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WEAKLY NULL SEQUENCES WITH AN UNCONDITIONAL SUBSEQUENCE

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Abstract. In the present paper we provide sufficient conditions such that a normalized pointwise convergent to zero sequence in $C(K,X)$ with K a compact space and X a Banach space has an unconditional subsequence.

As a consequence we obtain that any such sequence of functions $(f_n)_n$ with finite and uniformly bounded cardinality of their range admits an unconditional subsequence.

1. Introduction

The following theorem is due to H. Rosenthal [9], and its initial proof uses transfinite induction.

Theorem 1.1 (H. Rosenthal). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of non-zero characteristic functions in $C(K)$ with K a compact Hausdorff space. If $(f_n)_n$ converges pointwise to zero, then it contains an unconditional basic subsequence.*

Maurey-Rosenthal's example [8] shows that there is a compact K and a pointwise null sequence $f_n \in C(K)$, $(f_n)_n$ has no unconditional basic subsequence. The range of every f_n is a countable set $\{0\}\cup\{r_k : k\in\mathbb{N}\}$ where $\{r_k\}_k$ is a strictly decreasing null sequence. Setting in this case

$$g_n(x) = \begin{cases} f_n(x), & \text{if } f_n(x) = r_1,\dots,r_n,\\ 0, & \text{otherwise,}\end{cases}$$

we get that $\|g_n - f_n\| \le r_{n+1} \to 0$. Therefore also $(g_n)_n$ has no unconditional basic subsequence yielding that Theorem 1.1 cannot be extended in this case. So it is natural to ask whether it can be extended in the case where the cardinality of the range of f_n is finite and uniformly bounded by some positive integer.

In this direction, we prove the following:

Theorem 1.2. *Let K be a Hausdorff compact space, let X be a Banach space and let $f_n : K \to X$, $n\in\mathbb{N}$, be a sequence of normalized pointwise null continuous functions with the property that their range is of finite cardinality uniformly bounded by some positive integer J. Then $(f_n)_n$ has an unconditional subsequence.*

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In the case where X is finite dimensional, we obtain the following stronger result.

Theorem 1.3. *Let K be a compact space and let $f_m : K \to \mathbb{R}^n$, $m \in \mathbb{N}$, be a uniformly bounded sequence of continuous functions on K which is pointwise convergent to zero. Assume moreover that there exists a sequence of positive numbers ϵ_m converging to zero and a positive number μ such that for every $m \in \mathbb{N}$ and $x \in K$ either $\|f_m(x)\| \le \epsilon_m$ or $\|f_m(x)\| \ge \mu$. Then $(f_m)_m$ has an unconditional basic subsequence.*

To derive these we make use of a combinatorial theorem also proved in this paper which states the following:

Theorem 1.4. *Assume that I is a set, n a positive integer and for any $i \in I$, $G_1^i, \dots, G_n^i$ finite subsets of $\mathbb{N}$, such that setting $G^i = \bigcup_{k=1}^n G_k^i$, the closure of the family $\{G^i : i \in I\}$ in the pointwise topology consists of finite subsets of $\mathbb{N}$. Then for any $L \in [\mathbb{N}]$, there is $N = \{n_1 < n_2 < \dots\} \in [L]$ such that the following holds:*

Given $i \in I$, $q \in \mathbb{N}$, $k \in \{1, \dots, n\}$ and $A \subset G_k^i \cap \{n_1, \dots, n_q\}$, there is $i' \in I$ such that $G_k^{i'} \cap \{n_1, \dots, n_p\} = A$ and for all $k' \neq k$, $G_{k'}^{i'} \cap \{n_1, \dots, n_q\} \subset A$.

Here by $[L]$ we denote the set of all infinite subsets of L.

The above theorem makes use of some ideas due to J. Elton [4] and it shares similar arguments with the proofs of two results concerning restricted forms of unconditionality that occur in weakly null sequences. Namely J. Elton's near unconditionality ([4]) and Argyros-Mercourakis-Tsarpalias convex unconditionality [2]. It extends an earlier result in [2] which was used for an alternative proof of the above-mentioned Rosenthal's Theorem. This proof also can be found in [1].

After submitting the present paper, two recent papers came to our attention: The first one is due to I. Gasparis, E. Odell and B. Wahl, [6], where it is proved among others, the case $n = 1$ of Theorem 1.3. Some additional information is also obtained about the norm of the space generated by the subsequence. The second, due to J. Lopez-Abad and S. Todorcevic [7] (see also [11]) also provides a proof for the case $n = 1$ of Theorem 1.3 and also a proof for Theorem 1.2 in the case $X = \mathbb{R}$. Both papers have a different approach to these problems than ours.

2. Some combinatorial results

We make use of the following important principle of infinite combinatorics, namely the infinite Ramsey theorem. For M an infinite subset of $\mathbb{N}$, we denote by $[M]$ the set of all infinite subsets of M. We endow $[\mathbb{N}]$ with the topology of pointwise convergence.

Theorem 2.1 ([3, 10, 5])**.** *Let $\mathcal{A}$ be an analytic subset of $[\mathbb{N}]$. For every $L \in [\mathbb{N}]$ there exists an $M \in [L]$ such that either $[M] \subset \mathcal{A}$ or else $[M] \subset [L] \setminus \mathcal{A}$.*

We pass now to the proof of Theorem 1.4.

Proof of Theorem 1.4. For $1 \le k \le n$ let P_k be the following property of $L_k = \{\ell_1 < \ell_2 < \dots\}$.

For all $i \in I$ and $q \in \mathbb{N}$ if $A \subset G_k^i \cap \{\ell_1, \dots, \ell_q\}$, then there is $i' \in I$ such that $G_k^{i'} \cap \{\ell_1, \dots, \ell_q\} = A$ and for all $k' \neq k$, $G_{k'}^{i'} \cap \{\ell_1, \dots, \ell_q\} \subset A$.

We first prove that for a fixed $k \in \{1, \dots, n\}$ and $L \in [\mathbb{N}]$ there is $N \in [L]$ satisfying property P_k. So, we fix k and prove the following claim.

Claim. For any $L \in [\mathbb{N}]$ and $J \subset I$ we partition $[L]$ into two disjoint sets $\mathcal{A}_{J,L}$ and $\mathcal{B}_{J,L}$ as follows:

$M = \{m_1 < m_2 < \dots\} \in \mathcal{A}_{J,L}$ if and only if for all $q \in \mathbb{N}$, if there exists an $i \in J$ such that $\{m_2, \dots, m_q\} \subset G^i_k$, then there exists an $i' \in J$ such that $\{m_2, \dots, m_q\} \subset G^{i'}_k$ and moreover $m_1 \notin \bigcup_{k'=1}^n G^{i'}_{k'}$.

We also set $\mathcal{B}_{J,L} = [L] \setminus \mathcal{A}$.

We claim that there is an $M \in [L]$ such that $[M] \subset \mathcal{A}_{J,L}$.

It is easy to check that if $M = \{m_1 < m_2 < \dots\} \in \mathcal{B}_{J,L}$, then there is a $q \in \mathbb{N}$ such that any other set starting with $m_1, m_2, \dots, m_q$ is also in $\mathcal{B}_{J,L}$. Therefore $\mathcal{B}_{J,L}$ is open, so we can find an $M = \{m_1 < m_2 < \dots\} \in [L]$ such that either $[M] \subset \mathcal{A}_{J,L}$ or $[M] \subset \mathcal{B}_{J,L}$. Next we argue that the second case is impossible, by showing that for any $p \in \mathbb{N}$, there is an $i \in I$ such that $\{m_1, \dots, m_p\} \subset \bigcup_{k'=1}^n G^i_{k'}$. This contradicts the fact that $\{G^i = \bigcup_{k'=1}^n G^i_{k'} : i \in I\}$ is relatively compact in the finite subsets of $\mathbb{N}$. So fix such a p and note that for all $\ell \in [1, p]$, the set $\{m_\ell, m_{p+1}, m_{p+2}, \dots\}$ is in $[M]$. Therefore, there exists a $q(\ell) \ge p$ and an $i(\ell) \in J$ so that $\{m_{p+1}, \dots, m_{q(\ell)}\} \subset G^{i(\ell)}_k$ and whenever for some $i' \in J$, $\{m_{p+1}, \dots, m_{q(\ell)}\} \subset G^{i'}_k$, we have that also $m_\ell \in \bigcup_{k'=1}^n G^{i'}_{k'}$. Let $q(\ell_r)$ be the greatest among $q(1), \dots, q(p)$. Then for all ℓ, $\{m_{p+1}, \dots, m_{q(\ell)}\} \subset G^{i(\ell_r)}_k$ and therefore $\{m_1, \dots, m_p\} \subset \bigcup_{k'=1}^n G^{i(\ell_r)}_{k'}$.

It follows therefore that for some $M = \{m_1 < m_2 < \dots\} \in [L]$, $[M] \subset \mathcal{A}_{J,L}$.

Using the previous claim, we define inductively $M_1 \supset M_2 \supset \dots$ infinite subsets of L and $n_1 = \min M_1 < n_2 = \min M_2 < \dots$, as follows:

M_1 is the appropriate set given by the claim such that $[M_1] \subset \mathcal{A}_{I,L}$.

Assume we have defined $M_1 \supset M_2 \supset \cdots \supset M_p$ and $n_1 = \min M_1 < n_2 = \min M_2 < \cdots < n_p = \min M_p$. Let $F_1, F_2, \dots, F_{2^p}$ be an enumeration of the subsets of $\{n_1, \dots, n_p\}$ and for $1 \le j \le 2^p$, let

$$I_{F_j} = \{i \in I : G^i_k \cap \{n_1, \dots, n_p\} = F_j \text{ and for all } k' \ne k,\ G^i_{k'} \cap \{n_1, \dots, n_p\} \subset F_j\}.$$

Repeatedly using the previous claim, let $M^1_p \in [M_p \setminus \{n_p\}]$ be such that $[M^1_p] \subset \mathcal{A}_{I_{F_1}, M_p \setminus \{n_p\}}$, $M^2_p \in [M^1_p]$ be such that $[M^2_p] \subset \mathcal{A}_{I_{F_2}, M^1_p}, \dots$, $M^{2^p}_p$ be such that $[M^{2^p}_p] \subset \mathcal{A}_{I_{F^{2^p}}, M^{2^p-1}_p}$. We set $M_{p+1} = M^{2^p}_p$ and $n_{p+1} = \min M_{p+1} > n_p$. Observe that since $M^1_p \supset M^2_p \supset \dots, \supset M^{2^p}_p = M_{p+1}$, we have that

$$[M_{p+1}] \subset \mathcal{A}_{I_{F_j}, M_p} \text{ for all } 1 \le j \le 2^p. \tag{1}$$

To prove that $N = \{n_1 < n_2 < \dots\}$ has property P_k, let $i \in I$, $q \in \mathbb{N}$ and $A \subset G^i_k \cap \{n_1, \dots, n_q\}$. We find inductively on $p \le q$, $i_p \in I$ such that

$$G^{i_p}_k \cap \{n_1, \dots, n_p\} = A \cap \{n_1, \dots, n_p\}, \tag{2}$$

$$\text{for } k' \ne k,\ G^{i_p}_{k'} \cap \{n_1, \dots, n_p\} \subset A \cap \{n_1, \dots, n_p\} \tag{3}$$

$$\text{and } G^{i_p}_k \cap \{n_{p+1}, \dots, n_q\} \supset A \cap \{n_{p+1}, \dots, n_q\}. \tag{4}$$

Then i_q is the required index in I, demonstrating that N indeed has property P_k.

To start with, set $i_0 = i$. Then (4) is the only non-void condition and is satisfied by our hypothesis that $A \subset G^i_k \cap \{n_1, \dots, n_q\}$. Assume we have defined i_p so that

(2), (3) and (4) are fulfilled, and $p < q$. Consider two cases:

If $n_{p+1} \in A$, then by (4) $n_{p+1} \in G_k^{i_p}$, thus setting $i_{p+1} = i_p$, (2), (3) and (4) are also fulfilled for $p+1$.

As a second case, let $n_{p+1} \notin A$. Set $F_1 = G_k^{i_p} \cap \{n_1, \dots, n_p\}$ and let

$$Q = \{n_{p+1}\} \cup \{n \in N : n \in A \cap \{n_{p+1}, \dots, n_q\} \text{ or } n > \max A \cap \{n_{p+1}, \dots, n_q\}\}$$

be the infinite subset of N beginning with $\{n_{p+1}\} \cup (A \cap \{n_{p+1}, \dots, n_q\})$. Since by (1) $[M_{p+1}] \subset \mathcal{A}_{I_{F_1}, M_p}$ and $Q \in [M_{p+1}]$, we get that there is an $i_{p+1} \in I_{F_1}$ such that $G_k^{i_{p+1}} \supset A \cap \{n_{p+1}, \dots, n_q\} = A \cap \{n_{p+2}, \dots, n_q\}$ (which demonstrates (4) for $p+1$), but now $n_{p+1} \notin \bigcup_{k'=1}^n G_{k'}^{i_{p+1}}$. Since moreover $i_{p+1} \in I_{F_1}$ and $F_1 = G_k^{i_p} \cap \{n_1, \dots, n_p\}$, we get that $G_k^{i_{p+1}} \cap \{n_1, \dots, n_p\} = A \cap \{n_1, \dots, n_p\}$ and for $k' \neq k$, $G_{k'}^{i_{p+1}} \cap \{n_1, \dots, n_p\} \subset A \cap \{n_1, \dots, n_p\}$. This immediately implies (2) and (3) for $p+1$, since n_{p+1} does not belong to any of the sets $G_1^{i_{p+1}}, \dots, G_n^{i_{p+1}}$ or A.

So, for a fixed $k \in \{1, \dots, n\}$ and $L \in [\mathbb{N}]$ there is $L_k \in [L]$ satisfying property P_k. Note that in this case, if $L' \in [L_k]$, then L' also satisfies property P_k. This is true since if $L' = \{\ell_{k_1}, \ell_{k_2}, \dots\}$ with $k_1 < k_2 < \dots$, and $A \subset G_k^i \cap \{\ell_{k_1}, \ell_{k_2}, \dots, \ell_{k_q}\}$, for some i and q, then also $A \subset G_k^i \cap \{\ell_1, \dots, \ell_{k_q}\}$ and we may apply property P_k for L_k to find the appropriate index i'. Using this observation we successively find $L_1 \in [L]$, $L_2 \in [L_1], \dots,$ $L_n \in [L_{n-1}]$ such that L_k satisfies property P_k. Then $M = L_n$ satisfies property P_k for any k which is exactly what we wanted to prove. □

Using stronger hypothesis about the compactness of $G_1^i, \dots, G_n^i$, $i \in I$, we obtain the following result:

Corollary 2.2. *Assume that I is a set, n a positive integer and $\{(G_k^i)_{k=1}^n : i \in I\}$ a compact family of $(\{0,1\}^{\mathbb{N}})^n$. Then for any $L \in [\mathbb{N}]$, there is $N \in [L]$ such that the following holds:*

Given $i \in I$, $k = 1, \dots, n$ and $A \subset G_k^i \cap N$, then there is $i' \in I$ such that $G_k^{i'} \cap N = A$ and $G_{k'}^{i'} \cap N \subset A$ whenever $k' \neq k$.

In particular, if for every i, $G_1^i, G_2^i, \dots, G_n^i$ are pairwise disjoint and for some i and k, $A \subset G_k^i$, then there is $i' \in I$ such that $G_k^{i'} \cap N = A$ and $G_{k'}^{i'} \cap N = \emptyset$ whenever $k' \neq k$.

Proof. In this case, $G^i = \bigcup_{k=1}^n G_k^i$ is also a compact family of the finite subsets of $\mathbb{N}$. To see this, choose a sequence $G^{i_m}, m \in \mathbb{N}$, out of $G^i : i \in I$ and find $M \in [\mathbb{N}]$ such that for any $k = 1, \dots, n$, $\{G_k^{i_m} : m \in M\}$ is convergent to some G_k^i. Then it is easy to check that $\{G^{i_m}\}_{m \in M}$ converges to $G^i = \bigcup_{k=1}^n G_k^i$.

Therefore, by Theorem 2.1, we can find $N = \{n_1 < n_2 < \dots\} \in [L]$ such that given $i \in I$, $q \in \mathbb{N}$, $k \in \{1, \dots, n\}$ and $A \subset G_k^i \cap \{n_1, \dots, n_q\}$, there is $i_q \in I$ such that $G_k^{i_q} \cap \{n_1, \dots, n_q\} = A$ and for all $k' \neq k$, $G_{k'}^{i_q} \cap \{n_1, \dots, n_q\} \subset A$. Now choosing $M \in [\mathbb{N}]$ such that $\{(G_{k'}^{i_q})_{k=1,\dots,n}\}_{q \in M}$ is convergent to some $(G_{k'}^j)_{k'=1,\dots,n}$, it is easy to check that $G_k^j \cap N = A$ and for all $k' \neq k$, $G_{k'}^j \cap N \subset A$. □

Remark 2.1. Using the notation of the above corollary, it is not clear to us even for $n = 2$, whether there exists an $N \in [L]$ such that the following holds:

Given $i \in I$ and $A_1 \subset G_1^i \cap N$, $A_2 \subset G_2^i \cap N$, there is $i' \in I$ such that $A_1 = G_1^i \cap N$ and $A_2 = G_2^i \cap N$.

3. Sequences with an unconditional basic subsequence

From Theorem 1.4 we obtain a proof of Theorem 1.3 as follows.

Proof of Theorem 1.3. Let $M > 0$ be such that $(f_m)_m$ is uniformly bounded by M. By passing if necessary to a subsequence of $\{f_m : m \in \mathbb{N}\}$, we may assume that for all $m \in \mathbb{N}$, $\epsilon_m < \mu$. Consider the sequence $g_m : K \to \mathbb{R}^n$ with

$$g_m(x) = \begin{cases} f_m(x), & \text{if } \|f_m(x)\| \geq \mu, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\epsilon_m < \mu$, we have that g_m is continuous for all $m \in \mathbb{N}$ and moreover $\|f_m - g_m\| \leq \epsilon_m \to 0$. Therefore it suffices to find an unconditional basic subsequence for $\{g_m\}_{m\in\mathbb{N}}$.

For any $m \in \mathbb{N}$ and $1 \leq k \leq n$, let g_m^k be the k-coordinate function of g_m. i.e. $g_m(x) = (g_m^1(x), \dots, g_m^n(x))$. Next for $x \in K$ and $1 \leq k \leq n$ we define the following subsets of $\mathbb{N}$:

$$G_{k,1}^x = \{m \in \mathbb{N} : g_m^k(x) \geq \mu\}, \qquad G_{k,-1}^x = \{m \in \mathbb{N} : g_m^k(x) \leq -\mu\}.$$

Also let $G^x = \{m \in \mathbb{N} : \|g_m(x)\| \geq \mu\}$. Considering the supremum norm on $\mathbb{R}^n$, we have that $G^x = \bigcup_{i=-1,1} \bigcup_{k=1}^n G_{k,i}^x$.

It is easy to see that since $(g_n)_n$ are continuous and pointwise null, $\{G^x : x \in K\}$ is a compact family of finite subsets of $\mathbb{N}$.

By Theorem 1.4, there is $N = \{n_1 < n_2 < \dots\} \in [\mathbb{N}]$ such that for all $q \in \mathbb{N}$, $x \in K$, $k = 1, \dots, n$, $i = -1, 1$ and $A \subset G_{k,i}^x \cap \{n_1, \dots, n_q\}$ there is $x' \in K$ such that $G_{k,i}^{x'} \cap \{n_1, \dots, n_q\} = A$ and for all other combinations $(k', i') \neq (k, i)$, $G_{k,i'}^{x'} \cap \{n_1, \dots, n_q\} \subset A$. Using this, we next prove that $g_{n_1}, g_{n_2}, \dots$ is an unconditional basic sequence. So let $q \in \mathbb{N}$, $a_{n_1}, \dots, a_{n_q} \in \mathbb{R}$ and $A \subset \{n_1, \dots, n_q\}$. We try to estimate the norm of $\sum_{m\in A} a_m g_m$ in terms of the norm of $\sum_{\ell=1}^q a_{n_\ell} g_{n_\ell}$.

For some $x \in K$, we have that $\|\sum_{m\in A} a_m g_m\| = \|\sum_{m\in A} a_m g_m(x)\|$ and since $\|\sum_{m\in A} a_m g_m(x)\| = \sup\{|\sum_{m\in A} a_m g_m^k(x)| : k = 1, \dots, n\}$, for some $k = 1, \dots, n$ $\|\sum_{m\in A} a_m g_m\| = |\sum_{m\in A} a_m g_m^k(x)|$.

Let

$$A_{1,1} = \{m \in A : a_m \geq 0, g_m^k(x) > 0\}, \qquad A_{1,-1} = \{m \in A : a_m \geq 0, g_m^k(x) < 0, \},$$
$$A_{-1,1} = \{m \in A : a_m < 0, g_m^k(x) > 0\}, \quad A_{-1,-1} = \{m \in A : a_m < 0, g_m^k(x) < 0\}.$$

Since $A = \bigcup_{i=-1,1} \bigcup_{j=-1,1} A_{j,i}$, we get that for appropriate choose of i and j,

$$\Big\| \sum_{m\in A} a_m g_m \Big\| = \Big| \sum_{m\in A} a_m g_m^k(x) \Big| \leq 4 \Big| \sum_{m\in A_{j,i}} a_m g_m^k(x) \Big|.$$

Therefore

$$\Big\| \sum_{m\in A} a_m g_m \Big\| \leq 4 \Big| \sum_{m\in A_{j,i}} a_m g_m^k(x) \Big|.$$

Obviously, $A_{j,i} \subset G_{k,i}^x \cap \{n_1, \dots, n_q\}$. Therefore we can find $x' \in K$ such that $G_{k,i}^{x'} \cap \{n_1, \dots, n_q\} = A_{j,i}$ and for all other $(k', i') \neq (k, i)$, $G_{k',i'}^{x'} \cap \{n_1, \dots, n_q\} \subset A_{j,i}$. Note that in particular $G_{k,-i}^{x'} \cap \{n_1, \dots, n_q\} = \emptyset$, since a value of a function cannot be both positive and negative. Therefore setting $T_q = \{n_1, \dots, n_q\}$, we have

$$\begin{aligned}\Big\| \sum_{\ell=1}^{q} a_{n_\ell} g_{n_\ell} \Big\| &\geq \Big| \sum_{\ell=1}^{q} a_{n_\ell} g^k_{n_\ell}(x') \Big| = \Big| \sum_{m \in G^{x'}_{k,i} \cap T_q} a_m g^k_m(x') + \sum_{m \in G^{x'}_{k,-i} \cap T_q} a_m g^k_m(x') \Big| \\ &= \Big| \sum_{m \in A_{j,i}} a_m g^k_m(x') \Big| = \sum_{m \in A_{j,i}} |a_m| |g^k_m(x')| \geq \sum_{m \in A_{j,i}} |a_m| \mu \\ &= \frac{\mu}{M} \sum_{m \in A_{j,i}} |a_m| M \geq \frac{\mu}{4M} 4 \Big| \sum_{m \in A_{j,i}} a_m g^k_m(x) \Big| \\ &\geq \frac{\mu}{4M} \Big\| \sum_{m \in A} a_m g_m \Big\| \end{aligned}$$

which completes the proof. □

Proof of Theorem 1.2. To avoid confusion, we denote by $\|\cdot\|_X$ the norm of X, and by $\|\cdot\|$ the norm on $C(K, X)$, where for an $f \in C(K, X)$, $\|f\| = \sup\{\|f(k)\|_X : k \in K\}$.

Let $f_n(K) = \{w^1_n, w^2_n, \dots, w^{J_n}_n\}$ with $J_n \leq J$ be an enumeration of $f_n(K)$ such that $w^1_n, w^2_n, \dots, w^{J_n}_n$ are different from each other and moreover $\|w^1_n\|_X \leq \|w^2_n\|_X \leq \dots \leq \|w^{J_n}_n\|_X$. By the Pigeonhole Principle, we get that for some $m \leq J$ and for infinitely many n, $J_n = m$. Since each f_n is normalized, for all those n, $\|w^{J_n}_n\|_X = 1$. Therefore there exists an m such that $\|w^{m-1}_n\|_X$ forms a null sequence, but $\|w^m_n\|_X$ does not. Using this observation we next argue that we may assume, passing if necessary to a different sequence, that there is a $\mu > 0$ such that for all n, $w^1_n = 0$ and $\|w^2_n\|_X \geq \mu$.

Set $\epsilon_n = \|w^{m-1}_n\|_X \to 0$. By passing if necessary to a subsequence of $(f_n)_n$ we may assume that there is a $\mu > 0$ such that for all n, $\epsilon_n < \mu \leq \|w^m_n\|_X$. Now define $g_n : K \to X$ as follows:

$$g_n(k) = \begin{cases} 0, & \text{if } \|f_n(k)\|_X \leq \epsilon_n, \\ f_n(k), & \text{otherwise.} \end{cases}$$

It is easy to see that in this case every g_n is continuous, the cardinality of $g_n(K)$ is also bounded by J and for every $k \in K$ either $g_n(k) = 0$ or $\|g_n(k)\|_X \geq \mu$. Moreover, all but finitely many g_n take the value 0, since$(f_n)_n$ is pointwise null. Observe that $\|f_n - g_n\| \leq \epsilon_n \to 0$. Therefore the existence of an unconditional subsequence of $(g_n)_n$ implies the existence of an unconditional subsequence of $(f_n)_n$. So we may indeed assume that for all n, $w^1_n = 0$ and $\|w^2_n\|_X \geq \mu$.

For any $r \in [2, J]$ and $k \in K$, let $G^k_r = \{n \in \mathbb{N} : r \leq J_n \text{ and } f_n(k) = w^r_n\}$. Also set $G^k = \bigcup_{r=2}^{J} G^k_r = \{n \in \mathbb{N} : \|f_n(k)\|_X \geq \mu\}$. Since $(f_n)_{n\in\mathbb{N}}$ is a pointwise null sequence of continuous functions and for every $k \in K$, either $f_n(k) = 0$ or $\|f_n(k)\|_X \geq \mu$, we get that $\{G^k : k \in K\}$ is a compact family of finite subsets of $\mathbb{N}$. We use Theorem 1.4 to obtain an $N = \{n_1 < n_2 < \dots\} \in [\mathbb{N}]$ such that for any $k \in K$, $2 \leq r \leq J$, $q \in \mathbb{N}$ and $A \subset G^k_r \cap \{n_1, \dots, n_q\}$, there is $k' \in K$ such that $G^{k'}_r \cap \{n_1, \dots, n_q\} = A$ and for all $r' \neq r$, $G^{k'}_{r'} \cap \{n_1, \dots, n_q\} \subset A$. Observe that since for $r' \neq r$, $w^{r'}_n \neq w^r_n$, we have that $G^{k'}_r \cap G^{k'}_{r'} = \emptyset$ and therefore in this case $G^{k'}_{r'} \cap \{n_1, \dots, n_q\} = \emptyset$.

We next prove that $(f_n)_{n\in N}$ is an unconditional basic sequence. So let $q \in \mathbb{N}$, $a_{n_1}, a_{n_2}, \ldots, a_{n_q} \in \mathbb{R}$ and $A \subset \{n_1, \ldots, n_q\}$. We try to estimate the norm of $\sum_{n\in A} a_n f_n$ in terms of the norm of $\sum_{\ell=1}^q a_{n_\ell} f_{n_\ell}$.

For some $k \in K$,

$$\Big\| \sum_{n\in A} a_n f_n \Big\| = \Big\| \sum_{n\in A} a_n f_n(k) \Big\|_X. \tag{5}$$

Let $A = \bigcup_{r=2}^J A_r$ where $A_r = \{n \in A : r \le J_n \text{ and } f_n(k) = w_n^r\}$. Since for $r \neq r'$, $w_n^r \neq w_n^{r'}$ we get that A_r, $r = 2, \ldots, J$, are pairwise disjoint. Therefore there is an $r_0 \in [2, J]$ such that

$$\Big\| \sum_{n\in A} a_n f_n(k) \Big\|_X \le (J-1) \Big\| \sum_{n\in A_{r_0}} a_n f_n(k) \Big\|_X = (J-1) \Big\| \sum_{n\in A_{r_0}} a_n w_n^{r_0} \Big\|_X. \tag{6}$$

Since obviously $A_{r_0} \subset G_{r_0}^k \cap \{n_1, \ldots, n_q\}$, by our hypothesis for N there exists $k' \in K$ such that $G_{r_0}^{k'} \cap \{n_1, \ldots, n_q\} = A_{r_0}$ and for $r' \neq r_0$, $G_{r'}^{k'} \cap \{n_1, \ldots, n_q\} = \emptyset$. In this case for all $n \in \{n_1, \ldots, n_q\}$ if $f_n(k') \neq 0$, then $f_n(k') = w_n^{r_0}$. Therefore

$$\Big\| \sum_{\ell=1}^q a_{n_\ell} f_{n_\ell}(k') \Big\|_X = \Big\| \sum_{n\in A_{r_0}} a_n w_n^{r_0} \Big\|_X,$$

so that by (6) and (5) we get

$$\Big\| \sum_{n\in A} a_n f_n \Big\| \le (J-1) \Big\| \sum_{\ell=1}^q a_{n_\ell} f_{n_\ell} \Big\|$$

and this proves immediately that $(f_n)_{n\in N}$ is an unconditional basic sequence. □

Remark 3.1. Maurey-Rosenthal's example [8] shows that there is a compact K such that for a pointwise null sequence $f_n \in C(K)$, $\{f_n\}_n$ has no unconditional basic subsequence. In this case the range of every f_n is a countable set $\{0\} \cup \{r_k : k \in \mathbb{N}\}$ where $\{r_k\}_k$ is a null sequence. As we have mentioned in the Introduction, a slight modification of $(f_n)_n$ derives a sequence $(g_n)_n$ pointwise convergent to zero with no unconditional subsequence and such that every g_n has a finite range.

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