing degree. The second is the intrinsic randomness of a real. In particular, we are interested in the 1-random reals, which were introduced by Martin-Löf [13] and represent the most widely studied randomness class. For the purposes of this introduction, we will assume that the reader is somewhat familiar with basic algorithmic randomness, as per Li-Vitányi [12], and with computability theory [18]. A review of notation and terminology will be given in Section 1.

Intuitively, a 1-random real is very complex. This complexity can be captured formally in terms of unpredictability or incompressibility, but is it reflected in the computational complexity of the real? To put this question more precisely: *which Turing degrees contain* 1*-random reals?* We call such degrees 1*-random.* A beautiful result here is the theorem of Kučera [9] and Gács [6] that every set is Turing reducible to a 1-random real. Therefore, 1-random reals can have arbitrarily high Turing degree. Moreover, Kučera proved that every degree $\mathbf{a} \geq \mathbf{0}'$ is 1-random. On the other hand, the distribution of 1-random degrees below $\mathbf{0}'$ is only partially understood.

It is well known that there is a nonempty Π_1^0 class which contains only 1-random reals. In particular, consider the complement of one of the Σ_1^0 classes in a universal Martin-Löf test. Hence, by the low basis theorem of Jockusch and Soare [7], there are low 1-random reals. Several other results on the distribution of 1-random degrees are known. For instance, minimal degrees and 1-generic degrees cannot be 1-random [11], so there are lots of Δ_2^0 degrees which do not contain 1-random reals. Furthermore, Kučera [9] proved that the 1-random degrees are not closed upwards. In particular, he constructed a Δ_2^0 PA degree $\mathbf{a}$ which is not 1-random. Recall that

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a degree is PA if it computes a member of every nonempty Π^0_1 class.[1] Applying this to a nonempty Π^0_1 class containing only 1-randoms, $\mathbf{a}$ must bound a 1-random degree. Thus, even below $\mathbf{0}'$, the degrees of 1-randoms are not closed upwards. Frank Stephan [19] recently clarified the situation by proving the following remarkable theorem: *if* $\mathbf{a}$ *is PA and* 1*-random, then* $\mathbf{0}' \leq_T \mathbf{a}$.

These results demonstrate that the distribution of the 1-random degrees is quite complicated, even below $\mathbf{0}'$. In the present paper, we approach the problem by considering the jumps of Δ^0_2 1-random reals. First, let us briefly consider the jumps of arbitrary 1-randoms. Jockusch and Soare [7] observed that if $\mathcal{P}$ is a nonempty Π^0_1 class with no computable members and $S \geq_T \emptyset'$, then there is an $A \in \mathcal{P}$ such that $A' \equiv_T A \oplus \emptyset' \equiv_T S$. This is an extension of the Friedberg completeness criterion [5] to the members of Π^0_1 classes, and its proof is a straightforward generalization of the low basis theorem. Because, as above, there is a nonempty Π^0_1 class containing only 1-randoms, we conclude that 1-random reals can have all possible jumps.

Our initial hope was to determine the jumps of Δ^0_2 1-random reals in the same way: simply prove that every Π^0_1 class $\mathcal{P}$ with no computable members had Δ^0_2 members of all possible jumps. Unfortunately this plan fails badly. Cenzer [2] proved that there exists such a class which contains only GL_1 (generalized low) elements, i.e., $A \in \mathcal{P}$ implies that $A' \equiv_T A \oplus \emptyset'$. Therefore, all Δ^0_2 elements of $\mathcal{P}$ are low.

We will instead prove a basis theorem for Π^0_1 classes *of positive measure*. Our result can be viewed as a generalization of the Shoenfield jump inversion theorem [17]. It is easy to see that the jump of a Δ^0_2 set is Σ^0_2. Shoenfield's theorem gives a converse: if S is Σ^0_2 and $S \geq_T \emptyset'$, then there is an $A \in \Delta^0_2$ such that $A' \equiv_T S$. We extend this result by requiring A to be a member of a given Π^0_1 class with nonzero measure.

Theorem 1. *Let $\mathcal{P}$ be a Π^0_1 class such that $\mu(\mathcal{P}) > 0$. For every Σ^0_2 set $S \geq_T \emptyset'$, there is a Δ^0_2 real $A \in \mathcal{P}$ such that $A' \equiv_T S$.*

The theorem implies jump inversion for Δ^0_2 1-randoms[2] because there is a Π^0_1 class with nonzero measure containing only 1-random reals. In fact, any nonempty Π^0_1 class of 1-randoms will suffice, because Kurtz [11] proved that every Π^0_1 class which contains a 1-random has positive measure.

Corollary 2. *For every Σ^0_2 set $S \geq_T \emptyset'$, there is a* 1*-random real $A \in \Delta^0_2$ such that $A' \equiv_T S$.*

We have already seen the two extreme cases of this corollary: there is a low 1-random real and a 1-random of degree $\mathbf{0}'$. We remark that any 1-random of c.e. degree—the existence of which is guaranteed by the Kreisel basis theorem—must have degree $\mathbf{0}'$. This was noted by Kučera [9], who proved that 1-random reals have fixed point free degree. Therefore, Arslanov's completeness criterion [1] implies that $\mathbf{0}'$ is the only 1-random c.e. degree.

Before we turn to the proof of our basis theorem, we would like to mention some other results which clarify the connection between computability and randomness, in particular, strong notions of randomness. If we replace the Σ^0_1 classes by Σ^0_n

[1] The PA degrees were originally defined to be the degrees of complete consistent extension of Peano Arithmetic. The equivalence of these two definitions follows from the work of Scott [16] and Solovay (unpublished).

[2] Kučera claims this result without proof in a remark at the end of [10].

classes in the definition (given below) of a Martin-Löf test, then the resulting randomness concept is called *n-randomness.* Kurtz [11] proved that it is equivalent to passing every Martin-Löf test relative to $\emptyset^{(n-1)}$. Kautz [8] proved that for these higher levels of randomness the Turing degrees of jumps are very constrained. In particular, if A is an n-random real, then $A^{(n-1)} \equiv_T A \oplus \emptyset^{(n-1)}$. For instance, all 2-random reals are GL_1. Finally, Miller and Yu [14] recently found the following very interesting connection between n-randomness and Turing reducibility: *every* 1*-random real Turing below an n-random is also n-random.* These results and similar ones are reported in the forthcoming survey paper [4], and the forthcoming book [3]. We believe that there are a number of extremely interesting connections between computability and randomness still waiting to be found.

1. Definitions, notation and terminology

We consider real numbers to be elements of the Cantor space 2^ω and denote the standard measure on 2^ω by μ. For convenience, we do not distinguish between a set of natural numbers and the infinite binary sequence representing that set. For $x, y \in 2^{<\omega}$, we write $x \preceq y$ if x is a prefix of y. Similarly, $x \prec A$ means that x is a prefix of $A \in 2^\omega$. For $x \in 2^{<\omega}$, let $[x] = \{A \in 2^\omega \mid x \prec A\}$; such sets form a clopen basis for the standard topology on Cantor space. To $V \subseteq 2^{<\omega}$ we associate the open set $[V] = \bigcup_{x \in V} [x]$. If V is computably enumerable, then we call $[V]$ a Σ^0_1 class. A Π^0_1 class is the complement of a Σ^0_1 class. Alternately, a Π^0_1 class is the set of infinite paths through a Π^0_1 tree $T \subseteq 2^{<\omega}$, where a *tree* is a subset of $2^{<\omega}$ closed downward under the prefix relation. Note that there is an effective enumeration $\{\mathcal{P}_e\}_{e\in\omega}$ of all Π^0_1 classes.

Martin-Löf [13] defined the random reals to be those which avoid certain effective sets of measure zero, sets representing properties satisfied by almost no real numbers. A *Martin-Löf test* is a computable sequence $\{\mathcal{V}_i\}_{i\in\omega}$ of Σ^0_1 classes such that $\mu(\mathcal{V}_i) \leq 2^{-i}$. A real $A \in 2^\omega$ *passes* a Martin-Löf test $\{\mathcal{V}_i\}_{i\in\omega}$ if $A \notin \bigcap_{i\in\omega} \mathcal{V}_i$. A real which passes all Martin-Löf tests is called 1*-random.* Martin-Löf proved that it is sufficient to consider a single *universal* test; i.e., there is a Martin-Löf test $\{\mathcal{U}_i\}_{i\in\omega}$ such that $\bigcap_{i\in\omega} \mathcal{U}_i$ is exactly the class of non-random reals. In particular, $2^\omega \setminus \mathcal{U}_1$ is a nonempty Π^0_1 class containing only 1-random reals.

2. The proof

The proof of Theorem 1 can be viewed as a finite injury construction relative to the halting problem. In that sense, it is similar to Sacks' construction of a minimal degree below $\mathbf{0}'$ [15]. We require two additional ideas from the literature. The first is the method of forcing with Π^0_1 classes, which was introduced by Jockusch and Soare [7] to prove the low basis theorem. This method is used to ensure that $A' \leq_T S$. The second is a version of a lemma of Kučera [9] which allows us to recursively bound the positions of branchings in a Π^0_1 class with nonzero measure. The lemma allows us to code S into A' using a variation of a process known as Kučera coding.

Lemma 3 (Kučera, 1985)**.** *Let $\mathcal{P}$ be a Π^0_1 class such that $\mu(\mathcal{P}) > 0$. Then there a Π^0_1 subclass $\mathcal{Q} \subseteq \mathcal{P}$ and a computable function $g\colon \omega \to \omega$ such that $\mu(\mathcal{Q}) > 0$ and*

$$(\forall e)\left[\, \mathcal{Q} \cap \mathcal{P}_e \neq \emptyset \implies \mu(\mathcal{Q} \cap \mathcal{P}_e) \geq 2^{-g(e)} \,\right].$$

To see why this lemma is true, let g be any computable function such that $\sum_{e\in\omega} 2^{-g(e)} < \mu(\mathcal{P})$. Let $\mathcal{Q}$ be the Π^0_1 subclass of $\mathcal{P}$ obtained by removing the reals in $\mathcal{P}_e[s]$ (the stage s approximation to $\mathcal{P}_e$) whenever $\mathcal{P}_e[s] \cap \mathcal{Q}[s]$ has measure less than $2^{-g(e)}$. The choice of g guarantees that $\mu(\mathcal{Q}) > 0$.[3]

Proof of Theorem 1. We are given a Π^0_1 class $\mathcal{P} \subseteq 2^\omega$ with nonzero measure and a Σ^0_2 set $S \geq_T \emptyset'$. Take the Π^0_1 class $\mathcal{Q} \subseteq \mathcal{P}$ and computable function $g\colon \omega \to \omega$ guaranteed by Lemma 3. We will construct a Δ^0_2 real $A \in \mathcal{Q}$ such that $A' \equiv_T S$.

Before describing the construction we must give a few preliminary definitions. For every $\sigma \in 2^{<\omega}$, define a Π^0_1 class

$$\mathcal{F}_\sigma = \{B \in \mathcal{Q} \mid (\forall e < |\sigma|)\, \sigma(e) = 0 \implies \varphi^B_e(e)\uparrow\}.$$

At each stage $s \in \omega$ of the construction, we will define a string $\sigma_s \in 2^s$ which is intended to approximate $A' \upharpoonright s$. We will tentatively restrict A to the class $\mathcal{F}_{\sigma_s}$ in order to *force its jump*. It is important to note that this restriction may be injured at a later stage by the enumeration of an $e < s$ into S.

Next we define a computable function $f\colon \omega \to \omega$ which grows fast enough to ensure that it (eventually) bounds the branchings between elements of $\mathcal{F}_\sigma$, for every $\sigma \in 2^{<\omega}$. We will use f to code elements of S into A (or more precisely, into A'). Let $h\colon 2^{<\omega} \times 2^{<\omega} \to \omega$ be a computable function such that $\mathcal{P}_{h(x,\sigma)} = [x] \cap \mathcal{F}_\sigma$, for all $x, \sigma \in 2^{<\omega}$. Set $f(0) = 0$. For $s \in \omega$, inductively define

$$f(s+1) > \max\{g(h(x,\sigma)) \mid x \in 2^{f(s)} \text{ and } \sigma \in 2^s\}.$$

Now take $x \in 2^{f(t)}$ and $\sigma \in 2^s$, for $t \geq s$, such that $[x] \cap \mathcal{F}_\sigma \neq \emptyset$. We claim that x has distinct finite extensions $y_0, y_1 \in 2^{f(t+1)}$ which extend to reals in $\mathcal{F}_\sigma$. Assume not. Let $\widehat{\sigma} = \sigma 1^{t-s}$ and note that $\mathcal{F}_{\widehat{\sigma}} = \mathcal{F}_\sigma$. Then $\mu(\mathcal{Q} \cap \mathcal{P}_{h(x,\widehat{\sigma})}) = \mu([x] \cap \mathcal{F}_\sigma) \leq 2^{-f(t+1)} < 2^{-g(h(x,\widehat{\sigma}))}$, which contradicts the lemma.

Kučera used the fact that we can bound branchings in a Π^0_1 class with nonzero measure to code information into members of such a class. The most basic form of Kučera coding constructs a real by extensions, choosing the leftmost or rightmost permissible extension to encode the next bit. For our construction, we only distinguish between the rightmost extension and any other permissible extension. Let $\mathcal{R}$ be a Π^0_1 class and let $x \in 2^{f(s+1)}$, for some $s \in \omega$. Define $H_f(\mathcal{R}; x)$ to be true iff

$$(\exists n) \left[\begin{array}{l} \text{if } y \in 2^{f(s+1)} \text{ extends } x \upharpoonright f(s) \text{ and is} \\ \text{to the right of } x, \text{ then } \mathcal{R}[n] \cap [y] = \emptyset \end{array} \right],$$

where $\mathcal{R}[n]$ is the approximation to $\mathcal{R}$ at stage n. Note that $H_f(\mathcal{R}; x)$ is a Σ^0_1 condition. By compactness, if $\mathcal{R} \cap [y] = \emptyset$, then there is an $n \in \omega$ such that $\mathcal{R}[n] \cap [y] = \emptyset$. This implies that if $\mathcal{R} \cap [x] \neq \emptyset$, then $H_f(\mathcal{R}; x)$ is true iff x is the rightmost length $f(s+1)$ extension of $x \upharpoonright f(s)$ which extends to an element of $\mathcal{R}$.

It will be useful to understand the interaction between f and H_f. Assume that we have $x \in 2^{f(t)}$, for some $t \geq s$, and $\sigma \in 2^s$ such that $[x] \cap \mathcal{F}_\sigma \neq \emptyset$. Let $\widehat{x} \in 2^{f(t+1)}$ be the leftmost extension of x such that $[\widehat{x}] \cap \mathcal{F}_\sigma \neq \emptyset$. By the definition of f, there are multiple extensions to choose from, so $H_f(\mathcal{F}_\sigma, \widehat{x})$ is false. In fact, if $\tau \preceq \sigma$, then $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$. Therefore, $H_f(\mathcal{F}_\tau, \widehat{x})$ is also false.

We are now ready to describe the construction. Let $\{S_s\}_{s\in\omega}$ be a $\emptyset'$-computable enumeration of the Σ^0_2 set S. We may assume that $S_0 = \emptyset$ and $|S_{s+1} \smallsetminus S_s| = 1$ for all $s \in \omega$. We construct A by initial segments using a $\emptyset'$ oracle. At each stage

[3]We thank the referee for suggesting this simple proof of Lemma 3.

$s \in \omega$, we find a string $x_s \in 2^{f(t)}$, for some $t \geq s$. Define $A = \bigcup_s x_s$. Each stage also produces a string $\sigma_s \in 2^s$, which is an approximation to A', although not necessarily an initial segment of it. For each $s \in \omega$, we require that

(1) $[x_s] \cap \mathcal{F}_{\sigma_s} \neq \emptyset$.
(2) If $e < s$ and $B \in [x_s] \cap \mathcal{F}_{\sigma_s \upharpoonright e+1}$, then $B'(e) = \sigma_s(e)$.

The Construction.

Stage 0. Let x_0 and σ_0 both be the empty string. Then $[x_0] \cap \mathcal{F}_{\sigma_0} = \mathcal{Q} \neq \emptyset$, so (1) is satisfied. Note that (2) is vacuous.

Stage $s+1$. Assume that we have already constructed $x_s \in 2^{f(t)}$, for some $t \geq s$, and $\sigma_s \in 2^s$ satisfying (1) and (2). Let $e \in S_{s+1} \smallsetminus S_s$ (this element is unique).

Case 1. If $e > s$, then let $x_{s+1} \in 2^{f(t+1)}$ be the leftmost extension of x_s such that $[x_{s+1}] \cap \mathcal{F}_{\sigma_s} \neq \emptyset$. Note that $\emptyset'$ can determine if $[y] \cap \mathcal{F}_{\sigma_s} = \emptyset$, for each $y \in 2^{<\omega}$, so $\emptyset'$ can find x_{s+1}. If $[x_{s+1}] \cap \mathcal{F}_{\sigma_s 0} \neq \emptyset$, then let $\sigma_{s+1} = \sigma_s 0$. Otherwise, let $\sigma_{s+1} = \sigma_s 1$. Again, this can be determined using the $\emptyset'$ oracle. Note that (1) and (2) are satisfied by our choices of x_{s+1} and σ_{s+1}.

Case 2. If $e \leq s$, then let $\tau = \sigma_s \upharpoonright e$. Consider the least number $m \in \omega$ such that $f(\langle e, m\rangle) \geq |x_s|$. First define $\widehat{x}_s \in 2^{f(\langle e,m\rangle)}$ to be the leftmost extension of x_s such that $[\widehat{x}_s] \cap \mathcal{F}_\tau \neq \emptyset$. Next let $x_{s+1} \in 2^{f(\langle e,m\rangle+1)}$ be the rightmost extension of $\widehat{x}_s$ such that $[x_{s+1}] \cap \mathcal{F}_\tau \neq \emptyset$. Let σ_{s+1} be the lexicographically least string of length $s+1$ which extends τ and satisfies $[x_{s+1}] \cap \mathcal{F}_{\sigma_{s+1}} \neq \emptyset$. Again, the construction is computable relative to $\emptyset'$ and we have ensured that (1) and (2) continue to hold.

End Construction.

We turn to the verification. The construction is computable from a $\emptyset'$ oracle, so A is Δ^0_2. Furthermore, (1) tells us that $[x_s] \cap \mathcal{F}_{\sigma_s} \neq \emptyset$, for each $s \in \omega$. Because $\mathcal{F}_{\sigma_s} \subseteq \mathcal{Q} \subseteq \mathcal{P}$, this implies that every x_s can be extended to an element of $\mathcal{P}$. But $\mathcal{P}$ is closed, so $A = \bigcup_s x_s \in \mathcal{P}$. All that remains to verify is that $A' \equiv_T S$.

First we prove that $A' \leq_T S$. To determine whether $e \in A'$, use S and $\emptyset'$ to find a stage $s > e$ such that $S_s \upharpoonright e+1 = S \upharpoonright e+1$. Let $\tau = \sigma_s \upharpoonright e+1$. We claim that $\sigma_t \upharpoonright e+1 = \tau$, for all $t \geq s$. This is because the only way that $\sigma_s \upharpoonright e+1$ can be injured during the construction is in Case 2, when an element $i \leq e$ is enumerated into S. But this will never happen after stage s. Therefore, for all $t \geq s$, we have $\tau \preceq \sigma_t$ and hence $\mathcal{F}_{\sigma_t} \subseteq \mathcal{F}_\tau$. So $[x_t] \cap \mathcal{F}_\tau \neq \emptyset$, for all $t \geq s$, which implies that $A \in \mathcal{F}_\tau$. By (2), we have $A'(e) = \tau(e)$. This proves that we can uniformly decide if $e \in A'$ using only $S \oplus \emptyset' \equiv_T S$. Therefore, $A' \leq_T S$.

Now we show that $S \leq_T A'$. Assume by induction that we have determined $S \upharpoonright e$, for some $e \in \omega$. Find the least $s \geq e$ such that $S_s \upharpoonright e = S \upharpoonright e$. Let $\tau = \sigma_s \upharpoonright e$ and note, as above, that $\tau \preceq \sigma_t$, for all $t \geq s$. Find the least $m \in \omega$ such that $f(\langle e, m\rangle) \geq |x_s|$. Of course, both s and m can be found by $\emptyset'$. We claim that $e \in S$ iff either $e \in S_s$ or $(\exists n \geq m)\ H_f(\mathcal{F}_\tau; A \upharpoonright f(\langle e,n\rangle+1))$. If $e \in S \smallsetminus S_s$, then Case 2 ensures that $H_f(\mathcal{F}_\tau; A \upharpoonright f(\langle e,n\rangle+1))$ holds for some $n \geq m$. So, assume that $e \notin S$. Then for every $n \geq m$, the construction chooses the leftmost extension of $A \upharpoonright f(\langle e,n\rangle)$ which is extendible in the appropriate Π^0_1 class. This class is of the form $\mathcal{F}_{\widehat{\tau}}$, where $\widehat{\tau} \preceq \sigma_t$ for some $t \geq s$ and $|\widehat{\tau}| \geq e$. This implies that $\tau \preceq \widehat{\tau}$, so $\mathcal{F}_{\widehat{\tau}} \subseteq \mathcal{F}_\tau$. The definition of f ensures that there are distinct length $f(\langle e,n\rangle+1)$ extensions of $A \upharpoonright f(\langle e,n\rangle)$ which can be extended to elements of $\mathcal{F}_{\widehat{\tau}}$. Therefore, the leftmost choice consistent with $\mathcal{F}_{\widehat{\tau}}$ must be left of the rightmost choice consistent with $\mathcal{F}_\tau$. Hence $H_f(\mathcal{F}_\tau; A \upharpoonright f(\langle e,n\rangle+1))$ is false. Finally, note that A' can decide if

$(\exists n \geq m)\ H_f(\mathcal{F}_\tau; A \upharpoonright f(\langle e, n\rangle + 1))$, because H_f is Σ^0_1. Therefore A' can determine if $e \in S$, proving that $S \leq_T A'$. □

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