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A STABILITY PROPERTY FOR LINEAR GROUPS

MARIAN F. ANTON

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ABSTRACT. For suitable rings of integers R, we show that the mod p group cohomology for $GL_{n+3p-5}(R)$ comes from $GL_\infty(R)$ when restricted to the diagonal matrices $D_n(R)$ for all ranks $n \geq 2$.

1. Introduction

For a prime p and a primitive p-th root of unity ζ_p, the mod p group cohomology of invertible $n \times n$ matrices GL_n over $R = \mathbb{Z}[\frac{1}{p}, \zeta_p]$ for $n = 1, 2, ..., \infty$ plays a central role in the algebraic K-theory of integers being one of the interfaces for Quillen-Lichtenbaum conjectures [5] (compare also [11]).

A somewhat surprising result [2] about this cohomology for p regular is the fact that its image on diagonal matrices D_n inside GL_n agrees with the one predicted by the above-mentioned conjectures as soon as a distinguished finite set of mod p homology classes vanish in H_*GL_2. The notation H_* stands for the mod p group homology functor. These classes were called "étale obstruction classes" due to the fact that they measure the difference between the classifying space of GL_n and its étale model in a sense made precise by the Theorem 2.4.

In this note we show that if these étale obstruction classes vanish only in the stable range, then the mod p group cohomology for GL_{n+3p-5} can be calculated when restricted to the diagonal matrices D_n instead of D_{n+3p-5}. Although this result is known in the stable range [10] and it is weaker than similar results [1, 9] for $p = 2$ and $p = 3$, it has the advantage that it holds in the unstable range for any regular prime $p \geq 5$, a case previously unknown. Whether the étale obstruction classes are vanishing already in H_*GL_2 or not for $p \geq 5$, it seems to be a difficult computational question which remains unsolved.

In more detail, by using Dirichlet's Unit Theorem and standard group cohomology calculations for p odd, we have

$$H^*D_n \approx \mathbb{F}_p[x_1, x_2, ..., x_n] \otimes \bigotimes_{i=0}^{\frac{p-1}{2}} \Lambda(y_{i,1}, y_{i,2}, ..., y_{i,n}), \tag{1.1}$$

where H^* denotes the mod p group cohomology functor and Λ the exterior algebra over the finite field $\mathbb{F}_p$ of order p. The generators have the cohomological degrees given by $|x_j| = 2$ and $|y_{i,j}| = 1$ for all $j = 1, 2, ..., n$ and $i = 0, 1, 2, ..., \frac{p-1}{2}$. Inside

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this ring we distinguish a subring of invariants defined by the formula

$$H^*D_n^{inv} = \mathbb{F}_p[c_1, c_2, ..., c_n] \otimes \bigotimes_{i=0}^{\frac{p-1}{2}} \Lambda(e_{i,1}, e_{i,2}, ..., e_{i,n}) \tag{1.2}$$

with c_j the symmetrization of $x_1...x_j$ and $e_{i,j}$ the symmetrization of $x_1...x_{j-1}y_{i,j}$ for all $j = 1, 2, ..., n$ and $i = 0, 1, 2, ..., \frac{p-1}{2}$. The main result is

Theorem 1.1. *If p is an odd regular prime, then the image of the restriction map $H^*GL_{n+3p-5} \to H^*D_n$ induced by the tail inclusion $D_n \to GL_{n+3p-5}$ (see (2.3)) agrees with $H^*D_n^{inv}$ for all $n \geq 2$.*

The idea for proving Theorem 1.1 is to use the case $n = \infty$ to show that the étale obstruction classes vanish in the stable range and to slightly modify our theorem, stating that the image of the restriction map is computable if these classes vanish. The étale obstruction classes will be defined and studied in section 2 and the proof of the theorem will be given in section 3.

Notation 1.2. For the rest of the paper we assume that p is an odd regular prime if not otherwise stated. Let H^* (resp. H_*) be the mod p cohomology (resp. homology) functor on groups or spaces and let SL_n be the subgroup of matrices of determinant 1 in GL_n for $n = 1, 2, ..., \infty$.

2. Étale obstruction classes

2.1. **Definition.** By setting $n = 1$, $r = \frac{p-1}{2}$, $x = x_1$ and $y_i = y_{i,1}$ in formula (1.1), we see that an additive basis for H^*D_1 is given by the monomials

$$x^I = x^k \prod_{i=0}^{r} y_i^{\epsilon_i}, \quad I = (k, \epsilon_0, \epsilon_1, ..., \epsilon_r), \tag{2.1}$$

where I runs over all the sequences of non-negative integers with $\epsilon_i \in \{0, 1\}$ for $i = 0, 1, 2, ..., r$. Let (u_I) be the basis of H_*D_1 dual to the basis (x^I) in H^*D_1. In what follows we tacitly assume that x is a Bockstein class.

Definition 2.1. We call $u_I \in H_*D_1$ with I as in (2.1) an *étale obstruction class* if

$$a(I) = \frac{1}{2}(\epsilon_0 + \epsilon_1 + ... + \epsilon_r - k)$$

is a positive integer.

Remark 2.2. In the original definition [2] we used SL_2 instead of D_1, but the two definitions agree with respect to a suitable group homomorphism $D_1 \to SL_2$.

2.2. **The classifying space étale model.** For each $n = 2, 3, ..., \infty$ we have the following commutative diagram:

$$\begin{array}{ccccc} H_*D_2 & \longrightarrow & H_*GL_2 & \xrightarrow{f_{2*}} & H_*GL_2^{et} \\ \downarrow & & \downarrow & & \downarrow \\ H_*D_n & \longrightarrow & H_*GL_n & \xrightarrow{f_{n*}} & H_*GL_n^{et} \end{array} \tag{2.2}$$

with the first two vertical maps induced by the tail inclusions

$$D_2 \to D_n, \quad GL_2 \to GL_n, \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.3}$$

and the unlabelled horizontal maps induced by the canonical inclusions

$$D_2 \subset GL_2, \quad D_n \subset GL_n.$$

The maps f_{2*} and f_{n*} are induced by the *natural* Dwyer-Friedlander maps

$$f_2 : BGL_2 \to BGL_2^{et}, \quad f_n : BGL_n \to BGL_n^{et}$$

from the classifying spaces BGL_2 and BGL_n to their "étale model spaces" BGL_2^{et} and BGL_n^{et}, respectively. The notation $H_*GL_n^{et}$ stands for $H_*BGL_n^{et}$. Their definition can be found elsewhere [4]. Finally, the last vertical map is induced by naturality. In this setting, the following result is known:

Theorem 2.3 ([6])**.** *The map $H^*GL_n^{et} \to H^*D_n$ dual to the composition map*

$$H_*D_n \to H_*GL_n \xrightarrow{f_{n*}} H_*GL_n^{et}$$

in diagram (2.2) *is injective, and its image agrees with $H^*D_n^{inv}$ as in* (1.2) *for all $n = 1, 2, ..., \infty$.*

2.3. **The kernel of the composition map $H_*D_n \to H_*GL_n^{et}$.** Let

$$t : D_1 \to D_2, \quad t(a) = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}, \text{ for } a \in D_1,$$

be a group homomorphism and consider the diagram

$$D_1 \times D_1 \xrightarrow{t \times Id} D_2 \times D_1 \xrightarrow{\mu} D_2,$$

where μ is the multiplication of a matrix by a scalar map. Now apply the mod p homology functor H_* to this diagram and get

$$H_*D_1 \otimes H_*D_1 \xrightarrow{t_* \otimes Id} H_*D_2 \otimes H_*D_1 \xrightarrow{\mu_*} H_*D_2. \tag{2.4}$$

H_*D_1 is a Pontryagin ring [3] with the identity element $1 \in H_0D_1 \approx \mathbb{F}_p$ corresponding to the identity element of the coefficient field. Then the last map in the above diagram gives H_*D_2 a module structure over the Pontryagin ring H_*D_1:

$$w \circ z = \mu_*(w \otimes z) \text{ for } w \in H_*D_2 \text{ and } z \in H_*D_1. \tag{2.5}$$

Now recall that $u_I \in H_*D_1$ is the dual basis of the additive basis $x^I \in H^*D_1$ as in (2.1). Let $z_1 \in H_*D_1$ be an arbitrary element and observe that the image of the element $u_I \otimes z_1 \in H_*D_1 \otimes H_*D_1$ via the composite map of the diagram (2.4) is $t_*(u_I) \circ z_1 \in H_*D_2$.

Theorem 2.4. *In the unstable range $2 \le n < \infty$, the kernel of the composition map $H_*D_n \to H_*GL_n^{et}$ in the diagram* (2.2) *is spanned by elements of the form*

$$(t_*(u_I) \circ z_1) \otimes z_2 \otimes ... \otimes z_{n-1} \in H_*D_2 \otimes H_*D_1 \otimes ... \otimes H_*D_1 \approx H_*D_n,$$

where $u_I \in H_*D_1$ runs over all étale obstruction classes and $z_i \in H_*D_1$ are arbitrary elements for $i = 1, 2, ..., n$.

The proof of this theorem is given in my previous paper [2]. If we specialize to $n = 2$ and $z_1 = 1$, then we deduce as a corollary of this theorem that

Corollary 2.5. *If $u_I \in H_*D_1$ is an étale obstruction class, then $t_*(u_I) \in H_*D_2$ maps to zero in $H_*GL_n^{et}$ via composite maps in the diagram* (2.2) *for $n = 2, 3, ..., \infty$.*

In the stable range, we have Mitchell's result:

Theorem 2.6 ([10])**.** *The kernel of the map $H_*D_\infty \to H_*GL_\infty^{et}$ is the same as the kernel of the map $H_*D_\infty \to H_*GL_\infty$.*

3. Proof of the main theorem

3.1. **Proof of Theorem 1.1.** In the following commutative diagram, in which the maps are defined as in (2.2), we observe that the map $H_*GL_n^{et} \to H_*GL_{n+3p-5}^{et}$ is injective by Theorem 2.3 and formula (1.2):

$$\begin{array}{ccccc} H_*D_n & \longrightarrow & H_*GL_n & \xrightarrow{f_{2*}} & H_*GL_n^{et} \\ \downarrow & & \downarrow & & \downarrow \\ H_*D_{n+3p-5} & \longrightarrow & H_*GL_{n+3p-5} & \xrightarrow{f_{n+3p-5*}} & H_*GL_{n+3p-5}^{et} \end{array}$$

Hence, Theorem 1.1 follows from Theorem 2.3 and the dual of the following statement: *the kernel of the composition map $H_*D_n \to H_*GL_n^{et}$ is included in the kernel of the map $H_*D_n \to H_*GL_{n+3p-5}$ for $n \geq 2$.* The statement is an immediate consequence of Theorem 2.4 and Lemma 3.4 below.

Let us name the tail inclusions (2.3) simply by "tail".

Lemma 3.1. *If $n \geq 3p-4$, then the element $\mathrm{tail}_*(t_*(u_I)) \in H_*D_n$ maps to zero in H_*GL_n via the map induced by inclusion if u_I is an étale obstruction class.*

Proof. The key observation is the fact that the homological degree of an element u_I with $a(I)$ a positive integer is bounded from above by $\frac{3}{2}(p-1)-1$ (coincidentally, this number is exactly one less the virtual cohomological dimension of SL_2). This is easily seen by inspection, where $a(I)$ is given in Definition 2.1. By Van der Kallen stability theorem [8], $H_iGL_\infty \approx H_iGL_n$ for $n \geq 2i+1$. Because $\mathrm{tail}_*(t_*(u_I)) \in H_*D_n$ maps to zero in $H_*GL_\infty^{et}$ by Corollary 2.5, then Theorem 2.6 now guarantees that $\mathrm{tail}_*(t_*(u_I))$ maps to zero in H_*GL_∞ via maps induced by tail and canonical inclusions. Combining this fact with the previous observation the conclusion follows. □

For each $k \geq 2$, the commutative diagram

$$\begin{array}{ccc} D_k \times D_1 & \longrightarrow & GL_k \times D_1 \\ \mu\downarrow & & \mu\downarrow \\ D_k & \longrightarrow & GL_k \end{array}$$

where μ is the matrix by scalar multiplication map and the horizontal maps are inclusions, induces a homomorphism $H_*D_k \to H_*GL_k$ of modules over the Pontryagin ring H_*D_1. Also, we recall that the Pontryagin ring H_*D_1 for abelian groups like D_1 is the tensor product of an exterior algebra in degree one generators and a divided polynomial algebra in even degree generators [3], p. 126.

Lemma 3.2. *If $w \in H_*D_2$, $z, z_1, ..., z_{k-1} \in H_*D_1$ with z primitive, and $d^k : D_1 \to D_k$ is the diagonal map, then we have the following formula:*

$$\begin{aligned} ((w \circ z_1) \otimes z_2 \otimes ... \otimes z_{k-1}) \circ z &= ((w \circ z_1) \otimes z_2 \otimes ... \otimes z_{k-1}) \cdot d_*^k(z) \\ &= (w \circ (z_1 \cdot z)) \otimes z_2 \otimes ... \otimes z_{k-1} + (w \circ z_1) \otimes (z_2 \cdot z) \otimes ... \otimes z_{k-1} \\ &\quad + ... + (w \circ z_1) \otimes z_2 \otimes ... \otimes (z_{k-1} \cdot z). \end{aligned}$$

Here $\circ$ is the module structure product and $\cdot$ is the Pontryagin product in H_*D_1 or H_*D_k from the context (compare (2.5)). The factor z in this summation is permuted cyclically due to the way the diagonal acts on primitives [7], pp. 284-286. The proof is evident and hence, omitted.

Lemma 3.3. *If* $n \geq 3p-3$, *then* $\mathrm{tail}_*(t_*(u_I)\circ z_1) \in H_*D_n$ *vanishes in* H_*GL_n, *where* $u_I \in H_*D_1$ *is an étale obstruction class and* $z_1 \in H_*D_1$ *is an arbitrary element.*

Proof. If we specialize the formula in Lemma 3.2 to $k = n \geq 3p-3$, $w = t_*(u_I)$, and $z_i = 1$ for $i = 1, ..., k-1$, then we see that the left-hand side vanishes in H_*GL_n by Lemma 3.1 and the observation before Lemma 3.2, while the right-hand side can be written as

$$\mathrm{tail}_*(t_*(u_I)\circ z) + t_*(u_I)\otimes d_*^{n-3}(z)\otimes 1 + \mathrm{tail}_*(t_*(u_I))\otimes z.$$

By applying Lemma 3.1 and using a block-multiplication argument, we see that $\mathrm{tail}_*(t_*(u_I))\otimes z$ vanishes in H_*GL_n for $n-1 \geq 3p-4$. Also, $t_*(u_I)\otimes d_*^{n-3}(z)\otimes 1$ vanishes in H_*GL_n because it is a sum of terms invariant under permutations (maps induced from conjugation in GL_n) and hence, each such summand has the same image as $\mathrm{tail}_*(t_*(u_I))\otimes z$ in H_*GL_n, i.e. zero. We conclude that $\mathrm{tail}_*(t_*(u_I)\circ z)$ vanishes in H_*GL_n for z primitive in H_*D_1, and the result follows by an inductive argument. □

Lemma 3.4. *If* $n \geq 2$, *then all the elements of the form*

$$(t_*(u_I)\circ z_1)\otimes z_2\otimes ...\otimes z_{n-1} \in H_*D_2\otimes H_*D_1\otimes ...\otimes H_*D_1 \approx H_*D_n$$

vanish in H_*GL_{n+3p-5}, *where* $u_I \in H_*D_1$ *is an étale obstruction class and* $z_i \in H_*D_1$ *are arbitrary elements for* $i = 1, 2, ..., n-1$.

Proof. The images in H_*GL_{n+3p-5} of the elements as in the lemma are the same as the images of the elements

$$z_2\otimes ...\otimes z_{n-1}\otimes \mathrm{tail}_*(t_*(u_I)\circ z_1) \in H_*D_{n-2}\otimes H_*D_{3p-3},$$

where $\mathrm{tail}_*(t_*(u_I)\circ z_1) \in H_*D_{3p-3}$ vanishes in H_*GL_{3p-3} by Lemma 3.3. The proof is now concluded by a block-multiplication argument. □

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Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506-0027 – and – I.M.A.R., P.O. Box 1-764, Bucharest, Romania 70700

E-mail address: anton@ms.uky.edu